

3. Projective and injective modules

Motivation: Free modules have some nice properties, that actually hold for a more general class of modules. Consider the following two results.

Prop 3.1: Suppose R is a ring with 1 , M is a unitary R -module, F is a free R -module, and $f \in \text{Hom}_R(M, F)$ is surjective. Then M has a free submodule E that is R -isomorphic with F such that $M = E \oplus \ker f$:

$$0 \longrightarrow E \hookrightarrow M \xrightarrow{f} F \longrightarrow 0$$

PF: If B is a basis for F , choose $x_b \in M$ s.t. $f(x_b) = b$ for each $b \in B$, and set $E = R\langle \{x_b : b \in B\} \rangle$.

If $\sum_{i=1}^k r_i x_{b_i} = 0$, then $0 = f\left(\sum_{i=1}^k r_i x_{b_i}\right) = \sum_{i=1}^k r_i b_i \Rightarrow r_i = 0 \forall i$,
i.e., $\{x_b : b \in B\}$ is linearly independent.

Thus, $\{x_b : b \in B\}$ is a basis for E , so E is free and clearly E is R -isomorphic with F .

If $x \in M$, write $f(x) = \sum_{i=1}^k r_i b_i$ and note that $x - \sum_{i=1}^k r_i x_{b_i} \in \ker f$.

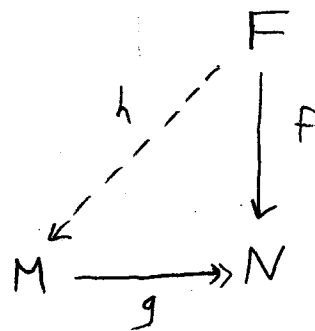
Thus, $M = E + \ker f$.

Since $E \cap \ker f = 0$, $M = E \oplus \ker f$ by Thm 2.3. \square

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Prop 3.2: Suppose R is a ring with 1 , and $M, N \in \mathcal{F}$ are unitary R -modules with F free.

If $f \in \text{Hom}_R(F, N)$ and $g \in \text{Hom}_R(M, N)$ is surjective, then $\exists h \in \text{Hom}_R(F, M)$ such that $f = gh$.



(The homomorphism $f: F \rightarrow N$ "lifts" to a homomorphism h .)

Pf: Let B be a basis of F .

For each $b_i \in B$, choose $m_i \in M$ such that $g(m_i) = f(b_i)$.

Define $h: B \rightarrow M$ by $h(b_i) = m_i$, and extend this to $h \in \text{Hom}_R(F, M)$. This clearly works.

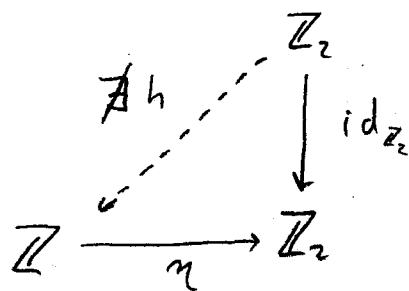
□

Remark: Props 3.1 & 3.2 do not necessarily hold if F is not free.

For example, take $M = \mathbb{Z}$, $N = F = \mathbb{Z}_2$, and $\eta: \mathbb{Z} \rightarrow \mathbb{Z}_2$ the natural quotient map, so $\ker \eta = 2\mathbb{Z}$.

Then $\mathbb{Z} \not\cong \mathbb{Z} \oplus 2\mathbb{Z}$, nor does there

exist $h \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_2 \rightarrow \mathbb{Z})$ such that $\text{id}_{\mathbb{Z}_2} = \eta h$.



Goal: Understand what class of modules these results do hold for. These will be precisely the "projective" modules.

First, we need to introduce the notion of an exact sequence.

Def: A sequence of R -modules and R -homomorphism

$$\dots \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \dots \quad (*)$$

is exact at M_i if $\text{im } f_i = \ker f_{i+1}$. The sequence is

exact if it is exact at each M_i .

Prop 3.3:

(i) $0 \longrightarrow L \xrightarrow{f} M$ is exact iff f is injective

(ii) $M \xrightarrow{g} N \longrightarrow 0$ is exact iff g is surjective.

(iii) $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ is exact iff

(a) f is injective

(b) g is surjective

(c) g induces an isomorphism $\text{coker } f := M/f(L) \cong N$

Pf: Exercise.

An exact sequence of type (iii) above is called a

short exact sequence.

Any long exact sequence (*) can be broken up into short

exact sequences via restriction.

Example: If $\xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1}$ is exact, then

$0 \longrightarrow \text{im } f_i \hookrightarrow M_i \twoheadrightarrow \text{im } f_{i+1} \longrightarrow 0$ is exact.

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Prop 3.4: Consider a sequence $L \xrightarrow{f} M \xrightarrow{g} N$ of homomorphisms.

(i) f is injective iff $f \circ h_1 = f \circ h_2 \Rightarrow h_1 = h_2 \quad \forall h_i \in \text{Hom}(M, N)$.

(ii) g is surjective iff $h_1 \circ g = h_2 \circ g \Rightarrow h_1 = h_2 \quad \forall h_i \in \text{Hom}(L, M)$.

Pf: HW #6, last semester.

Prop 3.5: Let $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ be an exact sequence. Then

(i) $0 \rightarrow \text{Hom}_R(D, L) \xrightarrow{f_*} \text{Hom}_R(D, M) \xrightarrow{g_*} \text{Hom}_R(D, N)$ is exact

(ii) $0 \rightarrow \text{Hom}_R(N, D) \xrightarrow{g^*} \text{Hom}_R(M, D) \xrightarrow{f^*} \text{Hom}_R(L, D)$ is exact.

Pf:

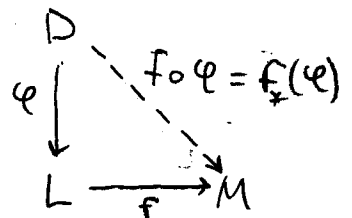
(i) • First, show exactness at $\text{Hom}_R(D, L)$, i.e., injectivity of f_* :

Recall $f_*: \varphi \mapsto f \circ \varphi$

Suppose $f_*(\varphi) = f_*(\theta)$.

Then $f \circ \varphi = f \circ \theta \Rightarrow \varphi = \theta$ since

f is injective (Prop 3.4 (i)). ✓



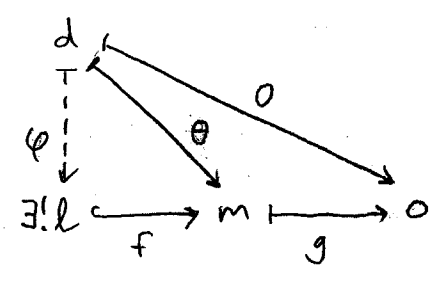
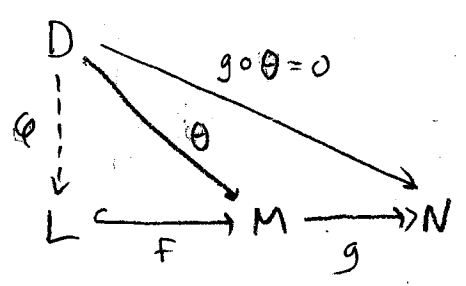
• Next, show exactness at $\text{Hom}_R(D, M)$, i.e., $\text{im } f_* = \ker g_*$

$\text{im } f_* \subseteq \ker g_*$ (equivalently, $g_* \circ f_* = 0$):

$f_*: \varphi \mapsto f \circ \varphi$, $g_*: \theta \mapsto g \circ \theta$

$\Rightarrow g_* \circ f_*: \varphi \mapsto g \circ f \circ \varphi = 0$ since $g \circ f = 0$ ($\text{im } f = \ker g$). ✓

$\text{Im } f_* \supseteq \ker g_*$



Suppose $\theta \in \ker g_*$. Then $g \circ \theta = 0 \in \text{Hom}_R(D, N)$.

We will construct $\varphi \in \text{Hom}_R(D, L)$ s.t. $f_*(\varphi) = f \circ \varphi = \theta$, i.e., a preimage of θ .

Choose $d \in D$, and let $m = \theta(d) \in M$.

Since f is injective, $\exists! l \in L$ s.t. $f(l) = m$.

Define $\varphi(d) = f^{-1}(m) = l$.

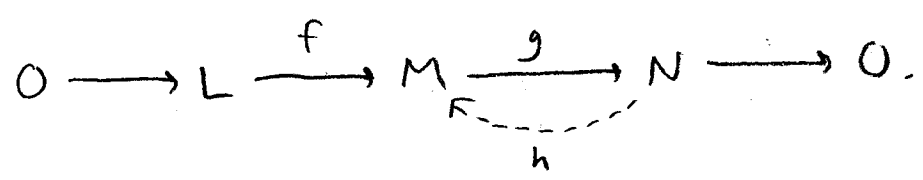
It is easy to check that $\varphi \in \text{Hom}_R(D, L)$, and that

$$f_*(\varphi) = f \circ \varphi = \theta \quad (\text{see above diagrams}), \text{ so}$$

$$\theta \in \text{Im } f_* \Rightarrow \text{Im } f_* \supseteq \ker g_* \quad \checkmark$$

This proves (i). The proof of (ii) is analogous (it's dual), and is an exercise (Hw). □

Def: A short exact sequence $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ splits if $\exists h \in \text{Hom}(N, M)$ s.t. $g \circ h = 1_N$, i.e.,



This is equivalent to $M \cong L \oplus N$, i.e., we have the following diagram:

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$$\begin{array}{ccccccccc}
 0 & \longrightarrow & L & \hookrightarrow & M & \twoheadrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
 0 & \longrightarrow & L & \xrightarrow{i} & L \oplus N & \xrightarrow{\pi} & N & \longrightarrow & 0
 \end{array}$$

Here L is the injection of the first summand, and π is the projection onto the second summand.

Thm 3.5: Let P be a unitary R -module. The following are equivalent:

(i) For any unitary R -modules L, M, N , if

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 \text{ is exact, then}$$

$$0 \longrightarrow \text{Hom}_R(P, L) \xrightarrow{f_*} \text{Hom}_R(P, M) \xrightarrow{g_*} \text{Hom}_R(P, N) \longrightarrow 0$$

is exact.

(ii) Prop 3.2 holds for P : If $M \xrightarrow{g} N \longrightarrow 0$ is exact

and $\varphi \in \text{Hom}_R(P, N)$, then $\exists h \in \text{Hom}_R(P, M)$

s.t. $\varphi = gh$. (h is a "lift" of φ).

$$\begin{array}{ccc}
 & & P \\
 & \swarrow h & \downarrow \varphi \\
 M & \xrightarrow{g} & N \longrightarrow 0
 \end{array}$$

(iii) Every short exact sequence $0 \longrightarrow L \hookrightarrow M \twoheadrightarrow P \longrightarrow 0$ splits (i.e., $M \cong L \oplus P$). Thus, Prop 3.1 holds for P .

(iv) P is a direct summand of a free R -module (i.e., for some free module F and R -module M , $F \cong M \oplus P$).

Pf:

(i) \Leftrightarrow (ii) Let $\varphi \in \text{Hom}_R(P, N)$. The condition of (i) holding is equivalent to $g_*: \text{Hom}_R(P, M) \rightarrow \text{Hom}_R(P, N)$ being surjective, i.e., there we have $g_*: h \longmapsto \varphi$ for some $h \in \text{Hom}_R(P, M)$, such that $g \circ h = \varphi$. In other words, given the following diagram, $\exists h: P \rightarrow M$ that makes it commute. This is precisely the condition of (ii) \checkmark

$$\begin{array}{ccc} & & P \\ & \swarrow h & \downarrow \varphi \\ M & \xrightarrow{g} & N \end{array}$$

(ii) \Rightarrow (iii) Let $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} P \rightarrow 0$ be exact.

By (ii), the identity map $\text{id}: P \rightarrow P$ lifts to a homomorphism $h: P \rightarrow M$ such that $g \circ h = \text{id}$.

$$\begin{array}{ccc} & & P \\ & \swarrow h & \downarrow \text{id} \\ M & \xrightarrow{g} & P \rightarrow 0 \end{array}$$

Thus, we have $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} P \rightarrow 0$, as desired. \checkmark

(iii) \Rightarrow (iv) Every module P is the quotient of a free module F , so we have an exact sequence

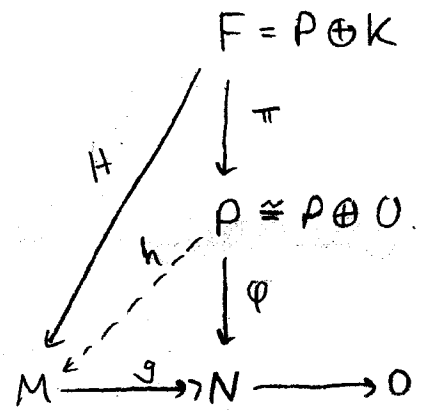
$$0 \rightarrow \ker \pi \xrightarrow{f} F \xrightarrow{\pi} P \rightarrow 0 \text{ for such an } f.$$

Since this sequence splits by (iii), $F = P \oplus \ker \pi$. \checkmark

(iv) \Rightarrow (ii) Suppose that P is a direct summand of a free module F , i.e., $F = P \oplus K$. Let $g \in \text{Hom}_R(M, N)$ be surjective, and $\varphi \in \text{Hom}_R(P, N)$. We must show that there is some $h \in \text{Hom}_R(P, M)$ s.t. $g \circ h = \varphi$.

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If F is based on B , then let $\pi: P \oplus K \rightarrow P$ be the natural projection map, and $H \in \text{Hom}_R(F, M)$ such that $g \circ H = \varphi \circ \pi$ (which exists by Prop 3.2).



Note that $P \cong P \oplus 0 \subseteq P \oplus K$, given

$\pi: (p, k) \mapsto p \in P$, define $h \in \text{Hom}_R(P, M)$ by $h: (p, 0) \mapsto H((p, 0))$.

Clearly, this is an R -homomorphism and makes the above diagram commute. \checkmark

□

Def: An R -module P is called projective if it satisfies any of the equivalent conditions of Thm 3.5.

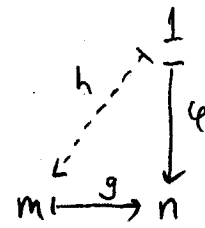
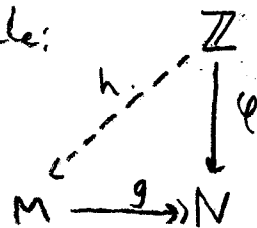
*Motivation for terminology: P is projective iff any R -module M that projects onto P has (an isomorphic copy of) P as a direct summand. (Condition (iii) in Thm 3.5).

Cor: Every module is a quotient of a projective module.

Examples:

1. \mathbb{Z} is a projective \mathbb{Z} -module:

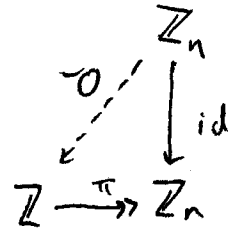
Define $h(1) = g^{-1}(\varphi(1))$, and extend additively.



\mathbb{Z} , \mathbb{Z}_n is not a projective \mathbb{Z} -module:

$$0 \rightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \rightarrow 0$$

$\downarrow \quad \quad \quad \downarrow$
 $1 \quad \quad \quad n \quad \quad \quad 0$



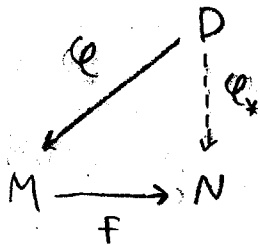
Similarly, a free \mathbb{Z} -module cannot have nonzero elements of finite order.

The "dual" to the notion of projective module are called injective modules. Consider $f \in \text{Hom}_R(M, N)$.

Compare how Hom-sets induce homomorphisms between them:

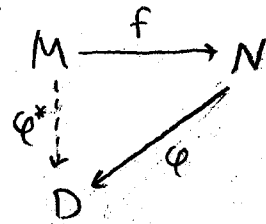
$$f_*: \text{Hom}_R(D, M) \rightarrow \text{Hom}_R(D, N)$$

$$\varphi \longmapsto \varphi_* = f \circ \varphi$$



$$f^*: \text{Hom}_R(N, D) \rightarrow \text{Hom}_R(M, D)$$

$$\varphi \longmapsto \varphi^* = \varphi \circ f$$



Thm 3.6: Let Q be a unitary R -module. The following are equivalent:

(i) For any unitary R -modules L, M, N , if

$$0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0 \text{ is exact, then}$$

$$0 \rightarrow \text{Hom}_R(N, Q) \xrightarrow{g^*} \text{Hom}_R(M, Q) \xrightarrow{f^*} \text{Hom}_R(L, Q) \rightarrow 0$$

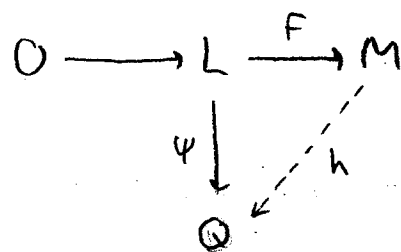
is exact.

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(ii) If $0 \rightarrow L \xrightarrow{f} M$ is exact and $\psi \in \text{Hom}_R(L, Q)$, then

$\exists h \in \text{Hom}_R(M, Q)$ s.t. $\psi = hf$.

(h is a "lift" of ψ).



(iii) If Q is a submodule of the R -module M , then Q

is a direct summand of M , i.e., every short exact sequence

$$0 \longrightarrow Q \longleftarrow M \longrightarrow N \longrightarrow 0 \text{ splits.}$$

Pf. Exercise.

Def. An R -module Q is called injective if it satisfies any of the equivalent conditions of Thm 3.6.

Example:

(1) \mathbb{Z} is not an injective \mathbb{Z} -module, since the exact sequence

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \\
 & & \downarrow 1 & & \downarrow 2 & &
 \end{array}$$

(2) \mathbb{Q} and \mathbb{Q}/\mathbb{Z} are injective (but not projective) \mathbb{Z} -modules.

(3) Fact: No non-zero finitely generated \mathbb{Z} -module is injective.

Cor: Every \mathbb{Z} -module is the submodule of an injective \mathbb{Z} -module.

Thm 3.7: Every unitary R -module M is contained in an injective R -module. (Exercise; see Dummit & Foote Ex. 10.5 #15-16).