4. Tensor Products

Motivating example 1: Consider a linear transformation $T: V \longrightarrow V$ where V is an n-dimensional real vector space, and T has minimal polynomial $m_T(x) = x^2+1$. We say that T has eigenvalues $\lambda_{1,2} = \pm i$, and so the eigenvectors will also have complex entries. But V is a real vector space, so we've just "extended" it to be a $\mathbb{C}$-vector space.

Note: $\mathbb{C}$ and V are both $\mathbb{R}$-modules, so what we're really doing is multiplying elements in two different $\mathbb{R}$-modules together. And we require associativity, e.g., $(i\,3)\vec{v} = i(3\vec{v})$.

Motivating example 2: Consider 2 rings $R \subseteq S$, $1_R = 1_S$, and let M be a left S-module.

Clearly, M can be considered as a left R-module, i.e., given an S-module structure, we can always define an R-module structure.

But the opposite direction fails in general.

Example: $\mathbb{Z} \subseteq \mathbb{Q}$. Recall that a ring R is trivially an R-module.

* We can think of $\mathbb{Q}$ as a $\mathbb{Q}$-module, but also as a $\mathbb{Z}$-module.
* We can think of $\mathbb{Z}$ as a $\mathbb{Z}$-module, but not as a $\mathbb{Q}$-module.

However, $\mathbb{Z}$ (as a $\mathbb{Z}$-module) naturally embeds into $\mathbb{Q}$ (as a $\mathbb{Q}$-module).

Question: When can an R-module be mapped into an S-module?

Def: Suppose R is a ring, M a right R-module, and N is a left R-module. If A is an abelian group, then a function $f: M \times N \longrightarrow A$ is a bilinear map if

(i) $f(x+u, y) = f(x,y) + f(u,y)$

(ii) $f(x, y+v) = f(x,y) + f(x,v)$

(iii) $f(xr, y) = f(x, ry)$

for all $x, u \in M$, $y, v \in N$, $r \in R$.

Def: A tensor product of a right R-module M and a left R-module N is an abelian group T, together with a bilinear map $t: M \times N \longrightarrow T$ such that given an abelian group A and any bilinear map $f: M \times N \longrightarrow A$, $\exists!$ homomorphism $g: T \longrightarrow A$ s.t $f = gt$.

$$\begin{tikzcd} & T \arrow[dr, dashed, "g"] & \\ M \times N \arrow[ur, "t"] \arrow[rr, "f"'] & & A \end{tikzcd}$$

Prop 4.1: (Uniqueness) If a tensor product exists, it is unique up to multiplication.

Pf: Exercise.

Example: Let $M = \mathbb{Q}$, $N = \mathbb{Z}_2$, $R = \mathbb{Z}$.

If $f: \mathbb{Q} \times \mathbb{Z}_2 \longrightarrow A$ is a bilinear map, then

$$f(x,1) = f\left(\tfrac{x}{2}, 2\right) = f\left(\tfrac{x}{2}, 0\right) = f\left(\tfrac{x}{2}, 0 \cdot 0\right) = f(0,0) = 0$$

$T = 0$ is a tensor product for $\mathbb{Q}$ & $\mathbb{Z}_2$ over $\mathbb{Z}$.

$$\begin{tikzcd} & 0 \arrow[dr, dashed] & \\ \mathbb{Q} \times \mathbb{Z}_2 \arrow[ur, "t"] \arrow[rr, "0"'] & & A \end{tikzcd}$$

Thm 4.2: (Existence) If R is a ring, M a right R-module, and N a left R-module, then a tensor product of M and N exists.

Pf: (Constructive): Let F be the free abelian group based on the set $M \times N$, i.e., elements of the form $\sum_{\text{finite}} k_i (m_i, n_i)$, $k_i \in \mathbb{Z}$, $m_i \in M$, $n_i \in N$.

Let $H \leq F$ be the subgroup generated by all elements of the form

(i) $(x+u, y) - (x, y) - (u, y)$

(ii) $(x, y+v) - (x, y) - (x, v)$

(iii) $(xr, y) - (x, ry)$

for all $x, u \in M$, $y, v \in N$, $r \in R$.

Set $T = F/H$, and define $t : M \times N \longrightarrow T$
$(x, y) \longmapsto (x, y) + H$. (clearly bilinear)

Claim: This works.

Check universal property: Suppose A is an abelian group and $f : M \times N \longrightarrow A$ is a bilinear map.

$$\begin{tikzcd}
 & F \arrow[dl, "\eta"'] \arrow[d] \arrow[dr, "h"] & \\
 & M \times N \arrow[dl, "t"'] \arrow[dr, "f"] & \\
F/H \arrow[rr, dashed, "g"'] & & A
\end{tikzcd}$$

Define $h : F \longrightarrow A$
$\sum k_i (m_i, n_i) \longmapsto \sum k_i f(m_i, n_i)$.

Let $\eta : F \longrightarrow F/H$ be the canonical quotient map.

Then $\ker \eta \leq \ker h$, so by the universal property of quotient groups, $\exists!\ g : F/H \longrightarrow A$ s.t. $gt = f$. $\square$

By Props 4.1 & 4.2, we may call the group T __the__ tensor product of M and N, over R.

__Notation:__ Write $x \otimes y$ for $(x,y)+H$, and $\sum_i x \otimes y$ for a finite sum (i.e., an arbitrary element of T).

Write $M \otimes_R N$ for the group T.

__Remark:__
$$(x+u)\otimes y = x \otimes y + u \otimes y$$
$$x \otimes (y+v) = x \otimes y + x \otimes y$$
$$xr \otimes y = x \otimes ry.$$

Thus, $M \otimes_R N$ is like the cross product $M \times N$, but $\otimes_R$ acts like a "gate" that allows scalars $r \in R$ to pass through.

* Big idea of the universal property of tensor products (which actually arises as a special case.

__Cor 4.3:__ Let $R \subseteq S$ be rings with $1_R = 1_S$, let N be a left R-module and $t: N \longrightarrow S \otimes_R N$, $t: n \longmapsto 1 \otimes n$. If L is any left S-module, and $f \in \mathrm{Hom}_R(N, L)$, then $\exists!\ g \in \mathrm{Hom}_S(S \otimes_R N, L)$ such that $f = gt$.

$$\begin{tikzcd} & S \otimes_R N \arrow[dr, dashed, "g"] & \\ N \arrow[ur, "t"] \arrow[rr, "f"'] & & L \end{tikzcd} \qquad \begin{tikzcd} & 1 \otimes n \arrow[dr, dashed, mapsto, "g"] & \\ n \arrow[ur, mapsto, "t"] \arrow[rr, mapsto, "f"'] & & f(n) \end{tikzcd}$$

__Note:__ t, f are R-homomorphism, g is an S-homomorphism.

* In other words, every R-module homomorphism from an R-module into an S-module (where $R \subseteq S$) can be factored through $S \otimes_R N$.

<u>Prop 4.4</u>: Suppose R is a ring, $M_{1,2}$ right R-modules, $N_{1,2}$ left R-modules, $f \in \mathrm{Hom}_R(M_1, M_2)$ and $g \in \mathrm{Hom}_R(N_1, N_2)$. Then:

(i) $\exists!\ h \in \mathrm{Hom}_{\mathbb{Z}}(M_1 \otimes_R N_1, M_2 \otimes_R N_2)$ s.t. $h(x \otimes y) = f(x) \otimes g(y)$ $\forall x \in M_1, y \in N_1$.

(ii) Furthermore, if $f \in \mathrm{Hom}_R(M_2, M_3)$ and $g' \in \mathrm{Hom}_R(N_2, N_3)$, then
$$(f' \otimes g')(f \otimes g) = f' \circ f \otimes g' \circ g.$$

<u>Pf</u> (sketch):

(i) Define the bilinear maps $t_1 : M_1 \times N_1 \longrightarrow M_1 \otimes_R N_1$, $(x, y) \longmapsto x \otimes y$

$t_2 : M_1 \times N_1 \longrightarrow M_2 \otimes_R N_2$, $\quad (x, y) \longmapsto f(x) \otimes g(y)$.

$$\begin{tikzcd}
 & M_1 \otimes_R N_1 \arrow[dr, dashed, "h = f \otimes g"] & \\
M_1 \times N_1 \arrow[ur, "t_1"] \arrow[rr, "t_2"] & & M_2 \otimes_R N_2
\end{tikzcd}
\qquad
\begin{tikzcd}
 & x \otimes y \arrow[dr, mapsto] & \\
(x, y) \arrow[ur, mapsto] \arrow[rr, mapsto] & & f(x) \otimes g(y)
\end{tikzcd}$$

(Check that this works)

(ii) Check that the following works:

$$\begin{tikzcd}
 & M_1 \otimes_R N_1 \arrow[dr, "f_1 \otimes g_1"] & & \\
M_1 \times N_1 \arrow[ur, "t_{11}"] \arrow[rr, "t_{12}"] \arrow[dr, "f_1 \times g_1"'] & & M_2 \otimes_R N_2 \arrow[dr, dashed, "f_2 \otimes g_2"] & \\
 & M_2 \times N_2 \arrow[ur, dashed, "t_{22}"] \arrow[rr, "t_{23}"] \arrow[dr, "f_2 \times g_2"'] & & M_3 \otimes_R N_3 \\
 & & M_3 \otimes_R N_3 \arrow[ur, dotted, "t_{33}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & x \otimes y \arrow[dr, mapsto, "f_1 \otimes g_1"] & & \\
(x, y) \arrow[ur, mapsto, "t_{11}"] \arrow[rr, mapsto, "t_{12}"] \arrow[dr, mapsto, "f_1 \times g_1"'] & & f_1(x) \otimes g_1(y) \arrow[dr, mapsto, "f_2 \otimes g_2"] & \\
 & (f_1(x), g_1(y)) \arrow[ur, mapsto, "t_{22}"] \arrow[rr, mapsto, "t_{23}"] \arrow[dr, mapsto, "f_2 \times g_2"'] & & f_2 f_1(x) \otimes g_2 g_1(y) \\
 & & (f_2 f_1(x), g_2 g_1(y)) \arrow[ur, mapsto, "t_{33}"'] &
\end{tikzcd}$$

$\square$

<u>Def</u>: If S and R are rings, then an abelian group M is an <u>S-R-bimodule</u> if it is simultaneously a left S-module and a right R-module, with the additional requirement that

$$s(xr) = (sx)r \quad \forall s \in S,\ x \in M,\ r \in R.$$

If M is an S-R-bimodule and N a left R-module, then we can view $M \otimes_R N$ as a left S-module, as follows:

$$s\Big(\sum_i x_i \otimes y_i\Big) = \sum_i s x_i \otimes y_i \qquad (*)$$

<u>Why this works</u> (sketch):

$$\begin{tikzcd} & M\otimes_R N \arrow[dr, dashed, "g_s"] & \\ M\times N \arrow[ur, "t"] \arrow[rr, "f_s"'] & & M\otimes_R N \end{tikzcd} \qquad \begin{tikzcd} & x\otimes y \arrow[dr, dashed, mapsto, "g_s"] & \\ (x,y) \arrow[ur, mapsto, "t"] \arrow[rr, mapsto, "f_s"'] & & sx\otimes y \end{tikzcd}$$

Since g_s exists, S acts on $M \otimes_R N$ by

$$s \cdot \sum_i x_i \otimes y_i = \sum_i g_s(x_i \otimes y_i) = \sum_i s x_i \otimes y_i,$$ thus the definition $(*)$ is well-defined.

<u>Prop 4.5</u>: Let N be a unitary left R-module, and view R as an R-R-bimodule. Then $R \otimes_R N$ is R-isomorphic with N.

<u>Pf</u>: The map $f : R \times N \longrightarrow N$, $f(r,n) = rn$ is bilinear.

Thus $\exists\ g : R \otimes_R N \longrightarrow N$ s.t. $g(r \otimes n) = rn$ $\forall r \in R,\ n \in N$.

If $\sum_i r_i \otimes n_i \in \ker g$, then $\sum_i r_i n_i = 0$ &

and $\sum_i r_i \otimes n_i = \sum_i 1 \cdot r_i \otimes n_i = \sum_i 1 \otimes r_i y_i = 1 \otimes 0 = 0 \Rightarrow \ker g = 0$

Picture of this:

$$\begin{tikzcd} & R\otimes_R N \arrow[dr, dashed, "g"] & \\ R\times N \arrow[ur, "t"] \arrow[rr, "f"'] & & N \end{tikzcd} \qquad \begin{tikzcd} & r\otimes n \arrow[dr, dashed, mapsto, "g"] & \\ (r,n) \arrow[ur, mapsto, "t"] \arrow[rr, mapsto, "f"'] & & rn \end{tikzcd}$$

(injective)

$\square$

<u>Prop 4.6</u>: Suppose M is a right R-module and $N_{1,2}$ are left R-modules. Then $M\otimes_R(N_1\oplus N_2)\cong(M\otimes_R N_1)\oplus(M\otimes_R N_2)$.

Define homomorphisms

$$p_{1,2}: N_1\oplus N_2 \longrightarrow N_1\oplus N_2$$

$$p_1:(n_1,n_2)\longmapsto(n_1,0)$$

$$p_2:(n_1,n_2)\longmapsto(0,n_2)$$

$$\begin{tikzcd} & M\otimes_R(N_1\oplus N_2) \arrow[ddddr, dashed, bend left, "\exists!\, k"] & \\ & K_1 \arrow[u, hook] \arrow[dddr, "g"] & \\ & M\times N_1 \arrow[u, "t_1"] \arrow[ddr, "f"] & \\ & & \\ M\times(N_1\oplus N_2) \arrow[uuuur, bend left, "t"] \arrow[uuur, "1\otimes p_i"] \arrow[uur, "(1,p_1)"'] \arrow[rr, "h"'] & & A \end{tikzcd}$$

By Prop 4.4, $\exists$ homomorphisms

$$f_i = 1\otimes p_i : M\otimes_R(N_1\oplus N_2)\longrightarrow M\otimes_R(N_1\oplus N_2)$$

$$1\otimes p_1: m\otimes(n_1,n_2)\longmapsto m\otimes(n_1,0)$$

$$1\otimes p_2: m\otimes(n_1,n_2)\longmapsto m\otimes(0,n_2)$$

Clearly, $f_1+f_2=1$, $f_1f_2=f_2f_1=0$, and $f_i^2=f_i$, $i=1,2$.

Thus if $K_i := \operatorname{Im} f_i$, then $M\otimes_R(N_1\oplus N_2)=K_1\oplus K_2$ (easy to check this!)

* It now suffices to show that $K_i\cong M\otimes_R N_i$ for $i=1,2$.

The map $t_1: M\times N_1\longrightarrow K_1$

$$(m,n_1)\longmapsto x\otimes(n_1,0)=f_1(m\otimes(n_1,0)) \quad \text{is bilinear.}$$

Let $f: M\times N_1\longrightarrow A$ be a bilinear map to an abelian group A.

Define $h : M \times (N_1 \oplus N_2) \longrightarrow A$

$$(m, (n_1, n_2)) \longmapsto f(m, n_1)$$ which is also bilinear.

Then $\exists!$ homomorphism $k : M \otimes_R (N_1 \oplus N_2) \longrightarrow A$ s.t. $h = k\epsilon$

(by univ. property of tensor products).

Since $K_1 = \operatorname{im} f_1 \subseteq M \otimes_R (N_1 \oplus N_2)$, let $g = k|_{K_1} \in \operatorname{Hom}_R(K_1, A)$.

<u>Claim:</u> $g t_1 = f$

<u>Pf</u> (see diagram for motivation)

$$\begin{aligned} g t_1(m, n_1) &= g(m \otimes (n_1, 0)) \\ &= k(m \otimes (n_1, 0)) \\ &= kt(m, (n_1, 0)) \\ &= h(m, (n_1, 0)) \\ &= f(m, n_1) \quad \checkmark \end{aligned}$$

$$\begin{tikzcd} & m \otimes (n_1, 0) \arrow[ddrr, mapsto, "k"] & & \\ & m \otimes (n_1, 0) \arrow[u, hook, "id"'] \arrow[ddrr, mapsto, "g"] & & \\ & (m, n_1) \arrow[u, mapsto, "t_1"'] \arrow[drr, mapsto, "f"] & & \\ (m, (n_1, 0)) \arrow[uuur, "t"] \arrow[rrr, mapsto, "h"] & & & f(m, n_1) \end{tikzcd}$$

Uniqueness follows because K_1 is generated by $\operatorname{im}(t_1)$, and so

$K_1 \cong M \otimes_R N_1$.

An analogous argument shows that $K_2 \cong M \otimes_R N_2$. $\square$

<u>Remark:</u> There is an analogous statement to Prop 4.6, when $M = M_1 \oplus M_2$, and M_i is an S-R-bimodule. In this case, the isomorphism $(M_1 \oplus M_2) \otimes_R (N_1 \oplus N_2) \cong [(M_1 \oplus M_2) \otimes_R N_1] \oplus [(M_1 \oplus M_2) \otimes_R N_2]$ is an S-isomorphism.

Recall "Motivating Example 1": Suppose V is an F-vector space, and K/F an extension field. We can now formalize the natural construction of V as a K-vector space.

For convenience, we'll make the (unnecessary) assumption that $\dim V = n < \infty$, and $\{x_1, \dots, x_n\}$ is an F-basis.

Now, $V = Fx_1 \oplus \dots \oplus Fx_n \cong F^n$,

and K is a unitary K-F-bimodule

Define $V^K := K \otimes_F V$, which is a unitary K-module, i.e, a K-vector space.

Note: $V^K := K \otimes_F V \cong K \otimes_F (F^n) \cong (K \otimes_F F)^n \cong K^n$

since $V \cong F^n$; Prop 4.6 ; Prop 4.5

Therefore, $\dim_K(V^K) = \dim_F V$.

An element $x \in K^n$ can therefore be written as

$$x = \sum_{i=1}^{m} a_i \otimes v_i \in V^K, \quad \text{where } v_i = \sum_{j=1}^{n} b_{ji} x_j \in F^n, \quad b_{ji} \in F$$

Therefore, $x = \sum_{i=1}^{m} a_i \otimes \left(\sum_{j=1}^{n} b_{ji} x_j\right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} b_{ji} a_i\right)(1 \otimes x_j)$,

and so $\{1 \otimes x_1, \dots, 1 \otimes x_n\}$ is a K-basis for V^K.

The function $v \mapsto 1 \otimes v$ is a 1-1 map $V \hookrightarrow V^K$, so we may identify V with the image, and view V as a subset (but not a K-subspace) of V^K.

__Exercise__: Show that $\dim_F(V^K) = [K:F] \dim_F V$.

__Example__: Let V be a 3-dim'l $\mathbb{R}$-vector space.

Then $V^{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} V \cong \mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}) \cong (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R})^3 \cong \mathbb{C}^3$.

Thus, $V^{\mathbb{C}}$ is a 3-dim'l $\mathbb{C}$-vector space, and $\mathbb{C}$ is a 2-dim'l $\mathbb{R}$-vector space, so by Exercise, $V^{\mathbb{C}}$ is a $[\mathbb{C}:\mathbb{R}] \cdot \dim_{\mathbb{R}} V = 2 \cdot 3 = 6$-dimensional $\mathbb{R}$-vector space.

An element $x \in V^{\mathbb{C}}$ has the form

$$x = \sum_{i=1}^{3} z_i \otimes \vec{e}_i \qquad z_i \in \mathbb{C}, \quad \vec{e}_i \in V \text{ (unit basis vector)}$$

and so
$$\begin{aligned} x &= (z_1 \otimes \vec{e}_1) + (z_2 \otimes \vec{e}_2) + (z_3 \otimes \vec{e}_3) \\ &= (x_1 + y_1 i) \otimes \vec{e}_1 + (x_2 + y_2 i) \otimes \vec{e}_2 + (x_3 + y_3 i) \otimes \vec{e}_3 \\ &= (x_1 \otimes \vec{e}_1) + (y_1 i \otimes \vec{e}_1) + (x_2 \otimes \vec{e}_2) + (y_2 i \otimes \vec{e}_2) + (x_3 \otimes \vec{e}_3) + (y_3 i \otimes \vec{e}_3) \end{aligned}$$

i.e, $\{1 \otimes \vec{e}_1, 1 \otimes \vec{e}_2, 1 \otimes \vec{e}_3\}$ is a $\mathbb{C}$-basis for $V^{\mathbb{C}}$

and $\{1 \otimes \vec{e}_1, i \otimes \vec{e}_2, 1 \otimes \vec{e}_2, i \otimes \vec{e}_2, 1 \otimes \vec{e}_3, i \otimes \vec{e}_3\}$ is an $\mathbb{R}$-basis for V.

The associativity, e.g., $\underbrace{(i3)}_{\in\mathbb{C}}\underbrace{\vec{v}}_{\in V} = \underbrace{i}_{\in\mathbb{C}}\underbrace{(3\vec{v})}_{\in V} \in V^{\mathbb{C}}$ is guaranteed by the bilinearity of the tensor product: $i3 \otimes \vec{v} = i \otimes 3\vec{v} \in \mathbb{C} \otimes_{\mathbb{R}} V$.

__Remark__: If F is a field, and V, W are F-vector spaces, then
$\dim_F(V \otimes_F W) = (\dim_F V)(\dim_F W)$.

Again, let V, W be F-vector spaces, and suppose $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$ are bases for V & W, respectively.

Then $\{x_i \otimes y_j : 1 \le i \le n,\ 1 \le j \le m\}$ is a basis for $V \otimes_F W$.

Suppose $S : V \to V$, $T : W \to W$ are linear maps, represented by matrices $A = (a_{ij})_{n \times n}$, $B = (b_{ij})_{m \times m}$ (relative to $\{x_i\}$ and $\{y_j\}$).

By Prop 4.4, there is a linear map:

$$S \otimes T : V \otimes_F W \longrightarrow V \otimes_F W$$
$$v \otimes w \longmapsto S(v) \otimes T(w)$$

Order the basis: $x_1 \otimes y_1,\ x_1 \otimes y_2, \dots, x_1 \otimes y_m,$
$x_2 \otimes y_1,\ x_2 \otimes y_2, \dots, x_2 \otimes y_m, \dots$
$\dots, x_n \otimes y_1,\ x_n \otimes y_2, \dots, x_n \otimes y_m.$

Now, the matrix $A \otimes B$ that represents $S \otimes T$ has the block form $(a_{ij} B)_{n \times n}$, called the <u>Kronecker product</u> of A and B.

This arises a lot in representation theory (of groups), and thus in quantum mechanics.

<u>Def</u>: If R is a commutative ring, then a ring A is called an <u>R-algebra</u> if A is a left R-module such that:

$$r(ab) = (ra)b = a(rb) \quad \text{for all } r \in R,\ a, b \in A.$$

<u>Motivation</u>: an R-algebra is an R-module (or R-vector space, if R is a field), where we can additionally multiply "vectors".

Examples:

(1) A $\mathbb{Z}$-algebra is just an arbitrary ring A.

(2) If R is a commutative ring with 1, then any polynomial ring $R[x_1, \dots, x_n]$ is an R-algebra.

(3) If F is a field and K/F an extension, then K is an F-algebra

(4) If V is an F-vector space, then the ring $A = \mathrm{Hom}_F(V, V)$ is an F-algebra.

Suppose R is commutative, and A & B are R-algebras, and A is an R-R-bimodule, then $A \otimes_R B$ is an R-module.

We can also make it into an R-algebra, as follows:

Fix $a \in A$, $b \in B$, define
$$\begin{aligned} f_{ab} : A \times B &\longrightarrow A \otimes_R B \\ (c, d) &\longmapsto ac \otimes bd \end{aligned}$$

Since f_{ab} is bilinear, $\exists!$

$g_{ab} : A \otimes_R B \to A \otimes_R B$

s.t. $g_{ab} : c \otimes d \longmapsto ac \otimes bd$

for all $c \in A$, $d \in B$.

$$\begin{tikzcd} & A \otimes_R B \arrow[dr, dashed, "g_{ab}"] & \\ A \times B \arrow[ur, "\otimes"] \arrow[rr, "f_{ab}"'] & & A \otimes_R B \end{tikzcd} \qquad \begin{tikzcd} & c \otimes d \arrow[dr, dashed, mapsto, "g_{ab}"] & \\ (c, d) \arrow[ur, mapsto, "\otimes"] \arrow[rr, "f_{ab}"'] & & ac \otimes bd \end{tikzcd}$$

Define the "R-algebra multiplication" as follows:

$$\Big(\sum_i a_i \otimes b_i\Big)\Big(\sum_j c_j \otimes d_j\Big) = \sum_i g_{a_i b_i}\Big(\sum_j c_j \otimes d_j\Big) = \sum_{i,j} a_i c_j \otimes b_i d_j$$

Need to check that this works (easy!). If R has 1 and A is unitary, then so is $A \otimes_R B$.

The following is in a sense a "generalization" of the Chinese Reminder Theorem:

<u>Thm 4.7</u>: Suppose R is commutative with 1, and I, J are ideals in R. Then $(R/I) \otimes_R (R/J) \cong R/(I+J)$ (as R-algebras).

<u>Pf</u>: We must show that $R/I+J$ is a tensor product for R/I and R/J, using the universal property.

$$
\begin{tikzcd}
 & R/(I+J) \arrow[dr, dashed, "g"] & \\
R/I \times R/J \arrow[ur, "t"] \arrow[rr, "f"'] & & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & rs+(I+J) \arrow[dr, dashed, mapsto, "g"] & \\
(r+I, s+J) \arrow[ur, mapsto, "t"] \arrow[rr, mapsto, "f"'] & & a
\end{tikzcd}
$$

Define the balanced map
$t: R/I \times R/J \to R/(I+J)$
$(r+I, s+I) \longmapsto rs+(I+J)$. If $f: R/I \times R/J \longrightarrow A$ is an arbitrary homomorphism to an abelian group A, then define

$$g: R/(I+J) \longrightarrow A$$
$$r+(I+J) \longmapsto f(r+I, 1+J).$$

We need to check that this works!

<u>Well-defined</u>: Suppose $r+(I+J) = r'+(I+J)$.
Then $r' = r+a+b$ for some $a \in I$, $b \in J$.
Thus, $r'+I = r+b+I \in R/I$.

$$
\begin{aligned}
\Rightarrow \underbrace{f(r'+I, 1+J)}_{g(r+(I+J))} = f(r+b+I, 1+J) &\overset{\text{bilinearity}}{=} f(r+I, 1+J) + f(b+I, 1+J) \\
&= f(r+I, 1+J) + f(1+I, b+J) \\
&= f(r+I, 1+J) + f(1+I, 0) \\
&= f(r+I, 1+J) \\
&= g(r'+(I+J)) \checkmark
\end{aligned}
$$

g is clearly a homomorphism of abelian groups ✓

Check $gt = f$: $gt(r+I,\ s+J) = g(rs+I+J)$

$= f(rs+I,\ 1+J)$ by def'n

$= f((r+I)s,\ 1+J)$

$= f(r+I,\ s(1+J))$ bilinearity

$= f(r+I,\ s+J)$ ✓

Uniqueness is clear as well (t is surjective) ✓

By Prop 4.1, $R/(I+J)$ is the unique tensor product for R/I and R/J over R, i.e, $R/(I+J) \cong (R/I) \otimes_R (R/J)$ as abelian groups.

Remark: The explicit isomorphism is $r+(I+J) \longmapsto (r+I)\otimes(1+J)$. □

Cor: $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n \cong \mathbb{Z}_{(m,n)}$

Pf: $\mathbb{Z}_m = \mathbb{Z}/(m)$, $\mathbb{Z}_n = \mathbb{Z}/(n)$, $(m)+(n) = (m,n)$. □

Note that the special case of when $(m,n)=1$ is guaranteed by the Chinese Remainder Theorem