6. Categories, products, and coproducts

If G_1, G_2 are groups, then $G_1 \times G_2$ is a group, where the binary operation is $(x_1, x_2)(y_1, y_2) = (x_1 y_1, x_2 y_2)$.

This notion can easily be extended to an arbitrary finite product $G_1 \times G_2 \times \cdots \times G_k$, or even a countable product $G_1 \times G_2 \times \cdots$.

But what if we have uncountably many (or more) groups?

<u>Big idea</u>: Everything works if we define products as (co)universal properties.

<u>Def</u>: If $\{G_\alpha : \alpha \in A\}$ is a non-empty family of groups, then a <u>product</u> of the G_α's is a group P with a family of homomorphisms $p_\alpha : P \longrightarrow G_\alpha$, $\alpha \in A$, with the following universal property:

Given any group H, and homomorphisms $f_\alpha : H \to G_\alpha$, $\alpha \in A$, $\exists !\ f : H \longrightarrow P$ s.t. $p_\alpha f = f_\alpha$ $\forall \alpha \in A$.

$$\begin{tikzcd} H \arrow[rr, "f_\alpha"] \arrow[dr, dashed, "f"'] & & G_\alpha \\ & P \arrow[ur, "p_\alpha"'] & \end{tikzcd}$$

<u>Prop 6.1</u>: Let $\{G_\alpha : \alpha \in A\}$ be a non-empty family of groups. If a product $(P, \{p_\alpha\})$ exists, it is unique up to isomorphism, and each $p_\alpha : P \to G_\alpha$ is an epi.

<u>Pf</u>: Let $(P', \{p_\alpha'\})$ be another product. We have

$$\begin{tikzcd} P' \arrow[rr, "p_\alpha'"] \arrow[dr, dashed, "f"'] \arrow[rr, phantom, shift right=4ex, "p_\alpha' = p_\alpha f"] & & G_\alpha \\ & P \arrow[ur, "p_\alpha"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} P \arrow[rr, "p_\alpha"] \arrow[dr, dashed, "g"'] \arrow[rr, phantom, shift right=4ex, "p_\alpha = p_\alpha' g"] & & G_\alpha \\ & P' \arrow[ur, "p_\alpha'"'] & \end{tikzcd}$$

for each $\alpha \in A$, Thus, $p_\alpha = p_\alpha' g = p_\alpha f g \quad \forall \alpha \in A.$

Now, we have

$$\begin{tikzcd} P \arrow[rr, "p_\alpha"] \arrow[dr, dashed, "fg"'] & & G_\alpha \\ & P \arrow[ur, "p_\alpha"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} P \arrow[rr, "p_\alpha"] \arrow[dr, dashed, "1"'] & & G_\alpha \\ & P \arrow[ur, "p_\alpha"'] & \end{tikzcd}$$

thus $fg = 1_P$ (by uniqueness). Similarly, $gf = 1_{P'}$, so f & g are inverse isomorphisms. ✓

Now, we'll show each p_α is an epi.

Fix G_α, define $f_\beta : G_\alpha \to G_\beta$,

$$f_\beta(x) = \begin{cases} 1 \in G_\beta & x \in G_\beta \neq G_\alpha \quad \text{(trivial)} \\ x \in G_\beta & x \in G_\beta = G_\alpha \quad \text{(identity)} \end{cases}$$

Now, we have, for all β,

$$\begin{tikzcd} G_\alpha \arrow[rr, "f_\beta"] \arrow[dr, dashed, "f"'] & & G_\beta \\ & P \arrow[ur, "p_\beta"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G_\alpha \arrow[rr, "f_\alpha = 1_\alpha"] \arrow[dr, dashed, "f"'] & & G_\alpha \\ & P \arrow[ur, "p_\alpha"'] & \end{tikzcd}$$

Since $1_\alpha = p_\alpha f$ and 1_α is an epi, so is p_α. $\square$

<u>Thm 6.2</u>: If $\{G_\alpha : \alpha \in A\}$ is a nonempty family of groups, the the product of $\{G_\alpha : \alpha \in A\}$ exists.

<u>Pf</u>: We'll show that the cartesian product is the product.

Let $P = \prod \{G_\alpha : \alpha \in A\}$. Write elts as $(x_\alpha)_{\alpha \in A}$.

Binary operation: $(x_\alpha)(y_\alpha) = (x_\alpha y_\alpha)$. This is a group.

Define $p_\alpha : P \longrightarrow G_\alpha$ as the projection map

$(x_\beta) \longmapsto x_\alpha$.

Suppose $f_\alpha : H \longrightarrow G_\alpha$ is a homomorphism (all $\alpha \in A$).

Define $f : H \longrightarrow P$, $f(h) = (f_\alpha(h))_{\alpha \in A}$.

<u>Check</u>: $p_\alpha f = f_\alpha$ $\forall \alpha \in A$.

<u>Uniqueness</u>: Consider $\begin{tikzcd} H \arrow[rr, "f_\alpha"] \arrow[dr, "f"'] & & G_\alpha \\ & P \arrow[ur, "p_\alpha"'] & \end{tikzcd}$ and $\begin{tikzcd} H \arrow[rr, "f_\alpha"] \arrow[dr, "g"'] & & G_\alpha \\ & P \arrow[ur, "p_\alpha"'] & \end{tikzcd}$

i.e, $p_\alpha f = p_\alpha g$. Then $f(x)_\alpha = p_\alpha f(x) = p_\alpha g(x) = g(x)_\alpha$

$\Rightarrow f(x) = g(x)$ $\forall x \in H$ ✓ $\square$

Thus, $f(x) = g(x)$ $\forall x \in H$, and so f is unique, and P is a product. $\square$

If $A = \{1, 2, \ldots, n\}$ or $\{1, 2, 3, \ldots\}$ we write $G_1 \times G_2 \times \cdots \times G_n$ or $G_1 \times G_2 \times G_3 \times \cdots$. This is also called the <u>direct product</u>. The homomorphism p_α is called the <u>projection</u> of $\prod G_\alpha$ on the <u>direct factor</u> G_α.

<u>Thm 6.3</u>: Suppose $G_1, G_2 \leq G$ satisfying:

(i) $G_1, G_2 \trianglelefteq G$
(ii) $G_1 \cap G_2 = 1$
(iii) $G_1 G_2 = G$.

then $G \cong G_1 \times G_2$.

More generally, if $G_1, \ldots, G_n \leq G$ satisfying:

(i) $G_1, \ldots, G_n \trianglelefteq G$
(ii) $G_i \cap \langle \bigcup_{j \neq i} G_j \rangle = 1$ for each i
(iii) $G_1 G_2 \cdots G_n = G$,

then $G \cong G_1 \times G_2 \times \cdots \times G_n$.

<u>Pf</u> (Sketch, for $n=2$).

Since $G_1 \cap G_2 = 1$, each $x = x_1 x_2$ uniquely, where $x_i \in G_i$.

Also, $[x_1, x_2] = x_1^{-1} x_2^{-1} x_1 x_2 \in G_1 \cap G_2 = 1$ (by normality), thus $x_1 x_2 = x_2 x_1$.

Define our family of homomorphisms by $p_i(x) = x_i$. (check homom!)

Now, for any family of homoms $f_i : H \longrightarrow G_i$, define $f : H \longrightarrow G$ by $f(h) = f_1(h) f_2(h)$.

Then $p_i f = f_i$ (check!)

If $g : H \longrightarrow G$ is a homom. satisfying $p_i g = f_i$ (for each i),

then $p_i f = p_i g$. Then, for any $x \in H$, we have

$f(x) = p_1(f(x)) p_2(f(x)) = p_1(g(x)) p_2(g(x)) = g(x)$,

thus $f = g$ is unique.

Since products exist and are unique, and G is a product,

it follows that $G \cong G_1 \times G_2$. $\square$

When the assumption of Thm 6.3 hold, we say that G is the <u>internal direct product</u> of its subgps $G_1, \dots, G_n$.

<u>Some basic category theory</u>.

<u>Def</u>: A category $\mathcal{C}$ consists of

(I) A class of objects $Ob(\mathcal{C})$

(II) A class of morphisms $Hom(\mathcal{C})$ between objects, with

(i) Identity morphism $1_A : A \longrightarrow A$ for all $A \in Ob(\mathcal{C})$

(ii) Composition: $f \in Hom_{\mathcal{C}}(A, B)$, $g \in Hom_{\mathcal{C}}(B, C)$

$\Rightarrow g \circ f \in Hom_{\mathcal{C}}(A, C)$.

(iii) Associative: $h \circ (g \circ f) = (h \circ g) \circ f$.

Think of a category as a directed graph

Vertices $\longleftrightarrow$ objects

Edges $\longleftrightarrow$ morphisms.

<u>Examples</u> Set: sets with functions

Grp: Groups with homomorphisms

Ab: Abelian groups with homomorphisms

Vect_k: k-vector spaces with k-linear maps.

Top: Topological spaces with continuous functions.

<u>Def</u>: A morphism $f \in \text{Hom}_{\mathcal{C}}(A,B)$ is a

- <u>monomorphism</u> if $fg_1 = fg_2 \Rightarrow g_1 = g_2$
- <u>epimorphism</u> if $g_1 f = g_2 f \Rightarrow g_1 = g_2$
- <u>isomorphism</u> if $\exists\, g \in \text{Hom}_{\mathcal{C}}(B,A)$ with $fg = 1_B$ and $gf = 1_A$.

In this case, we say that A and B are <u>equivalent</u>.

<u>Def</u>: Let $\mathcal{C}$ be a category and $\{A_i : i \in I\}$ a family of objects of $\mathcal{C}$. A <u>product</u> for $\{A_i \mid i \in I\}$ is an object P of $\mathcal{C}$ with a family of morphisms $\{p_i : P \longrightarrow A_i \mid i \in I\}$ such that for any object B and family of morphisms $\{f_i : B \longrightarrow A_i \mid i \in I\}$, $\exists!$ $f \in \text{Hom}_{\mathcal{C}}(B,P)$ s.t. $p_i f = f_i$ $\forall i \in I$.

Denote this as $P = \prod_{i \in I} A_i$

Def: A <u>coproduct</u> (or <u>sum</u>) for $\{A_i \mid i \in I\}$ is an object $S \in Ob(\mathcal{C})$ with a family of morphisms $\{\iota_i : A_i \to S \mid i \in I\}$ such that for any object $B \in Ob(\mathcal{C})$ and family of morphisms $\{f_i : A_i \to B \mid i \in I\}$, $\exists!\ F \in Hom_{\mathcal{C}}(S, B)$ such that $F\iota_i = f_i \quad \forall i \in I$

Denote this by $S = \coprod_{i \in I} X_i$, or $S = \bigoplus_{i \in I} X_i$.

$$\begin{tikzcd}
 & B \arrow[dl, "f_1"'] \arrow[d, dashed, "F"] \arrow[dr, "f_2"] & \\
X_1 & X_1 \times X_2 \arrow[l, two heads, "p_1"] \arrow[r, two heads, "p_2"'] & X_2
\end{tikzcd}$$

Product of X_1 & X_2

$$\begin{tikzcd}
 & B & \\
X_1 \arrow[ur, "f_1"] \arrow[r, hook, "\iota_1"'] & X_1 \oplus X_2 \arrow[u, dashed, "f"] & X_2 \arrow[ul, "f_2"'] \arrow[l, hook', "\iota_2"]
\end{tikzcd}$$

Coproduct of X_1 & X_2.

<u>Examples</u>:

<u>Category</u>	<u>Objects</u>	<u>Morphisms</u>	<u>Product</u>	<u>Coproduct</u>
Set	Sets	Functions	Cartesian prod.	Disjoint union
Top	Top. spaces	Cont. maps	Cartesian prod.	Disjoint union
Grp	Groups	Homom.	Direct product	Free product
Ab	Abelian gps	Homom.	Direct product	Direct sum

Products and coproducts are defined via <u>universal mapping properties</u>, i.e., by the existence of certain uniquely determined morphisms. This notion can be generalized.

<u>Def</u>:
- An object $I \in Ob(\mathcal{C})$ is <u>universal</u> (or initial) if for each $C_i \in Ob(\mathcal{C})$, $\exists!\, p_i \in Hom_{\mathcal{C}}(I, C_i)$.
- An object $T \in Ob(\mathcal{C})$ is <u>couniversal</u> (or terminal) if for each $C_i \in Ob(\mathcal{C})$, $\exists!\, \iota_i \in Hom_{\mathcal{C}}(C_i, T)$.
- An universal and initial object is a <u>zero object</u>.

<u>Examples</u>

	<u>Category</u>	<u>Universal (initial)</u>	<u>Couniversal (terminal)</u>
	Set	$\emptyset$	$\{x\}$ (any x)
	Top	$\emptyset$	$\{x\}$
	Grp	1	1

<u>Thm 6.4</u>: Any two universal objects are equivalent.

<u>Pf</u>: (sketch) Let I and J be universal.

$$\begin{tikzcd}
I \arrow[rr, "g \circ f = 1_I"] \arrow[dr, "\exists! f"'] & & I \\
& J \arrow[ur, "\exists! g"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
J \arrow[rr, "f \circ g = 1_J"] \arrow[dr, "g"'] & & J \\
& I \arrow[ur, "f"'] &
\end{tikzcd}$$

Thus $fg = 1_J$ and $gf = 1_I \Rightarrow I$ & J are equivalent $\square$

<u>Note</u>: Similarly, we can show that any two couniversal objects are equivalent.

* For suitably chosen categories (co)products are the (co)universal objects.

<u>Example</u> Let $\{A_i \mid i \in I\}$ be a family of objects in $\mathcal{C}$.

Define a new category $\mathcal{D}$ as follows:

- Objects: Pairs $(B, \{f_i \mid i \in I\})$ where $f_i \in \mathrm{Hom}_{\mathcal{C}}(B, A_i)$.
- Morphisms: Elements $h \in \mathrm{Hom}_{\mathcal{C}}(B, D)$ s.t. $g_i \circ h = f_i \quad \forall i \in I$

$$\begin{tikzcd} B \arrow[rr, "f_i"] \arrow[dr, dashed, "h"'] & & A_i \\ & D \arrow[ur, "g_i"'] & \end{tikzcd}$$

<u>Check</u>: In this category, the couniversal (terminal) object is $\left(\prod_{i \in I} A_i, \{p_i \mid i \in I\}\right)$. (If $\prod_{i \in I} A_i$ exists in $\mathcal{C}$).

<u>Cor</u>: Products and coproducts are unique up to equivalence (when they exist).

<u>Example</u>: Category Ab (abelian groups)

The <u>direct product</u> (Cartesian product) of abelian groups exists, and is an abelian group, thus it is a product in Ab.

Def: The <u>weak direct product</u> of a family of groups $\{G_i \mid i \in I\}$, denoted $\prod_{i \in I}^{w} G_i$ is the set of $(x_\alpha)_\alpha$ such that $x_i = e_i$ (the identity) in G_i for all but a finite number of $i \in I$. If all G_i's are abelian, we write this as $\sum_{i \in I} G_i$ and call it the <u>direct sum</u>.

Thm 6.5: Let $\{A_i : i \in I\}$ be a family of abelian groups. Then $\sum_{i \in I} A_i$ is a coproduct in the category of abelian groups.

Pf: HW.

Note. If $|I| < \infty$, then the direct sum & direct product of abelian groups $\{A_i : i \in I\}$ coincide.

Example Category Grp (groups).

Again, the product of groups is simply the direct product.

Def: Given a family of groups $\{G_i : i \in I\}$, let $X = \coprod_{i \in I} G_i$.
A word on X is any sequence $(a_1, a_2, \ldots)$ such that $a_i \in X \amalg \{1\}$ and for some index n, $a_i = 1$ $\forall i \geq n$. A word is reduced if

(i) No $a_i \in X_i$ is the identity (in G_i)
(ii) a_i & a_{i+1} are never in the same group G_j.
(iii) $a_k = 1 \Rightarrow a_i = 1$ $\forall i \geq k$.

Note:
- $(1, 1, 1, \ldots)$ is reduced
- We may write a reduced word uniquely as $a_1 a_2 a_3 \cdots a_n = (a_1, a_2, a_3, \ldots, a_n, 1, 1, \ldots)$, $a_i \in X$.

Def: Let $\prod^*_{i \in I} G_i$ (or $G_1 * G_2 * \cdots * G_n$ if $|I| < \infty$) be the set of reduced words on X. This is the free product of $\{G_i : i \in I\}$, with binary operation concatenation (and possibly cancelation, e.g., $a_i a_i^{-1}$).

Example: Let $G_1 = \mathrm{Perm}(\{a,b,c\})$, $G_2 = \mathrm{Perm}(\{1,2,3\})$.

$g = (abc)(12)(ac)(321)(ab)(13) \in G_1 * G_2$

$h = (13)(abc) \in G_1 * G_2$

$gh = (abc)(12)(ac)(321)\overbrace{(ab)\cancel{(13)}\cancel{(13)}(abc)}^{=(bc)} = (abc)(12)(ac)(321)(bc)$

Note that $|G_1 \times G_2| = 36$ but $|G_1 * G_2| = \infty$.

Also, $\iota_1 : G_1 \hookrightarrow G_1 * G_2$, $\iota_1(x) = x$ (e.g., $\iota_1((ab)) = ab$) is a monomorphism.

Def: For each $k \in I$, $\iota_k : G_k \hookrightarrow \prod_{i \in I}^{*} G_i$, defined by $\iota_k(a) = (a, 1, 1, \ldots)$ is a monomorphism of groups.

Thm 6.6: $\prod_{i \in I}^{*} G_i$ is a coproduct in the category of groups.

Pf: HW.

Def: Let A, A_1, A_2, be objects in a category $\mathcal{C}$ and let $f_i \in \mathrm{Hom}_{\mathcal{C}}(A, A_i)$ for $i = 1, 2$. A pushout (or fiber coproduct) for (A, A_1, A_2, f_1, f_2) is a commutative diagram with the following property:

$$\begin{tikzcd}
B & A_1 \arrow[l, "g_1"'] \\
A_2 \arrow[u, "g_2"] & A \arrow[l, "f_2"] \arrow[u, "f_1"']
\end{tikzcd}$$

(*) For any object $C \in Ob(\mathcal{C})$ and morphisms $h_i \in Hom_{\mathcal{C}}(A_i, C)$ s.t. if $h_1 f_1 = h_2 f_2$, $\exists!$ $h \in Hom_{\mathcal{C}}(B, C)$ such that $hg_i = h_i$ for $i = 1, 2$.

$$\begin{tikzcd}
C & & \\
& B \arrow[ul, dashed, "h"] & A_1 \arrow[l, "g_1"'] \arrow[ull, bend right, "h_1"'] \\
& A_2 \arrow[u, "g_2"] \arrow[uul, bend left, "h_2"] & A \arrow[l, "f_2"] \arrow[u, "f_1"']
\end{tikzcd}$$

Prop: Suppose

$$\begin{tikzcd}
B' & A_1 \arrow[l, "g_1'"'] \\
A_2 \arrow[u, "g_2'"] & A \arrow[l, "f_2"] \arrow[u, "f_1"']
\end{tikzcd}$$

is another pushout for (A, A_1, A_2, f_1, f_2). Then $B' \cong B$.

Pf: HW.

Examples

[1] $\mathcal{C} = \mathbf{Set}$. Let $A = A_1 \cap A_2$, $f_i : A \hookrightarrow A_i$ inclusions.

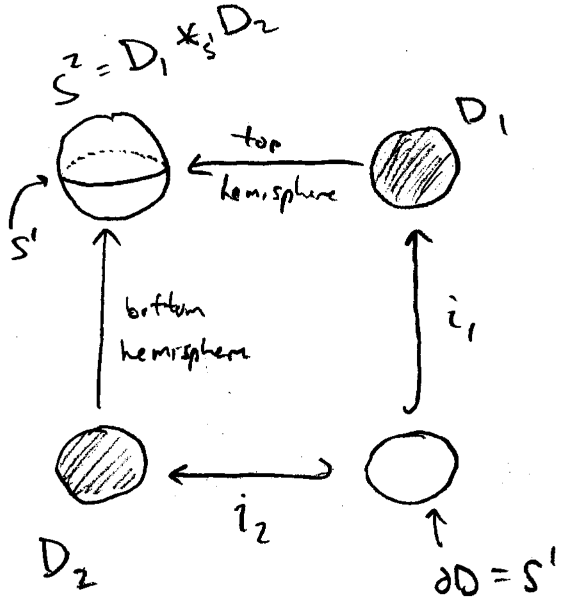
Then the pushout of (A, A_1, A_2, f_1, f_2) is

$$\begin{tikzcd}
A_1 \cup A_2 & A_1 \arrow[l, "g_1"'] \\
A_2 \arrow[u, "g_2"] & A_1 \cap A_2 \arrow[l, "f_2"] \arrow[u, "f_1"']
\end{tikzcd}$$

Think: $A_1 \sqcup A_2$ with $f_1(A)$ & $f_2(A_2)$ identified.

12 $\underline{\mathcal{C} = \text{Top}}$ (or Set)

Consider 2 disjoint closed disks, D_1 & D_2. They have boundary circle $\partial D_i = S^1$. The pushout of $(S^1, D_1, D_2, i_1, i_2)$, where $i_i : S^1 \hookrightarrow D_i$ are inclusion maps, is the 2-sphere.



Think: "glue 2 disks along their boundary circle."

Question: what do we get if $i_2 : S^1 \to D_2$ is the map that "wraps S^1 around ∂D_2 twice"?

13 $\underline{\mathcal{C} = \text{Grp}}$ If the maps $f_{1,2}$ are injective, then the pushout is the <u>free product with amalgamation</u>, and

$$\begin{tikzcd} B & G_1 \arrow[l] \\ G_2 \arrow[u] & H \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

is the quotient $G_1 *_H G_2 = (G_1 * G_2)/N$ where $N = \langle f_1(h) f_2(h)^{-1} : h \in H \rangle$.

<u>Note</u>: If $H = 1$, then this is just the free product $G_1 * G_2$. Think how this generalizes

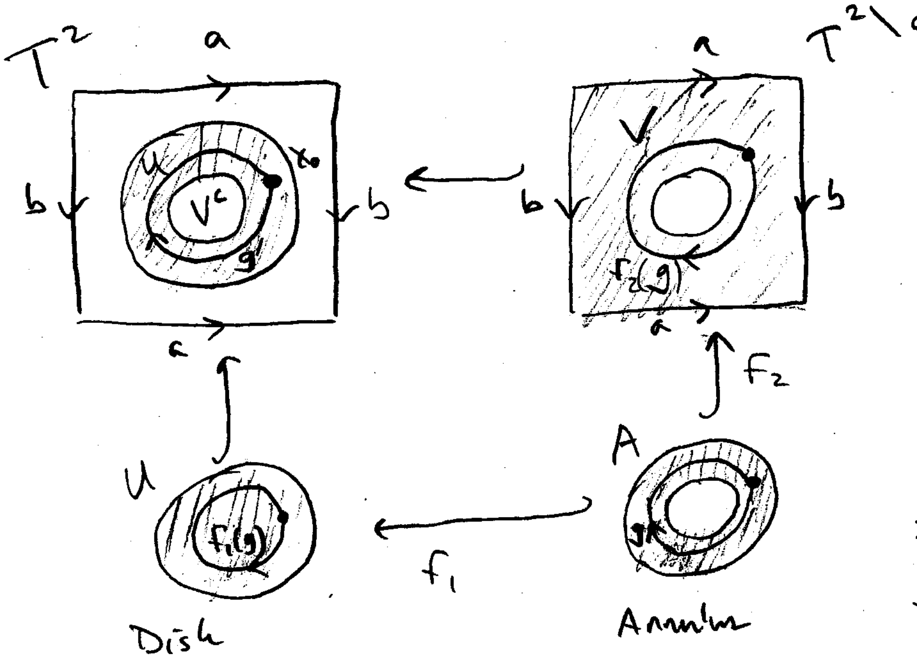
"The pushout, without A and $f_i : A \to A_i$, simply reduces to the coproduct of $\{A_i : i \in I\}$"

Application: Seifert-Van Kampen Theorem (algebraic topology).

Sketch of main idea: let $X = U \cup V$ and $A = U \cap V$ be path-connected top. spaces. The fundamental group of X is $\pi_1(X, x_0) \cong \pi_1(U, x_0) *_{\pi_1(A, x_0)} \pi_1(V, x_0)$.

Example: T^2 = torus 

$T^2 \setminus$ disk. g retracts to $aba^{-1}b^{-1}$ in V



$T^2 = U \cup V$, where U is a disk, and V deformation-retracts to the boundary of the square, which is $S^1 \vee S^1$ 

The annulus A def-retracts to S^1. $\pi_1(S^1, x_0) \cong \mathbb{Z} = \langle g \rangle$ and $\pi_1(S^1 \vee S^1, x_0) \cong \mathbb{Z} * \mathbb{Z} = \langle a, b \rangle$.

Seifert-Van Kampen: This pushout of top. spaces carries over to a pushout of groups (This is a functor: Top $\to$ Grp).

$$\begin{tikzcd}
\underset{\langle a,b \mid aba^{-1}b^{-1}\rangle \cong \mathbb{Z}\times\mathbb{Z}}{\pi_1(T^2, x_0)} & \underset{\langle a,b \mid \rangle \cong \mathbb{Z}*\mathbb{Z}}{\pi_1(S^1 \vee S^1, x_0)} \arrow[l] \\
\underset{1}{\pi_1(D^2, x_0)} \arrow[u] & \underset{\langle g \rangle \cong \mathbb{Z}}{\pi_1(A, x_0)} \arrow[l] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
1 & aba^{-1}b^{-1} \arrow[l, mapsto] \\
1 \arrow[u, mapsto] & g \arrow[l, mapsto] \arrow[u, mapsto]
\end{tikzcd}$$