8. Free groups, free objects, and group presentations

Goal of this section: You may have seen a group presentation like the following:

(i) $D_4 \cong \langle r, s \mid r^4 = 1,\ s^2 = 1,\ rs = sr^3 \rangle$.

(ii) $\mathbb{Z} \times \mathbb{Z} \cong \langle a, b \mid ab = ba \rangle$

(iii) $\mathbb{Z}_5 \cong \langle x \mid x^5 = 1 \rangle$.

But what does this "really" mean?

For example, take $G = \{1\}$, $r = s = 1$. Then r & s certainly "satisfy" the presentation in (i), but clearly, $D_4 \not\cong \{1\}$.

Or take $G = \{-1, 1\} \cong \mathbb{Z}_2$, $r = 1$, $s = -1$. Again, $r^4 = 1$, $s^2 = 1$, $rs = sr^3$, but $D_4 \not\cong \{-1, 1\}$.

Thus, we need to formalize what a group presentation really is. To do this, we'll need to introduce the notion of a free group, and we'll have to start from scratch with a free semigroup.

Recall: A semigroup is a non-empty set with an associative binary operation (Think: "group w/o identity or inverses").

A homomorphism between semigroups X, Y is a function $f: X \to Y$ such that $f(x_1 x_2) = f(x_1) f(x_2)$ $\forall x_{1,2} \in X$.

X and Y are isomorphic if $\exists$ 1-1 & onto homom. $f: X \to Y$.

Def: A semigroup X is free on a set S if there is a function $j: S \longrightarrow X$ s.t. for any other semigroup Y and function $k: S \longrightarrow Y$, there is a unique homom. $f: X \longrightarrow Y$ s.t. $fj = k$.

$$\begin{tikzcd} S \arrow[rr, "j"] \arrow[dr, "k"'] & & X \arrow[dl, dashed, "f"] \\ & Y & \end{tikzcd}$$

Prop 8.1: If a free semigroup exists on S, it is unique up to isomorphism.

Pf: Exercise (HW #7).

Example: Let $S = \{1\}$ and $\mathbb{N} = \{1, 2, 3, \dots\}$. Then $(\mathbb{N}, +)$ is a free semigroup on S with $j(1) = 1$. (Check!)

Thm 8.2: If $S \neq \emptyset$, then there exists a free semigroup X on S.

Pf: We will construct it explicitly.

Set $X = S \cup (S \times S) \cup (S \times S \times S) \cup \cdots$ ("all finite words over S")

Define a binary operation of concatenation:

$$(a_1, a_2, \dots, a_m)(b_1, \dots, b_k) = (a_1, \dots, a_m, b_1, \dots, b_k)$$

This is associative, thus X is a semigroup.

Let $j: S \longrightarrow X$ be the inclusion map: $j(x) = x$.

Claim: X is free on S.

To show this, let Y be a semigroup and $k: S \longrightarrow Y$ a function.

Define $f: X \to Y$ by

$$f(a_1, \dots, a_m) = k(a_1) \cdots k(a_m)$$

$$\begin{tikzcd} S \arrow[rr, hook, "j"] \arrow[dr, "k"'] & & X \arrow[dl, dashed, "f"] \\ & Y & \end{tikzcd}$$

<u>Check</u>: f is a homom. & $fj = k$. ✓

<u>Uniqueness</u>: Say $g: X \to Y$ is another homom s.t. $fj = gj = k$.

Then
$$\begin{aligned} g(a_1, \dots, a_m) &= g(ja_1, ja_2, \dots, ja_m) \\ &= g(ja_1) \cdots g(ja_m) \\ &= k(a_1) \cdots k(a_m) \\ &= f(ja_1) \cdots f(ja_m) = f(a_1, \dots, a_m) \Rightarrow f = g. \checkmark \end{aligned}$$

Since $\exists!$ f s.t. $fj = k$, X is a free semigroup on S. $\square$

<u>Prop 8.3</u>: (Quotient semigroups & their universal property).

Suppose Y is a semigroup and R an equiv. relation such that $\boxed{xRy \text{ \& } zRw \Rightarrow xz R yw}$. Then Y/R is a semigroup if we define $cl_R(x) \cdot cl_R(y) = cl_R(xy)$.

$$\begin{tikzcd} Y \arrow[rr, "q"] \arrow[dr, "h"'] & & Y/R \arrow[dl, dashed, "f"] \\ & Z & \end{tikzcd}$$

<u>Universal property</u>: If Z is another semigroup and $h: Y \to Z$ a homom., then $\exists!$ homom $f: Y/R \to Z$ s.t. $fq = h$ iff "$xRy \Rightarrow h(x) = h(y)$."

i.e., "every homomorphism respecting R factors through Y/R"

<u>Pf</u>: Exercise. (Define f in the obvious way & show that it works).

Def: A group F is <u>free</u> on a nonempty set S if $\exists$ function $\phi : S \to F$ s.t. if G is any other group and $\theta : S \to G$ any function, then $\exists!$ homom $f : F \to G$ s.t. $f\phi = \theta$.

$$\begin{tikzcd} S \arrow[rr, "\phi"] \arrow[dr, "\theta"'] & & F \arrow[dl, dashed, "f"] \\ & G & \end{tikzcd}$$

* We will show that free groups (if they exist), are unique up to isomorphism, and then we'll show they exist by constructing them from free semigroups, as quotients.

<u>Prop 8.4</u>: (Uniqueness). If a free group F exists on a nonempty set S, then F is unique up to isomorphism, and ϕ is 1-1.

<u>Pf</u>: <u>Uniqueness</u>: Exercise.

<u>Injectivity of ϕ</u>: Suppose ϕ were not 1-1, and $a \neq b$ but $\phi(a) = \phi(b)$.

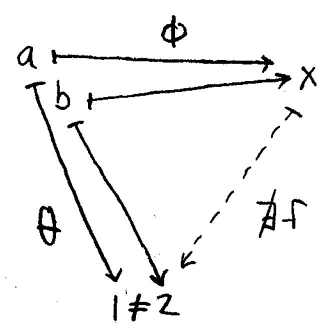
Define $\theta : S \to \mathbb{Z}$

as:
$$a \mapsto 1$$
$$b \mapsto 2$$
$$c \mapsto 0 \quad c \neq a, b.$$

$$\begin{tikzcd} S \arrow[rr, "\phi"] \arrow[dr, "\theta"'] & & F \arrow[dl, dashed, "f?"] \\ & \mathbb{Z} & \end{tikzcd}$$

Then $1 = \theta(a) = f\phi(a) = f\phi(b) = \theta(b) = 2$. $\lightning$

Thus, no such $f : F \to \mathbb{Z}$ can exist with $f\phi = \theta$, so F is not free.



Thm 8.5: (Existence) If $S \neq \emptyset$, then there is a free group F on S.

Pf: Choose a set S' with $|S'| = |S|$, $S \cap S' = \emptyset$, and put $T = S \cup S'$ (S' will serve as the "inverses" of elts in S).

Let $s \longmapsto s'$ be a 1-1 correspondence b/w S and S', and $s' \longmapsto (s')' = s'' = s$ the inverse map $S' \to S$.

Thus, $t \longmapsto t'$ is a bijection $T \longrightarrow T$.

Let X be the free semigroup on T (exists by Thm 8.2).

If G is a group and $g: X \to G$ a homomorphism, call g proper if $g(s') = g(s)^{-1}$ for all $s \in S$.

It follows easily that $g(t') = g(t)^{-1}$ for all $t \in T$.

[Motivation: For any group homomorphism $g: G \to H$, $g(x^{-1}) = g(x)^{-1}$ $\forall x \in G$]

Define a relation R on X where

$$xRy \quad \text{iff} \quad \text{"} g: X \to G \text{ proper} \implies g(x) = g(y) \text{"}$$

Check:
- R is an equiv. relation on X
- xRy and $zRw \Rightarrow xz\,R\,yw$.

Therefore, $F = X/R$ is a semigroup and $\mathfrak{z}: X \longrightarrow X/R$ is a homom. (by Prop 8.3). Write $\bar{x} = \mathfrak{z}(x)$.

Claim: (i) F is a group
(ii) F is free on S.

Pf of claim:

(i) F is a group:

Choose $s \in S$, $x \in X$ (under natural inclusion, $s, s' \in X$).

If $g: X \longrightarrow G$ is proper, then $g(ss') = 1$, so $g(xss') = g(x)$.

By definition of R, $xRxss' \iff \bar{x} = \overline{xss'} = \bar{x}\,\overline{ss'}$.

Similarly, $\overline{ss'}\,\bar{x} = \bar{x}$, so $\overline{ss'} \in 1_F = 1$. ✓

For $x = a_1 a_2 \cdots a_k$, $a_i \in T$, write $y = a_k' a_{k-1}' \cdots a_2' a_1'$

If $g: X \longrightarrow G$ is proper, then

$$g(xy) = g(a_1) \cdots g(a_k)\, g(a_k') \cdots g(a_1')$$
$$= g(a_1) \cdots g(a_k)\, g(a_k)^{-1} \cdots g(a_1)^{-1}$$

Thus, $xyRaa' \Rightarrow \overline{xy} = \bar{x}\bar{y} = \overline{aa'} = 1_F$.

Similarly, $\bar{y}\bar{x} = 1_F$ ✓

Since F contains an identity and inverses exist, F is a group. ✓

* Think of F as the group of finite words over $S \cup S^{-1}$ (the set of generators S and their inverses, S^{-1}), where the binary operation is concatenation.

(ii) <u>F is free on S</u>:

Let $S \overset{i}{\hookrightarrow} T \overset{j}{\hookrightarrow} X$ be inclusion maps.

Define $\phi : S \longrightarrow F$ by $\phi = qji$.

Now, let G be a group and $\theta : S \longrightarrow G$ any function.

Extend θ to $\theta' : T \longrightarrow G$ by setting $\theta'(s') = \theta(s)^{-1}$ $\forall s \in S$.

<u>Goal</u>: Show $\exists!$ homom. $f : X \longrightarrow G$ s.t. $f\phi = \theta$.

Since X is free on T, $\exists!$ homom. $\gamma : X \longrightarrow G$ s.t. $\gamma j = \theta'$

$$
\begin{tikzcd}
S \arrow[r, hook, "i"] \arrow[rrr, bend left, "\phi"] \arrow[dr, "\theta"'] & T \arrow[r, hook, "j"] \arrow[d, "\theta'"] & X \arrow[r, two heads, "q"] \arrow[dl, dashed, "\gamma"] & F = X/R \arrow[dll, dashed, "f"] \\
& G & &
\end{tikzcd}
$$

By Prop. 8.3, $\exists!$ homom. $f : F \longrightarrow G$ s.t. $fq = \gamma$.

<u>Note</u>: $f\phi = fqji = \gamma ji = \theta' i = \theta$.

Need uniqueness of f: Suppose $\exists\, h : F \longrightarrow G$ s.t. $h\phi = \theta$.

<u>Note</u>: $hqj = \theta'$, because

$$hqj(s') = hq(s)^{-1} = (hqj(a))^{-1} = \theta(a)^{-1} = \theta'(a').$$

Now, $hq = \gamma = fq$, so by Prop. 8.3, $h = f$. $\square$

Since ϕ is 1-1, we identify $s \in S$ with $\phi(s) \in F$, and just say $S \subset F$.

The elements of S are the <u>generators</u> of F, and we write $F = F_S = \langle S \rangle$.

<u>Note</u>: By definition, any function $\theta : S \longrightarrow G$ (for arbitrary G) can be extended <u>uniquely</u> to a homom $f : F_S \longrightarrow G$ s.t. $f\phi = \theta$.

$$\begin{tikzcd} S \arrow[rr, "\phi"] \arrow[dr, "\theta"'] & & F_S \arrow[dl, dashed, "f"] \\ & G & \end{tikzcd}$$

<u>Examples</u>:

(1) $|S| = 1$, say $S = \{s\}$, and let $\phi : S \hookrightarrow \mathbb{Z}$, $s \longmapsto 1$.

Let $\theta : S \longrightarrow G$ be <u>any</u> map;

say $\theta(s) = g$.

Then the homom. $f : \mathbb{Z} \longrightarrow G$, $1 \longmapsto g$

is the unique homom. s.t. $f\phi = \theta$.

Thus, $\mathbb{Z}$ is free on S.

$$\begin{tikzcd} S \arrow[rr, "\phi"] \arrow[dr, "\theta"'] & & \mathbb{Z} \arrow[dl, dashed, "f"] \\ & G & \end{tikzcd} \qquad \begin{tikzcd} s \arrow[rr, mapsto, "\phi"] \arrow[dr, mapsto, "\theta"'] & & 1 \arrow[dl, dashed, mapsto, "f"] \\ & g & \end{tikzcd}$$

<u>Note</u>: $\langle g \rangle \cong \mathbb{Z}$ or $\mathbb{Z}_n$, thus every cyclic group is the <u>quotient</u> of the <u>free group</u> on $\{s\}$ (<u>one generator</u>).

(2) Let $S = \{a, b\}$.

$$\begin{tikzcd} \{a,b\} \arrow[rr, hook, "\phi"] \arrow[dr, "\theta"'] & & F_S \arrow[dl, dashed, "f"] \\ & \langle (12), (123) \rangle = S_3 & \end{tikzcd}$$

Note that S_3 is not cyclic; it has <u>two</u> generators: $S_3 = \langle (12), (123) \rangle$.

Let $\phi : \{a, b\} \to F_S$ be the inclusion map.

Define
$$\begin{aligned} \theta : \{a,b\} &\longrightarrow S_3 = \langle (12), (123) \rangle. \\ a &\longmapsto (12) \\ b &\longmapsto (123). \end{aligned}$$

The free group F_S is the set of all words over $S = \{a, b\}$ under concatenation, which we write as $\langle a, b \mid \ \rangle$ (2 generators, no relations).

The map $\theta : S \to S_3$ extends to a unique homom. $f : F_S \to S_3$.

<u>Big idea</u>: The group S_3 is <u>generated</u> by 2 elements, and is a <u>quotient</u> of the free group F_S, where $|S| = 2$.

More generally, if $G = \langle S \rangle$ and $|S| = n$, then $\exists$ homom. $F_S \to G$ i.e, every group is the quotient of a free group.

<u>Thm 8.6</u>: Suppose $S, U \neq \emptyset$. Then $F_S \cong F_U$ iff $|S| = |U|$.

<u>Pf</u>: ($\Rightarrow$). <u>Case 1</u>: $|S| < \infty$.

$$\begin{tikzcd} S \arrow[rr, hook, "\phi"] \arrow[dr, "\theta"'] & & F_S \arrow[dl, dashed, "f"] \\ & \{\bar{0}, \bar{1}\} \cong \mathbb{Z}_2 & \end{tikzcd}$$

Since $F_S \cong F_U$, they have the same number of index-2 subgroups.

Every surjection $S \twoheadrightarrow \mathbb{Z}_2$ uniquely defines an index-2 subgroup (the kernel of $F_S \xrightarrow{f} \mathbb{Z}_2$).

Thus F_S has $2^{|S|} - 1$ index-2 subgps, & F_U has $2^{|U|} - 1$ $\Rightarrow |S| = |U|$. ✓

Case 2: $|S| = \infty$. Set $T = S \cup S^{-1}$, so $|T| = |S|$.

$$|F_S| \leq 1 + |T| + |T \times T| + |T \times T \times T| + \dots = \aleph_0 |T| = |S|.$$

Therefore, $|F_S| = |S|$, and so $|S| = |F_S| = |F_U| = |U|$ ✓

$(\Leftarrow)$ Suppose $h: S \to U$ is a bijection,

$$\begin{tikzcd}
F_S \arrow[r, dashed, "f"] & F_U \arrow[r, dashed, "g"] & F_S \\
S \arrow[u, "\phi_S"] \arrow[r, "h"'] \arrow[ur, "\phi_U h"] & U \arrow[u, "\phi_U"] \arrow[r, "h^{-1}"'] \arrow[ur, "\phi_S h^{-1}"] & S \arrow[u, "\phi_S"']
\end{tikzcd}$$

$\phi_S : S \hookrightarrow F_S$, $\phi_U : U \hookrightarrow F_U$.

Then $\phi_U f : S \hookrightarrow F_U$, so $\exists!$

homom. $f : F_S \to F_U$ s.t. $f\phi_S = \phi_U h$.

Similarly, $\phi_S h^{-1} : U \hookrightarrow F_S$, so $\exists!$

homom $g : F_U \to F_S$ s.t. $g\phi_U = \phi_S h^{-1}$.

$$\begin{tikzcd}
S \arrow[r, "\phi_S"] \arrow[dr, "\phi_S"'] & F_S \arrow[d, dashed, "1_{F_S} = gf"] \\
 & F_S
\end{tikzcd}$$

Now, we have $gf : F_S \to F_S$ satisfying

$\phi_S = gf\phi_S$, but also $1_{F_S} : F_S \to F_S$ satisfying

$\phi_S = 1_{F_S}\phi_S$. By uniqueness, $gf = 1_{F_S}$.

Similarly, $fg = 1_{F_U}$, so f & g are inverse isomorphisms, and

$F_S \cong F_U$. □

Def: The <u>rank</u> of a free group is the cardinality of any generating set.

Thm: Subgroups of free groups are free.

Thm: If $1 < |S|, |U| \leq \aleph_0$, then $\exists$ embedding $F_S \hookrightarrow F_U$.

Proofs: Require algebraic topology (covering spaces).

Not surprisingly, the concept of a "free" object can be defined in a categorical setting.

Def: A concrete category is a category $\mathcal{C}$ where the objects $A \in Ob(\mathcal{C})$ have an underlying set structure, $\sigma(A)$, and

(i) Every $f \in Hom_{\mathcal{C}}(A,B)$ is a function on the underlying sets:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathfrak{F}"'] & B \arrow[d, "\mathfrak{F}"] \\ \mathfrak{F}(A) \arrow[r, "\mathfrak{F}(f)"] & \mathfrak{F}(B) \end{tikzcd}$$

(ii) The identity morphism is the identity function on the sets:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "\mathfrak{F}"'] & A \arrow[d, "\mathfrak{F}"] \\ \mathfrak{F}(A) \arrow[r, "id"] & \mathfrak{F}(A) \end{tikzcd}$$

(iii) Composition of functions agree with composition of functions on the sets:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathfrak{F}"'] & B \arrow[r, "g"] \arrow[d, "\mathfrak{F}"] & C \arrow[d, "\mathfrak{F}"] \\ \mathfrak{F}(A) \arrow[r, "\mathfrak{F}(f)"'] & \mathfrak{F}(B) \arrow[r, "\mathfrak{F}(g)"'] & \mathfrak{F}(C) \end{tikzcd}$$

Most categories we encounter are concrete categories.

Think: Objects: Sets with extra structure (e.g. groups)
Morphisms: Functions with extra structure (e.g. homomorphisms).

Non-examples:

(1) Any directed graph defines a non-concrete category (with assumption of loops & transitive edges).

(2) Let G be any group. $Ob(\mathcal{C}) := \{G\}$,
$Hom(\mathcal{C}) = Hom(G,G) = G$.
Composition of morphisms $a \circ b = ab$.
Every morphism is an equivalence, & e is the identity morphism.

Note: A morphism is associated with a function b/w sets, but not always vice-versa.

Def: Let F be an object of a concrete category $\mathcal{C}$, S a non-empty set, and $\phi: S \longrightarrow F$ a map of sets. Then F is free on S if for any $A \in Ob(\mathcal{C})$ and map (of sets) $\theta: S \longrightarrow A$, $\exists!\ f \in Hom_{\mathcal{C}}(F, A)$ s.t. $f\phi = \theta$ (as maps of sets $S \to A$).

Thm 8.7: Let $F, F' \in Ob(\mathcal{C})$ be free objects on S, S' resp, and $|S| = |S'|$. Then F and F' are equivalent.

Pf: HW #7 (Mimic the "$\Leftarrow$" direction of proof of Thm 8.6).

Free objects are universal (i.e, initial) objects in an appropriately constructed category (like products were).

Example: Let $F \in Ob(\mathcal{C})$ be free on S, $\phi: S \longrightarrow F$.

Define a new category $\mathcal{D}$ as follows:

$Ob(\mathcal{D})$: maps of sets $S \longrightarrow A \in Ob(\mathcal{C})$

$Hom(\mathcal{D})$: $h \in Hom_{\mathcal{D}}(f: S \to A,\ g: S \to B)$ s.t. $hf = g$.

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, "g"'] & A \arrow[d, dashed, "h"] \\ & B \end{tikzcd}$$

Check: • $1_A: A \to A$ is identity morphism from $f \to f$:

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, "f"'] & A \arrow[d, dashed, "1_A"] \\ & A \end{tikzcd}$$

• h is an equivalence in $\mathcal{D}$ iff h is an equivalence in $\mathcal{C}$.

• If F is free on S, then $\phi: S \longrightarrow F$ is the universal (i.e, initial) object in $\mathcal{D}$.

Return to the setting of groups.

The existence of free groups implies the following:

<u>Prop 8.8</u>: If $G = \langle S \rangle$, then $\exists$ homom $F_S \twoheadrightarrow G$.

<u>Pf</u>: Let $\phi : S \hookrightarrow F_S$ and $\theta : S \hookrightarrow G$.

Since F_S is free on S, ϕ extends to a homom. $f : F_S \longrightarrow G$ with $f(s) = s$. Since $\langle S \rangle = G$, f is surjective. $\square$

$$\begin{tikzcd} S \arrow[r, hook, "\phi"] \arrow[dr, "\theta"'] & F_S \arrow[d, dashed, "f"] \\ & \langle S \rangle \end{tikzcd}$$

By Prop 8.8 & FHT, if $G = \langle S \rangle$, then $G \cong F_S / K$ for some $K \trianglelefteq F_S$.

<u>Note</u>: If $T \subseteq K$, then each $t \in K$ is a word in F_S (i.e, in $S \cup S^{-1}$). The quotient $q : F_S \longrightarrow F_S/K$ maps $t \longmapsto tK = K$, i.e, it "sets $t = 1$."

We say that G has a set S of <u>generators</u> subject to a set of <u>relations</u> $\{ t = 1 : t \in T \}$.

Thus, all groups $G = \langle S \rangle$ can be described as follows:

Let S be a set, $T \subseteq F_S$, and $K := \bigcap_{T \in N_i \trianglelefteq G} N_i$.

Define $\langle S \mid t = 1 \ \forall t \in T \rangle := F_S / K$, called a <u>presentation</u> of G.

<u>Ex</u>: The cyclic group of order n has presentation $\langle a \mid a^n = 1 \rangle$.

Note: Often, we omit the "$=1$" because it is understood, or just write, e.g., $G = \langle S \mid T \rangle$.

Prop 8.9: Suppose $G_1 = \langle S \mid T_1 \rangle$, $T_1 \subseteq T_2$, and $G_2 = \langle S \mid T_2 \rangle$. Then $\exists$ homom $g : G_1 \twoheadrightarrow G_2$.

$$\begin{tikzcd} S \arrow[r, hook, "\phi"] \arrow[d, "\theta"'] & F_S \arrow[d, "k"] \arrow[dl, dashed, "f"'] \\ G_2 & G_1 \arrow[l, dashed, "g"] \end{tikzcd}$$

Pf: Assume $k : F_S \longrightarrow F_S / K_1 = G_1$ is the canonical quotient map, where

$$K_1 = \bigcap_{T \in N_i \trianglelefteq G} N_i.$$

Let $\phi : S \hookrightarrow F_S$ and $\theta : S \hookrightarrow G_2$ be the inclusion maps. Since F_S is free, $\exists!\ f : F_S \twoheadrightarrow G_2$ s.t. $\theta = f\phi$.

Since $T_1 \subseteq T_2$, $\ker k \leq \ker f$.

Thus, $\exists$ homom $g : G_1 \twoheadrightarrow G_2$ s.t. $f = gk$. $\square$

* To summarize Prop 8.9: "Adding relations induces a homomorphism."

Note: Removing a generator s_i is equivalent to adding the relation $s_i = 1$.

Thus, if $S_1 \supseteq S_2$ and $T_1 \subseteq T_2$, then $\exists$ homom

$$\langle S_1, T_1 \rangle \twoheadrightarrow \langle S_2, T_2 \rangle.$$

Note: We actually don't need $S_1 \geq S_2$, but rather just a surjection $S_1 \twoheadrightarrow S_2$ that "respects relations."

More precisely, we have the following more general corollary:

Cor 8.10: Suppose G_1 is a group with presentation $\langle S \mid t=1 \ \forall t\in T\rangle$ and $G_2 = \langle S' \mid t'=1 \ \forall t'\in T'\rangle$ such that

(i) $\exists\, \theta : S \twoheadrightarrow S'$, say $\theta(s)=s'$ and extending $\theta: T \longrightarrow T'$

(ii) $t'=1 \ \forall t'\in T'$ (i.e., $\theta(t)=1 \ \forall t\in T$).

Then $\exists$ homom $g: G_1 \twoheadrightarrow G_2$.

Examples:

(1) Let $G_1 = \langle a \mid a^n = 1\rangle$ and $G_2 = \langle b \mid b^m = 1\rangle$, $m \mid n$.
Then $b^n = 1$, so the map $\theta: a \mapsto b$ extends to a homom. $G_1 \twoheadrightarrow G_2$. (In fact, $\mathbb{Z}_n \twoheadrightarrow \mathbb{Z}_m$)

(2) Let $G = \langle a, b \mid a^3 = 1,\ b^2 = 1,\ abab = 1\rangle$

Note: $\left.\begin{array}{l} a^3 = 1 \Rightarrow a^{-1} = a^2 \\ b^2 = 1 \Rightarrow b^{-1} = b \end{array}\right\}$ So, $abab = 1 \Leftrightarrow ab = ba^{-1} = ba^2$.

Thus, every element can be written in the form $b^i a^j$ where $i = 0,1$ and $j = 0,1,2$ $\Rightarrow |G| \leq 6$.

(So, $G \cong 1,\ \mathbb{Z}_2,\ \mathbb{Z}_3,\ \mathbb{Z}_4,\ \mathbb{Z}_2\oplus\mathbb{Z}_2,\ \mathbb{Z}_5,\ \mathbb{Z}_6$ or S_3).

Which one is it?

Consider $\sigma = (123)$ $\tau = (12)$ in S_3. Note that $S_3 = \langle \sigma, \tau\rangle$.

Let $\theta(a) = \sigma$ and $\theta(b) = \tau$. Check: $\sigma^3 = 1$, $\tau^2 = 1$, $\sigma\tau = \tau\sigma^2$.

By Cor 8.10 $\exists\, g: G \twoheadrightarrow S_3 \Rightarrow |G| \geq 6$.

Therefore, $G \cong S_3$, i.e, S_3 has presentation $\langle a, b \mid a^3, b^2, abab\rangle$.

(3) Let $G = \langle x, y \mid xy = y^2x,\ yx = x^2y \rangle$.

Note: $xy = y^2x \Rightarrow y^{-1}(xy) = yx = x^2y = x(xy) \Rightarrow x = y^{-1}$

Thus, $1 = xy = y^2x = y(yx) = y \Rightarrow y = 1 \Rightarrow x = 1$.

Hence, $G = 1$.

(4) Let $G = \langle a, b \mid a^n = b^2 = 1,\ ab = ba^{-1} \rangle$.

Consider D_n, with $\sigma = 2\pi/n$-rotation, and τ a reflection.

Define $\theta(a) = \sigma$, $\theta(b) = \tau$, and check $\sigma^n = \tau^2 = 1$, $\sigma\tau = \tau\sigma^{-1}$.

Thus $\exists$ homom $G \twoheadrightarrow D_n \Rightarrow |G| \geq |D_n| = 2n$

But also note: $a^{-1} = a^{n-1}$, $b^{-1} = b$, $ab = ba^{n-1}$, thus every elt can be written as $a^i b^j$, $0 \leq i < n$, $0 \leq j < 2$, hence $|G| \leq 2n$

Together, we conclude that $G \cong D_n$.

Fact: Given two finitely presented groups $G_1 = \langle S_1 \mid T_1 \rangle$ and $G_2 = \langle S_2 \mid T_2 \rangle$ (i.e., $|S_i|, |T_i| < \infty$), determining whether $G_1 \cong G_2$ is, in general, computationally undecidable!