3. Projective and injective modules

Motivation: Free modules have some nice properties, that actually hold for a more general class of modules. Consider the following two results.

Prop 3.1: Suppose R is a ring with 1, M is a unitary R-module, F is a free R-module, and $f \in \mathrm{Hom}_R(M, F)$ is surjective. Then M has a free submodule E that is R-isomorphic with F such that $M = E \oplus \ker f$:

$$0 \longrightarrow E \hookrightarrow M \overset{f}{\twoheadrightarrow} F \longrightarrow 0$$

Pf: If B is a basis for F, choose $x_b \in M$ s.t. $f(x_b) = b$ for each $b \in B$, and set $E = R\langle \{x_b : b \in B\} \rangle$.

If $\sum_{i=1}^{k} r_i x_{b_i} = 0$, then $0 = f\left(\sum_{i=1}^{k} r_i x_{b_i}\right) = \sum_{i=1}^{k} r_i b_i \Rightarrow r_i = 0 \ \forall i$,

i.e., $\{x_b : b \in B\}$ is linearly independent.

Thus, $\{x_b : b \in B\}$ is a basis for E, so E is free and clearly E is R-isomorphic with F.

If $x \in M$, write $f(x) = \sum_{i=1}^{k} r_i b_i$ and note that $x - \sum_{i=1}^{k} r_i x_{b_i} \in \ker f$.

Thus, $M = E + \ker f$.

Since $E \cap \ker f = 0$, $M = E \oplus \ker f$ by Thm 2.3. $\square$

Prop 3.2: Suppose R is a ring with 1, and M, N & F are unitary R-modules with F free.

If $f \in \mathrm{Hom}_R(F, N)$ and $g \in \mathrm{Hom}_R(M, N)$ is surjective, then $\exists\ h \in \mathrm{Hom}_R(F, M)$ such that $f = gh$.

$$\begin{tikzcd} & F \arrow[d, "f"] \arrow[dl, dashed, "h"'] \\ M \arrow[r, two heads, "g"'] & N \end{tikzcd}$$

(The homomorphism $f : F \to N$ "lifts" to a homomorphism h.)

Pf: Let B be a basis of F.

For each $b_i \in B$, choose $m_i \in M$ such that $g(m_i) = f(b_i)$.

Define $h : B \longrightarrow M$ by $h(b_i) = m_i$, and extend this to $h \in \mathrm{Hom}_R(F, M)$. This clearly works.

$\square$

Remark: Props 3.1 & 3.2 do not necessarily hold if F is not free.

For example, take $M = \mathbb{Z}$, $N = F = \mathbb{Z}_2$, and $\eta : \mathbb{Z} \twoheadrightarrow \mathbb{Z}_2$ the natural quotient map, so $\ker \eta = 2\mathbb{Z}$.

Then $\mathbb{Z} \not\cong E \oplus 2\mathbb{Z}$, nor does there exist $h \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2 \to \mathbb{Z})$ such that $id_{\mathbb{Z}_2} = \eta h$.

$$\begin{tikzcd} & \mathbb{Z}_2 \arrow[d, "id_{\mathbb{Z}_2}"] \arrow[dl, dashed, "\nexists h"'] \\ \mathbb{Z} \arrow[r, "\eta"'] & \mathbb{Z}_2 \end{tikzcd}$$

Goal: Understand what class of modules these results do hold for. These will be precisely the "projective" modules.

First, we need to introduce the notion of an exact sequence.

Def: A sequence of R-modules and R-homomorphism

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots \qquad (*)$$

is exact at M_i if $\operatorname{im} f_i = \ker f_{i+1}$. The sequence is exact if it is exact at each M_i.

Prop 3.3:

(i) $0 \longrightarrow L \xrightarrow{f} M$ is exact iff f is injective

(ii) $M \xrightarrow{g} N \longrightarrow 0$ is exact iff g is surjective.

(iii) $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ is exact iff

(a) f is injective

(b) g is surjective

(c) g induces an isomorphism $\operatorname{coker} f := M/f(L) \cong N$

Pf: Exercise.

An exact sequence of type (iii) above is called a short exact sequence.

Any long exact sequence $(*)$ can be broken up into short exact sequences via restriction.

Example: If $\xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1}$ is exact, then

$$0 \longrightarrow \operatorname{im} f_i \hookrightarrow M_i \twoheadrightarrow \operatorname{im} f_{i+1} \longrightarrow 0$$ is exact.

<u>Prop 3.4</u>: Consider a sequence $L \xrightarrow{f} M \xrightarrow{g} N$ of homomorphisms.

(i) f is injective iff $f \circ h_1 = f \circ h_2 \Rightarrow h_1 = h_2 \quad \forall h_i \in Hom(M,N)$.

(ii) g is surjective iff $h_1 \circ g = h_2 \circ g \Rightarrow h_1 = h_2 \quad \forall h_i \in Hom(L,M)$.

<u>Pf</u>: HW #6, last semester.

<u>Prop 3.5</u>: Let $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ be an exact sequence. Then

(i) $0 \longrightarrow Hom_R(D,L) \xrightarrow{f_*} Hom_R(D,M) \xrightarrow{g_*} Hom_R(D,N)$ is exact

(ii) $0 \longrightarrow Hom_R(N,D) \xrightarrow{g^*} Hom_R(M,D) \xrightarrow{f^*} Hom_R(L,D)$ is exact.

<u>Pf</u>:

(i) • First, show exactness at $Hom_R(D,L)$, i.e, injectivity of f_*:

<u>Recall</u> $f_*: \varphi \longmapsto f \circ \varphi$

$$\begin{tikzcd} D \arrow[d, "\varphi"'] \arrow[dr, dashed, "f \circ \varphi = f_*(\varphi)"] & \\ L \arrow[r, "f"'] & M \end{tikzcd}$$

Suppose $f_*(\varphi) = f_*(\theta)$.

Then $f \circ \varphi = f \circ \theta \Rightarrow \varphi = \theta$ since f is injective (Prop 3.4 (i)). ✓

• Next, show exactness at $Hom_R(D,M)$, i.e, $\text{im } f_* = \ker g_*$

<u>$\text{im } f_* \subseteq \ker g_*$</u> (equivalently, $g_* \circ f_* = 0$):

$f_*: \varphi \longmapsto f \circ \varphi \quad , \quad g_*: \theta \longmapsto g \circ \theta$

$\Rightarrow \quad g_* \circ f_*: \varphi \longmapsto g \circ f \circ \varphi = 0$ since $g \circ f = 0$ ($\text{im } f = \ker g$). ✓

<u>$\operatorname{im} f_* \supseteq \ker g_*$:</u>

$$\begin{tikzcd} D \arrow[d, dashed, "\varphi"'] \arrow[dr, "\theta"] \arrow[drr, "g\circ\theta=0"] & & \\ L \arrow[r, hook, "f"'] & M \arrow[r, two heads, "g"'] & N \end{tikzcd} \qquad \begin{tikzcd} d \arrow[d, dashed, mapsto, "\varphi"'] \arrow[dr, mapsto, "\theta"] \arrow[drr, mapsto, "0"] & & \\ \exists!\, \ell \arrow[r, hook, "f"'] & m \arrow[r, mapsto, "g"'] & 0 \end{tikzcd}$$

Suppose $\theta \in \ker g_*$. Then $g\circ\theta = 0 \in \mathrm{Hom}_R(D,N)$.

We will construct $\varphi \in \mathrm{Hom}_R(D,L)$ s.t. $f_*(\varphi) := f\circ\varphi = \theta$, i.e., a preimage of θ.

Choose $d\in D$, and let $m = \theta(d) \in M$.

Since f is injective, $\exists!\ \ell \in L$ s.t. $f(\ell) = m$.

Define $\varphi(d) = f^{-1}(m) = \ell$.

It is easy to check that $\varphi \in \mathrm{Hom}_R(D,L)$, and that $f_*(\varphi) = f\circ\varphi = \theta$ (see above diagrams), so $\theta \in \operatorname{im} f_* \Rightarrow \operatorname{im} f_* \supseteq \ker g_*$. ✓

This proves (i). The proof of (ii) is analogous (it's dual), and is an exercise (HW). □

<u>Def</u>: A short exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ <u>splits</u> if $\exists\, h \in \mathrm{Hom}(N,M)$ s.t. $g\circ h = 1_N$, i.e.,

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] & N \arrow[r] \arrow[l, dashed, bend left, "h"] & 0. \end{tikzcd}$$

This is equivalent to $M \cong L \oplus N$, i.e., we have the following diagram:

$$
\begin{tikzcd}
0 \arrow[r] & L \arrow[r, hook] \arrow[d, "\cong"] & M \arrow[r, two heads] \arrow[d, "\cong"] & N \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & L \arrow[r, hook, "\iota"] & L \oplus N \arrow[r, two heads, "\pi"] & N \arrow[r] & 0
\end{tikzcd}
$$

Here ι is the <u>injection</u> of the first summand, and π is the <u>projection</u> onto the second summand.

<u>Thm 3.5</u>: Let P be a unitary R-module. The following are equivalent:

(i) For any unitary R-modules L, M, N, if

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 \quad \text{is exact, then}$$

$$0 \longrightarrow \mathrm{Hom}_R(P, L) \xrightarrow{f_*} \mathrm{Hom}_R(P, M) \xrightarrow{g_*} \mathrm{Hom}_R(P, N) \longrightarrow 0$$

is exact.

(ii) Prop 3.2 holds for P: If $M \xrightarrow{g} N \longrightarrow 0$ is exact and $\varphi \in \mathrm{Hom}_R(P, N)$, then $\exists\, h \in \mathrm{Hom}_R(P, M)$ s.t. $\varphi = gh$ (h is a "lift" of φ).

$$
\begin{tikzcd}
 & P \arrow[dl, dashed, "h"'] \arrow[d, "\varphi"] & \\
M \arrow[r, two heads, "g"] & N \arrow[r] & 0
\end{tikzcd}
$$

(iii) Every short exact sequence $0 \longrightarrow L \hookrightarrow M \twoheadrightarrow P \longrightarrow 0$ splits (i.e, $M \cong L \oplus P$). Thus, Prop 3.1 holds for P.

(iv) P is a direct summand of a free R-module (i.e, for some free module F and R-module M, $F \cong M \oplus P$).

Pf:

(i) $\Leftrightarrow$ (ii) Let $\varphi \in \mathrm{Hom}_R(P, N)$. The condition of (i) holding is equivalent to $g_*: \mathrm{Hom}_R(P,M) \longrightarrow \mathrm{Hom}_R(P,N)$ being surjective, i.e, there we have $g_*: h \longmapsto \varphi$ for some $h \in \mathrm{Hom}_R(P,M)$, such that $g \circ h = \varphi$. In other words, given the following diagram, $\exists\, h: P \to M$ that makes it commute. This is precisely the condition of (ii) ✓

$$\begin{tikzcd} & P \arrow[d, "\varphi"] \arrow[dl, dashed, "h"'] \\ M \arrow[r, two heads, "g"] & N \end{tikzcd}$$

(ii) $\Rightarrow$ (iii) Let $0 \longrightarrow L \xhookrightarrow{f} M \xrightarrow{g} P \longrightarrow 0$ be exact.

By (ii), the identity map $id: P \longrightarrow P$ lifts to a homomorphism $h: P \longrightarrow M$ such that $g \circ h = id$.

$$\begin{tikzcd} & P \arrow[d, "id"] \arrow[dl, dashed, "h"'] & \\ M \arrow[r, two heads, "g"] & P \arrow[r] & 0 \end{tikzcd}$$

Thus, we have $\begin{tikzcd} 0 \arrow[r] & L \arrow[r, hook] & M \arrow[r, two heads, "g"] & P \arrow[r] \arrow[l, dashed, bend left, "h"] & 0 \end{tikzcd}$, as desired. ✓

(iii) $\Rightarrow$ (iv) Every module P is the quotient of a free module F, so we have an exact sequence

$\begin{tikzcd} 0 \arrow[r] & \ker \pi \arrow[r, hook] & F \arrow[r, two heads, "\pi"] & P \arrow[r] \arrow[l, dashed, bend left] & 0 \end{tikzcd}$ for such an F.

Since this sequence splits by (iii), $F = P \oplus \ker \pi$. ✓

(iv) $\Rightarrow$ (ii) Suppose that P is a direct summand of a free module F, i.e, $F = P \oplus K$. Let $g \in \mathrm{Hom}_R(M, N)$ be surjective, and $\varphi \in \mathrm{Hom}_R(P, N)$. We must show that there is some $h \in \mathrm{Hom}_R(P,M)$ s.t. $g \circ h = \varphi$.

If F is based on B, then let $\pi : P \oplus K \to P$ be the natural projection map, and $H \in \mathrm{Hom}_R(F, M)$ such that $g \circ H = \varphi \circ \pi$ (which exists by Prop 3.2).

$$\begin{tikzcd} & F = P \oplus K \arrow[dl, "H"'] \arrow[d, "\pi"] & \\ & P \cong P \oplus 0 \arrow[dl, dashed, "h"'] \arrow[d, "\varphi"] & \\ M \arrow[r, two heads, "g"] & N \arrow[r] & 0 \end{tikzcd}$$

Note that $P \cong P \oplus 0 \leq P \oplus K$, given $\pi : (p,k) \longmapsto p \in P$, define $h \in \mathrm{Hom}_R(P, M)$ by $h : (p, 0) \longmapsto H((p,0))$.

Clearly, this is an R-homomorphism and makes the above diagram commute. ✓

$\square$

Def: An R-module P is called <u>projective</u> if it satisfies any of the equivalent conditions of Thm 3.5.

* <u>Motivation for terminology:</u> P is projective iff any R-module M that projects onto P has (an isomorphic copy of) P as a direct summand. (Condition (iii) in Thm 3.5).

<u>Cor:</u> Every module is a quotient of a projective module.

<u>Examples:</u>

1. $\mathbb{Z}$ is a projective $\mathbb{Z}$-module: Define $h(1) = g^{-1}\varphi(1)$, and extend additively.

$$\begin{tikzcd} & \mathbb{Z} \arrow[dl, dashed, "h"'] \arrow[d, "\varphi"] \\ M \arrow[r, two heads, "g"] & N \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[dl, dashed, mapsto, "h"'] \arrow[d, mapsto, "\varphi"] \\ m \arrow[r, mapsto, "g"] & n \end{tikzcd}$$

2. $\mathbb{Z}_n$ is not a projective $\mathbb{Z}$-module:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, hook, "n"] & \mathbb{Z} \arrow[r, two heads, "\pi"] & \mathbb{Z}/n\mathbb{Z} \arrow[r] \arrow[l, dashed, bend left, "0"] & 0 \end{tikzcd}$$

$1 \longmapsto n$

$$\begin{tikzcd} & \mathbb{Z}_n \arrow[d, "id"] \arrow[dl, dashed, "0"'] \\ \mathbb{Z} \arrow[r, two heads, "\pi"] & \mathbb{Z}_n \end{tikzcd}$$

Similarly, a free $\mathbb{Z}$-module cannot have nonzero elements of finite order.

The "dual" to the notion of projective module are called <u>injective</u> modules. Consider $f \in \mathrm{Hom}_R(M,N)$.

<u>Compare</u> how Hom-sets induce homomorphisms between them:

$f_*: \mathrm{Hom}_R(D,M) \longrightarrow \mathrm{Hom}_R(D,N)$

$\varphi \longmapsto \varphi_* = f \circ \varphi$

$$\begin{tikzcd} & D \arrow[dl, "\varphi"'] \arrow[d, dashed, "\varphi_*"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

$f^*: \mathrm{Hom}_R(N,D) \longrightarrow \mathrm{Hom}_R(M,D)$

$\varphi \longmapsto \varphi^* = \varphi \circ f$

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, dashed, "\varphi^*"'] & N \arrow[dl, "\varphi"] \\ D & \end{tikzcd}$$

<u>Thm 3.6:</u> Let Q be a unitary R-module. The following are equivalent:

(i) For any unitary R-modules L, M, N, if

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$ is exact, then

$$0 \longrightarrow \mathrm{Hom}_R(N,Q) \xrightarrow{g^*} \mathrm{Hom}_R(M,Q) \xrightarrow{f^*} \mathrm{Hom}_R(L,Q) \longrightarrow 0$$

is exact.

(ii) If $0 \longrightarrow L \xrightarrow{f} M$ is exact and $\psi \in \mathrm{Hom}_R(L,Q)$, then $\exists\, h \in \mathrm{Hom}_R(M,Q)$ s.t. $\psi = hf$. (h is a "lift" of φ).

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "\psi"'] & M \arrow[dl, dashed, "h"] \\ & Q & \end{tikzcd}$$

(iii) If Q is a submodule of the R-module M, then Q is a direct summand of M, ie, every short exact sequence

$$0 \longrightarrow Q \hookrightarrow M \twoheadrightarrow N \longrightarrow 0 \quad \text{splits.}$$

Pf: Exercise.

Def: An R-module Q is called injective if it satisfies any of the equivalent conditions of Thm 3.6.

Examples:

(1) $\mathbb{Z}$ is not an injective $\mathbb{Z}$-module, since the exact sequence

$$0 \longrightarrow \mathbb{Z} \xhookrightarrow{2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \quad \text{does not split.}$$
$$1 \longmapsto 2$$

(2) $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective (but not projective) $\mathbb{Z}$-modules.

(3) Fact: No non-zero finitely generated $\mathbb{Z}$-module is injective.

Cor: Every $\mathbb{Z}$-module is the submodule of an injective $\mathbb{Z}$-module.

Thm 3.7: Every unitary R-module M is contained in an injective R-module. (Exercise; see Dummit & Foote Ex. 10.5 #15-16).