

4. Tensor Products

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Motivating example 1: Consider a linear transformation $T: V \rightarrow V$ where V is an n -dimensional real vector space, and T has minimal polynomial $m_T(x) = x^2 + 1$. We say that T has eigenvalues $\lambda_{1,2} = \pm i$, and so the eigenvectors will also have complex entries. But V is a real vector space, so we've just "extended" it to be a \mathbb{C} -vector space.

Note: \mathbb{C} and V are both \mathbb{R} -modules, so what we're really doing is multiplying elements in two different \mathbb{R} -modules together. And we require associativity, e.g., $(i3)\vec{v} = i(3\vec{v})$.

Motivating example 2: Consider 2 rings $R \subseteq S$, $1_R = 1_S$, and let M be a left S -module.

Clearly, M can be considered as a left R -module, i.e., given an S -module structure, we can always define an R -module structure.

But the opposite direction fails in general.

Example: $\mathbb{Z} \subseteq \mathbb{Q}$. Recall that a ring R is trivially an R -module.

• We can think of \mathbb{Q} as a \mathbb{Q} -module, but also as a \mathbb{Z} -module.

• We can think of \mathbb{Z} as a \mathbb{Z} -module, but not as a \mathbb{Q} -module.

However, \mathbb{Z} (as a \mathbb{Z} -module) naturally embeds into \mathbb{Q} (as a \mathbb{Q} -module).

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Question: When can an R -module be mapped into an S -module?

Def: Suppose R is a ring, M a right R -module, and N is a left R -module. If A is an abelian group, then a function $f: M \times N \rightarrow A$ is a bilinear map if

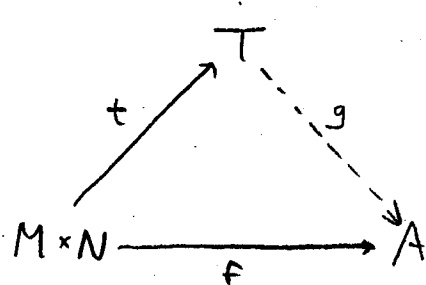
$$(i) f(x+u, y) = f(x, y) + f(u, y)$$

$$(ii) f(x, y+v) = f(x, y) + f(x, v)$$

$$(iii) f(xr, y) = f(x, ry)$$

for all $x, u \in M, y, v \in N, r \in R$.

Def: A tensor product of a right R -module M and a left R -module N is an abelian group T , together with a bilinear map $t: M \times N \rightarrow T$ such that given an abelian group A and any bilinear map $f: M \times N \rightarrow A$, $\exists!$ homomorphism $g: T \rightarrow A$ s.t. $f = gt$.



Prop 4.1: (Uniqueness) If a tensor product exists, it is unique up to multiplication.

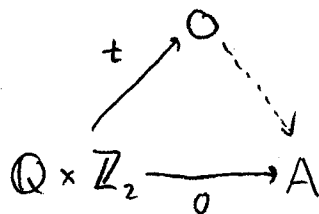
Pf: Exercise.

Example: Let $M = \mathbb{Q}, N = \mathbb{Z}_2, R = \mathbb{Z}$.

If $f: \mathbb{Q} \times \mathbb{Z}_2 \rightarrow A$ is a bilinear map, then

$$f(x, 1) = f\left(\frac{x}{2}, 2\right) = f\left(\frac{x}{2}, 0\right) = f\left(\frac{x}{2}, 0 \cdot 0\right) = f(0, 0) = 0$$

$T = 0$ is a tensor product for $\mathbb{Q} \otimes \mathbb{Z}_2$ over \mathbb{Z} .



Thm 4.2: (Existence) If R is a ring, M a right R -module, and N a left R -module, then a tensor product of M and N exists.

Pf: (Constructive): let F be the free abelian group based on the set $M \times N$, i.e., elements of the form $\sum_{\text{finite}} k_i (m_i, n_i)$, $k_i \in \mathbb{Z}$, $m_i \in M$, $n_i \in N$.

let $H \leq F$ be the subgroup generated by all elements of the form

- (i) $(x+u, y) - (x, y) - (u, y)$
- (ii) $(x, y+v) - (x, y) - (x, v)$
- (iii) $(xr, y) - (x, ry)$

for all $x, u \in M$, $y, v \in N$, $r \in R$.

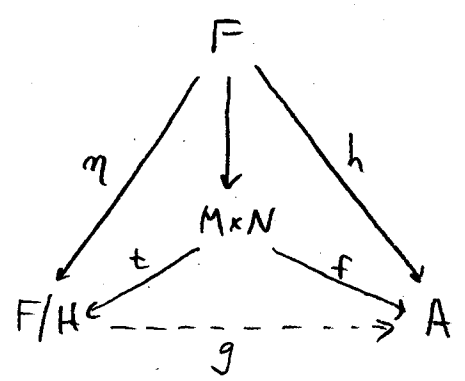
Set $T = F/H$, and define $t: M \times N \rightarrow T$
 $(x, y) \mapsto (x, y) + H$. (clearly bilinear)

Claim: This works.

check universal property: Suppose A is an abelian group and $f: M \times N \rightarrow A$ is a bilinear map.

Define $h: F \rightarrow A$
 $\sum k_i (m_i, n_i) \mapsto \sum k_i f(m_i, n_i)$.

let $\eta: F \rightarrow F/H$ be the canonical quotient map.



Then $\ker \eta \leq \ker h$, so by the universal property of quotient groups, $\exists!$ $g: F/H \rightarrow A$ s.t. $gt = f$. □

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By Props 4.1 & 4.2, we may call the group T the tensor product of M and N , over R .

Notation: Write $x \otimes y$ for $(x, y) + H$, and $\sum_i x \otimes y$ for a finite sum (i.e., an arbitrary element of T).

Write $M \otimes_R N$ for the group T .

Remark: $(x+u) \otimes y = x \otimes y + u \otimes y$

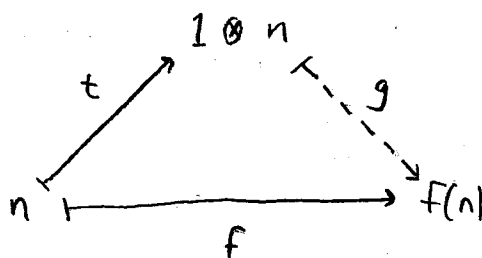
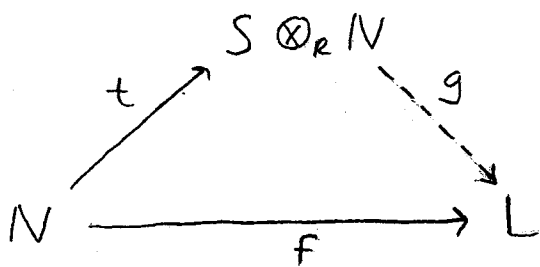
$x \otimes (y+v) = x \otimes y + x \otimes v$

$xr \otimes y = x \otimes ry$.

Thus, $M \otimes_R N$ is like the cross product $M \times N$, but \otimes_R acts like a "gate" that allows scalars $r \in R$ to pass through.

* Big idea of the universal property of tensor products (which actually arises as a special case).

Cor 4.3: Let $R \subseteq S$ be rings with $1_R = 1_S$, let N be a left R -module and $t: N \rightarrow S \otimes_R N$, $t: n \mapsto 1 \otimes n$. If L is any left S -module, and $f \in \text{Hom}_R(N, L)$, then $\exists!$ $g \in \text{Hom}_S(S \otimes_R N, L)$ such that $f = gt$.



Note: t, f are R -homomorphism, g is an S -homomorphism.

* In other words, every R -module homomorphism from an R -module into an S -module (where $R \subseteq S$) can be factored through $S \otimes_R N$.

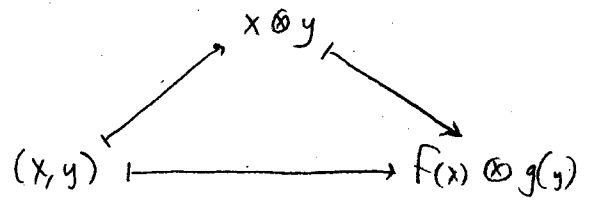
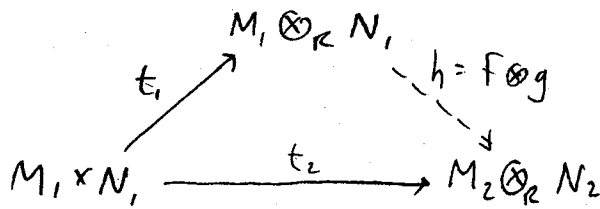
Prop 4.4: Suppose R is a ring, $M_{1,2}$ right R -modules, $N_{1,2}$ left R -modules, $f \in \text{Hom}_R(M_1, M_2)$ and $g \in \text{Hom}_R(N_1, N_2)$. Then:

(i) $\exists!$ $h \in \text{Hom}_Z(M_1 \otimes_R N_1, M_2 \otimes_R N_2)$ s.t. $h(x \otimes y) = f(x) \otimes g(y) \quad \forall x \in M_1, y \in N_1$.

(ii) Furthermore, if $f' \in \text{Hom}_R(M_2, M_3)$ and $g' \in \text{Hom}_R(N_2, N_3)$, then $(f' \otimes g')(f \otimes g) = f' \circ f \otimes g' \circ g$.

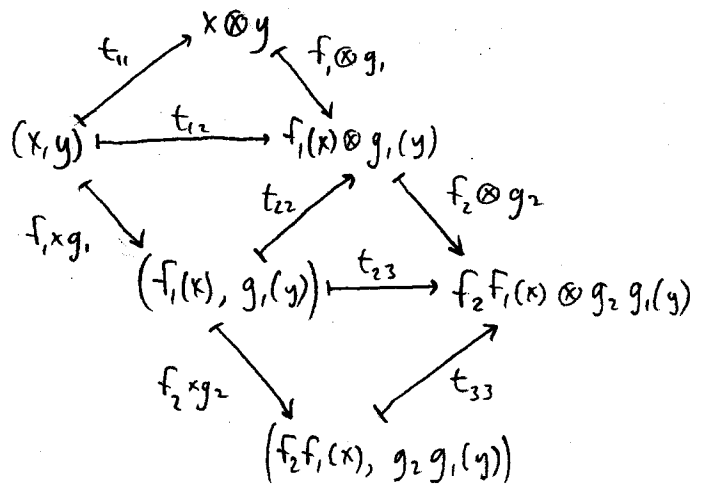
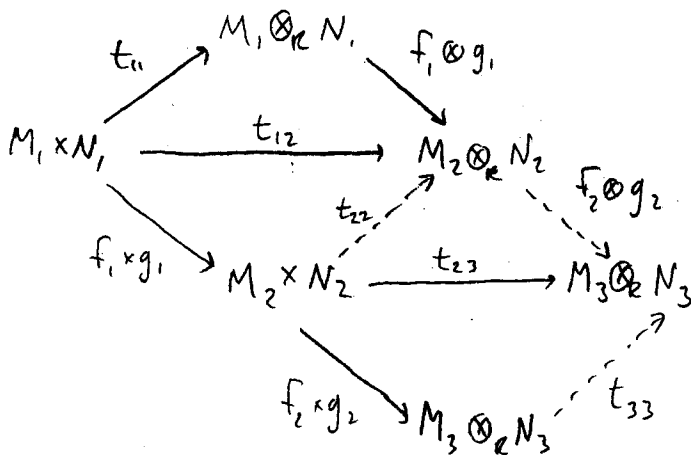
PF (sketch):

(i) Define the bilinear maps $t_1: M_1 \times N_1 \rightarrow M_1 \otimes_R N_1, (x, y) \mapsto x \otimes y$
 $t_2: M_1 \times N_1 \rightarrow M_2 \otimes_R N_2, (x, y) \mapsto f(x) \otimes g(y)$.



(Check that this works)

(ii) Check that the following works:



□

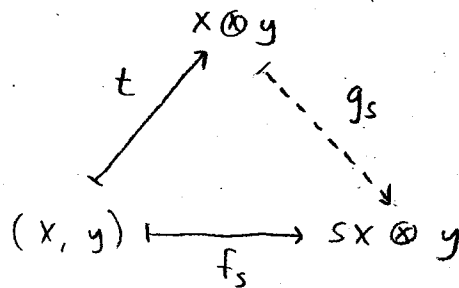
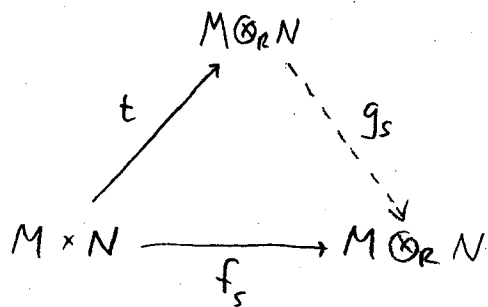
[6]

Def: If S and R are rings, then an abelian group M is an S - R -bimodule if it is simultaneously a left S -module and a right R -module, with the additional requirement that $s(xr) = (sx)r \quad \forall s \in S, x \in M, r \in R$.

If M is an S - R -bimodule and N a left R -module, then we can view $M \otimes_R N$ as a left S -module, as follows:

$$S \left(\sum_i x_i \otimes y_i \right) = \sum_i Sx_i \otimes y_i \quad (*)$$

Why this works (sketch):



Since g_s exists, S acts on $M \otimes_R N$ by

$$s \cdot \sum_i x_i \otimes y_i = \sum_i g_s(x_i \otimes y_i) = \sum_i Sx_i \otimes y_i, \text{ thus the definition } (*)$$

is well-defined.

Prop 4.5: Let N be a unitary left R -module, and view R as an R - R -bimodule. Then $R \otimes_R N$ is R -isomorphic with N .

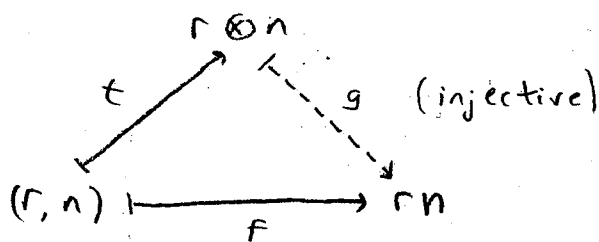
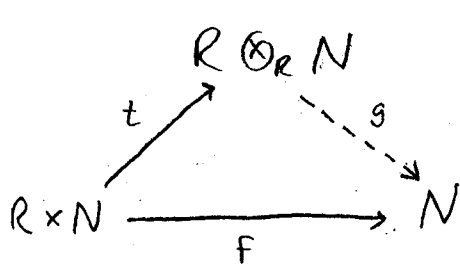
Pf: The map $f: R \times N \rightarrow N$, $f(r, n) = rn$ is bilinear.

Thus $\exists g: R \otimes_R N \rightarrow N$ s.t. $g(r \otimes n) = rn \quad \forall r \in R, n \in N$.

If $\sum_i r_i \otimes n_i \in \ker g$, then $\sum_i r_i n_i = 0$

$$\text{and } \sum_i r_i \otimes n_i = \sum_i 1 \cdot r_i \otimes n_i = \sum_i 1 \otimes r_i n_i = 1 \otimes 0 = 0 \Rightarrow \ker g = 0$$

Picture of this:



□

Prop 4.6: Suppose M is a right R -module and $N_{1,2}$ are left R -modules. Then $M \otimes_R (N_1 \oplus N_2) \cong (M \otimes_R N_1) \oplus (M \otimes_R N_2)$.

Define homomorphisms

$$p_{1,2} : N_1 \oplus N_2 \longrightarrow N_1 \oplus N_2$$

$$p_1 : (n_1, n_2) \longmapsto (n_1, 0)$$

$$p_2 : (n_1, n_2) \longmapsto (0, n_2)$$

By Prop 4.4, \exists homomorphisms

$$f_i = 1 \otimes p_i : M \otimes_R (N_1 \oplus N_2) \longrightarrow M \otimes_R (N_1 \oplus N_2)$$

$$1 \otimes p_1 : m \otimes (n_1, n_2) \longmapsto m \otimes (n_1, 0)$$

$$1 \otimes p_2 : m \otimes (n_1, n_2) \longmapsto m \otimes (0, n_2)$$

Clearly, $f_1 + f_2 = 1$, $f_1 f_2 = f_2 f_1 = 0$, and $f_i^2 = f_i$, $i=1,2$.

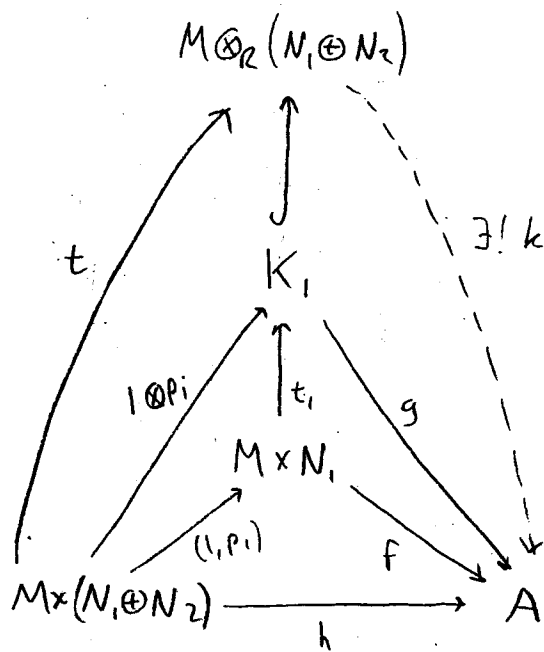
Thus if $K_i := \text{Im } f_i$, then $M \otimes_R (N_1 \oplus N_2) = K_1 \oplus K_2$ (easy to check this!)

* It now suffices to show that $K_i \cong M \otimes_R N_i$ for $i=1,2$.

The map $t_1 : M \times N_1 \longrightarrow K_1$

$$(m, n_1) \longmapsto m \otimes (n_1, 0) = f_1(m \otimes (n_1, 0)) \text{ is bilinear.}$$

Let $f : M \times N_1 \longrightarrow A$ be a bilinear map to an abelian group A .



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Define $h: M \times (N_1 \oplus N_2) \longrightarrow A$

$$(m, (n_1, n_2)) \longmapsto f(m, n_1) \quad \text{which is also bilinear.}$$

Then $\exists!$ homomorphism $k: M \otimes_{\mathbb{R}} (N_1 \oplus N_2) \longrightarrow A$ s.t. $h = k \circ t$

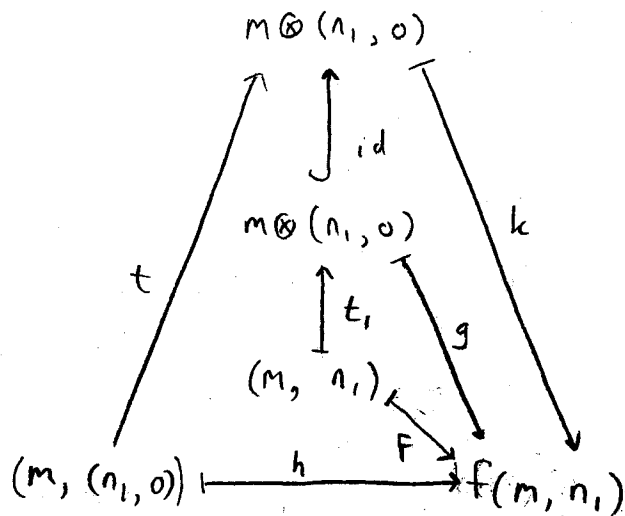
(by univ. property of tensor products).

Since $K_1 = \text{im } t_1 \subseteq M \otimes_{\mathbb{R}} (N_1 \oplus N_2)$, let $g = k|_{K_1} \in \text{Hom}_{\mathbb{R}}(K_1, A)$.

Claim: $g \circ t_1 = f$

Pf (see diagram for motivation)

$$\begin{aligned} g \circ t_1(m, n_1) &= g(m \otimes (n_1, 0)) \\ &= k(m \otimes (n_1, 0)) \\ &= k \circ t(m, (n_1, 0)) \\ &= h(m, (n_1, 0)) \\ &= f(m, n_1) \quad \checkmark \end{aligned}$$



Uniqueness follows because K_1 is generated by $\text{im}(t_1)$, and so

$$K_1 \cong M \otimes_{\mathbb{R}} N_1.$$

An analogous argument shows that $K_2 \cong M \otimes_{\mathbb{R}} N_2$. □

Remark: There is an analogous statement to Prop 4.6, when

$M = M_1 \oplus M_2$, and M_i is an S - R -bimodule. In this case,

$$\text{the isomorphism } (M_1 \oplus M_2) \otimes_{\mathbb{R}} (N_1 \oplus N_2) \cong [(M_1 \oplus M_2) \otimes_{\mathbb{R}} N_1] \oplus [(M_1 \oplus M_2) \otimes_{\mathbb{R}} N_2]$$

is an S -isomorphism.

Recall "Motivating Example 1": Suppose V is an F -vector space, and K/F an extension field. We can now formalize the natural construction of V as a K -vector space.

For convenience, we'll make the (unnecessary) assumption that $\dim V = n < \infty$, and $\{x_1, \dots, x_n\}$ is an F -basis.

$$\text{Now, } V = Fx_1 \oplus \dots \oplus Fx_n \cong F^n,$$

and K is a unitary K - F -bimodule

Define $V^k := K \otimes_F V$, which is a unitary k -module, i.e., a K -vector space.

$$\text{Note: } V^k := K \otimes_F V \cong K \otimes_F (F^n) \cong (K \otimes_F F)^n \cong K^n$$

$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{Since } V \cong F^n & \text{Prop 4.6} & \text{Prop 4.5} \end{array}$

Therefore, $\dim_k(V^k) = \dim_F V$.

An element $x \in K^n$ can therefore be written as

$$x = \sum_{i=1}^m a_i \otimes v_i \in V^k, \quad \text{where } v_i = \sum_{j=1}^n b_{ji} x_j \in F^n, \quad b_{ji} \in F.$$

$$\text{Therefore, } x = \sum_{i=1}^m a_i \otimes \left(\sum_{j=1}^n b_{ji} x_j \right) = \sum_{j=1}^n \left(\sum_{i=1}^m b_{ji} a_i \right) (1 \otimes x_j),$$

and so $\{1 \otimes x_1, \dots, 1 \otimes x_n\}$ is a k -basis for V^k .

The function $v \mapsto 1 \otimes v$ is a 1-1 map $V \rightarrow V^k$, so we

may identify V with the image, and view V as a subset (but not a k -subspace) of V^k .

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Exercise: Show that $\dim_F(V^K) = [K:F] \dim_F V$.

Example: Let V be a 3-dim \mathbb{R} -vector space.

$$\text{Then } V^{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} V \cong \mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}) \cong (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R})^3 \cong \mathbb{C}^3.$$

Thus, $V^{\mathbb{C}}$ is a 3-dim \mathbb{C} -vector space, and \mathbb{C} is a 2-dim \mathbb{R} -vector space, so by Exercise, $V^{\mathbb{C}}$ is a

$$[\mathbb{C}:\mathbb{R}] \cdot \dim_{\mathbb{R}} V = 2 \cdot 3 = 6\text{-dimensional } \mathbb{R}\text{-vector space.}$$

An element $x \in V^{\mathbb{C}}$ has the form

$$x = \sum_{i=1}^3 z_i \otimes \vec{e}_i \quad z_i \in \mathbb{C}, \quad \vec{e}_i \in V \text{ (unit basis vector)}$$

$$\begin{aligned} \text{and so } x &= (z_1 \otimes \vec{e}_1) + (z_2 \otimes \vec{e}_2) + (z_3 \otimes \vec{e}_3) \\ &= (x_1 + y_1 i) \otimes \vec{e}_1 + (x_2 + y_2 i) \otimes \vec{e}_2 + (x_3 + y_3 i) \otimes \vec{e}_3 \\ &= (x_1 \otimes \vec{e}_1) + (y_1 i \otimes \vec{e}_1) + (x_2 \otimes \vec{e}_2) + (y_2 i \otimes \vec{e}_2) + (x_3 \otimes \vec{e}_3) + (y_3 i \otimes \vec{e}_3) \end{aligned}$$

i.e., $\{1 \otimes \vec{e}_1, 1 \otimes \vec{e}_2, 1 \otimes \vec{e}_3\}$ is a \mathbb{C} -basis for $V^{\mathbb{C}}$

and $\{1 \otimes \vec{e}_1, i \otimes \vec{e}_1, 1 \otimes \vec{e}_2, i \otimes \vec{e}_2, 1 \otimes \vec{e}_3, i \otimes \vec{e}_3\}$ is an \mathbb{R} -basis for V .

The associativity, e.g., $\underbrace{(i3)}_{\in \mathbb{C}} \underbrace{\vec{v}}_{\in V} = i \underbrace{(3\vec{v})}_{\in V} \in V^{\mathbb{C}}$ is guaranteed by the

bilinearity of the tensor product: $i3 \otimes \vec{v} = i \otimes 3\vec{v} \in \mathbb{C} \otimes_{\mathbb{R}} V$.

Remark: If F is a field, and V, W are F -vector spaces, then

$$\dim_F(V \otimes_F W) = (\dim_F V)(\dim_F W).$$

Again, let V, W be F -vector spaces, and suppose $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$ are bases for V & W , respectively.

Then $\{x_i \otimes y_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ is a basis for $V \otimes_F W$.

Suppose $S: V \rightarrow V$, $T: W \rightarrow W$ are linear maps, represented by matrices $A = (a_{ij})_{n \times n}$, $B = (b_{ij})_{m \times m}$ (relative to $\{x_i\}$ and $\{y_j\}$).

By Prop 4.4, there is a linear map:

$$\begin{aligned} S \otimes T: V \otimes_F W &\longrightarrow V \otimes_F W \\ v \otimes w &\longmapsto S(v) \otimes T(w) \end{aligned}$$

Order the basis: $x_1 \otimes y_1, x_1 \otimes y_2, \dots, x_1 \otimes y_m,$
 $x_2 \otimes y_1, x_2 \otimes y_2, \dots, x_2 \otimes y_m, \dots$
 $\dots, x_n \otimes y_1, x_n \otimes y_2, \dots, x_n \otimes y_m.$

Now, the matrix $A \otimes B$ that represents $S \otimes T$ has the block form $(a_{ij} B)_{n \times n}$, called the Kronecker product of A and B .

This arises a lot in representation theory (of groups), and thus in quantum mechanics.

Def: If R is a commutative ring, then a ring A is called an R -algebra if A is a left R -module such that:

$$r(ab) = (ra)b = a(rb) \quad \text{for all } r \in R, a, b \in A.$$

Motivation: an R -algebra is an R -module (or R -vector space, if R is a field), where we can additionally multiply "vectors."

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Examples:

- (1) A \mathbb{Z} -algebra is just an arbitrary ring A .
- (2) If R is a commutative ring with 1, then any polynomial ring $R[x_1, \dots, x_n]$ is an R -algebra.
- (3) If F is a field and K/F an extension, then K is an F -algebra.
- (4) If V is an F -vector space, then the ring $A = \text{Hom}_F(V, V)$ is an F -algebra.

Suppose R is commutative, and A & B are R -algebras, and A is an R - R -bimodule, then $A \otimes_R B$ is an R -module.

We can also make it into an R -algebra, as follows:

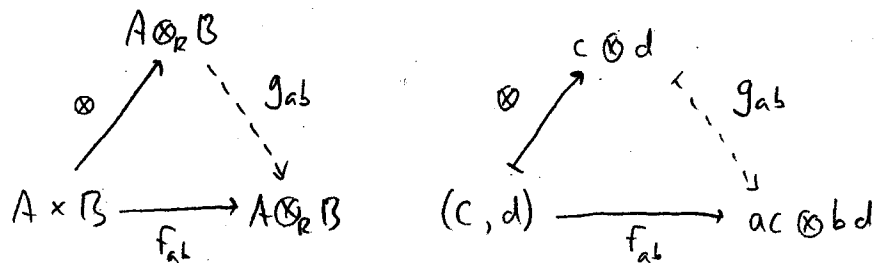
Fix $a \in A$, $b \in B$, define $f_{ab}: A \times B \rightarrow A \otimes_R B$
 $(c, d) \mapsto ac \otimes bd$

Since f_{ab} is bilinear, $\exists!$

$g_{ab}: A \otimes_R B \rightarrow A \otimes_R B$

s.t. $g_{ab}(c \otimes d) \mapsto ac \otimes bd$

for all $c \in A$, $d \in B$.



Define the " R -algebra multiplication" as follows:

$$\left(\sum_i a_i \otimes b_i\right) \left(\sum_j c_j \otimes d_j\right) = \sum_i g_{a_i, b_i} \left(\sum_j c_j \otimes d_j\right) = \sum_{ij} a_i c_j \otimes b_i d_j$$

Need to check that this works (easy!). If R has 1 and A is unital, then so is $A \otimes_R B$.

The following is in a sense a "generalization" of the Chinese Remainder Theorem:

Thm 4.7: Suppose R is commutative with 1, and I, J are ideals in R . Then $(R/I) \otimes_R (R/J) \cong R/(I+J)$ (as R -algebras).

PF: We must show that $R/(I+J)$ is a tensor product for R/I and R/J , using the universal property.

Define the balanced map

$$t: R/I \times R/J \rightarrow R/(I+J)$$

$$(r+I, s+J) \mapsto rs+(I+J). \quad \text{If } f: R/I \times R/J \rightarrow A \text{ is an arbitrary}$$

homomorphism to an abelian group A , then define

$$g: R/(I+J) \rightarrow A$$

$$r+(I+J) \mapsto f(r+I, 1+J).$$

We need to check that this works!

Well-defined: Suppose $r+(I+J) = r'+(I+J)$.

Then $r' = r+a+b$ for some $a \in I, b \in J$.

Thus, $r'+I = r+b+I \in R/I$.

$$\begin{aligned} \Rightarrow \underbrace{f(r'+I, 1+J)}_{g(r+(I+J))} &= f(r+b+I, 1+J) \stackrel{\text{bilinearity}}{=} f(r+I, 1+J) + f(b+I, 1+J) \\ &= f(r+I, 1+J) + f(1+I, b+J) \\ &= f(r+I, 1+J) + f(1+I, 0) \\ &= f(r+I, 1+J) \\ &= g(r'+(I+J)) \quad \checkmark \end{aligned}$$

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g is clearly a homomorphism of abelian groups ✓

$$\begin{aligned}
 \text{Check } g \circ t = f: \quad g(t(r+I, s+J)) &= g(rs+I+J) \\
 &= f(rs+I, 1+J) && \text{by def'n} \\
 &= f((r+I)s, 1+J) \\
 &= f(r+I, s(1+J)) && \text{bilinearity} \\
 &= f(r+I, s+J) \quad \checkmark
 \end{aligned}$$

Uniqueness is clear as well (t is surjective) ✓

By Prop 4.1, $R/(I+J)$ is the unique tensor product for R/I and R/J over R , i.e., $R/(I+J) \cong (R/I) \otimes_R (R/J)$ as abelian groups.

Remark: The explicit isomorphism is $r+(I+J) \longmapsto (r+I) \otimes (1+J)$. □

Cor: $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n \cong \mathbb{Z}_{(m,n)}$

Pf: $\mathbb{Z}_m = \mathbb{Z}/(m)$, $\mathbb{Z}_n = \mathbb{Z}/(n)$, $(m) + (n) = (m, n)$. □

Note that the special case of when $(m, n) = 1$ is guaranteed by the Chinese Remainder Theorem.