MTHSC 851 (Abstract Algebra)
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HW 10
Due Friday April 24th, 2009

(1) Let R and S be commutative rings, and let $f : R \to S$ be a ring homomorphism.
  (a) If f is surjective and I is an ideal of R, show that $f(I)$ is an ideal of S.
  (b) Show that part (a) is not true in general when f is not surjective.
  (c) Show that if f is surjective and R is a field, then S is a field as well.

(2) Let p be a fixed prime number, and consider the ring
$$R = \{\tfrac{a}{b} \colon a, b \in \mathbb{Z},\ (a,b) = 1,\ p \nmid b\}$$
with the usual operations of addition and multiplication of rational numbers.
  (a) Determine the group of units of R.
  (b) Prove that the principal ideal $(p) = pR$ is a maximal ideal of R, and in fact the only maximal ideal of R.

(3) Suppose R is an integral domain and $P \subseteq R$ is a prime ideal.
  (a) Show that both P and $R \setminus P$ are multiplicative semigroups.
  (b) If $S = R \setminus P$ show that $U(R_S) = R_S \setminus R_S P$. Conclude that $R_S P$ is the unique maximal ideal in R_S.

(4) (a) How many roots does $x^3 - x$ have in the ring $\mathbb{Z}/6\mathbb{Z}$?
  (b) What condition must hold for a commutative ring R so that the number of roots of a polynomial in $R[x]$ cannot exceed its degree?
  (c) Does your condition from (b) still hold if R is not commutative? Prove or give a counterexample.

(5) Let p be prime, and $k \geq 1$. Find $U(R)$ if $R = \mathbb{Z}_{p^k}[x]$. Prove all your claims.

(6) If R is a commutative ring with 1, $f(x) \in R[x]$, and $a \in R$, then we may substitute a for x and obtain $f(a) \in R$. Thus $f(x)$ determines a *polynomial function* $f \colon R \to R$.
  (a) If R is finite show that there must exist polynomials $f(x)$ and $g(x)$ in $R[x]$, with $f(x) \neq g(x)$, such that the associated polynomial functions f and g are identical, i.e., $f(a) = g(a)$ for all $a \in R$.
  (b) Find explicit examples of the phenomenon in (a) when $R = \mathbb{Z}_n$.
  (c) If R is an infinite integral domain show that the mapping $f(x) \mapsto f$ assiging to each polynomial in $R[x]$ its corresponding polynomial function is 1–1.

(7) (a) Prove that the ideal $I = \langle 2, x \rangle$ in $\mathbb{Z}[x]$ is not a principal ideal.
  (b) What is the quotient ring $\mathbb{Z}[x]/\langle 2, x \rangle$ isomorphic to?

(8) (a) Suppose that R is a commutative ring with identity, let $a_1, \ldots, a_n \in R$, and denote by I the ideal in the polynomial ring $R[x_1, \cdots, x_n]$ generated by the polynomials $x_1 - a_1, \ldots, x_n - a_n$. Formulate and prove necessary and sufficient conditions on R that will ensure that I is a maximal ideal.
  (b) What are the maximal ideals of $\mathbb{Z}[x]$? Of $\mathbb{Z}[x, y]$? (No proofs necessary)?