

5. Modules over a PID

* Throughout, R is a PID, and all modules are unitary.

Def: If M is an R -module, then $x \in M$ is a torsion element if $rx = 0$ for some $r \neq 0$ in R . (generalizes the concept of "finite order")

Def: If $x \in M$ is a torsion element, then its annihilator (or order ideal) is $\text{Ann}(x) = \{r \in R : rx = 0\}$.

Def: The ideal $\text{Ann}(x)$ is principal, and the generator S_x (i.e., $\text{Ann}(x) = (S_x)$) is called its order (unique up to associates).

Example: Let $R = \mathbb{Q}[x]$, and $M = V_T$ where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Then $\text{Ann}(u)$ and $\text{Ann}(v)$ are ideals of $R = \mathbb{Q}[x]$.

Note: If $f(x) = x - 1$, then $f(x) \cdot u = Tu - u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$$\Rightarrow |u| = x - 1$$

If $g(x) = x^2 - 2x + 1$, then $g(x) \cdot v = T^2 v - 2Tv + v$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

check: $(ax + b) \cdot v \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $x^2 - 2x + 1 \in \text{Ann}(v)$

$$\Rightarrow |v| = x^2 - 2x + 1$$

Def: Let $\text{Tor}(M)$ be the set of torsion elements of M .

(i) If $\text{Tor}(M) = M$, then M is a torsion module.

(ii) If $\text{Tor}(M) = \{0\}$, then M is torsion-free.

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Prop 5.1 $\text{Tor}(M)$ is a submodule of M , and $M/\text{Tor}(M)$ is torsion-free.

Pf: Pick $x, y \in T = \text{Tor}(M)$; $|x| = a$, $|y| = b$.

We must show: $\underline{x-y} \in T$ and $\underline{rx} \in T \quad \forall r \in R$.

Consider $ab(x-y) = b \cdot ax - a \cdot by = 0 \Rightarrow x-y \in T \quad \checkmark$

Also, $a(rx) = r(ax) = 0 \Rightarrow rx \in T$ for any $r \in R$. \checkmark

Therefore, T is a submodule of M . \checkmark

Now, consider $r(x+T) = rx+T = T \in M/T$. [Goal: show $x \in T$]

Then $rx \in T \Rightarrow \text{Ann}(rx) \neq 0$.

Pick $s \in \text{Ann}(rx)$. Then $s \cdot rx = sr \cdot x = 0 \Rightarrow x \in T \quad \checkmark$

□

Def: $\text{Tor}(M)$ is a submodule of M .

Remark 1: If A is an abelian group, then $\text{Tor}(A)$ is the subgroup of elements of finite order.

Remark 2: Prop 5.1 actually holds if R is any integral domain.

Prop 5.2: If M is a finitely generated torsion-free R -module, then M is free of finite rank.

Pf: Let $M = R\langle x_1, \dots, x_n \rangle$, and let $\{b_1, \dots, b_k\}$ be a maximal linearly independent subset of $\{x_1, \dots, x_n\}$.

Claim: $M \hookrightarrow N := R\langle b_1, \dots, b_k \rangle \Rightarrow M$ is free (Thm 2.12).

For each x_i , $\exists s_i \neq 0 \in R$ and $r_{i1}, \dots, r_{ik} \in R$ s.t. $s_i x_i + \underbrace{\sum_{j=1}^k r_{ij} b_j}_{\in N} = 0$

$\Rightarrow s_i x_i \in N$.

Put $s = \prod_{i=1}^n s_i$. Clearly, $s x_i \in N \ \forall i \Rightarrow s m \in N \ \forall m \in M$.

Define $f: M \rightarrow N$, $f(m) = sm$.

Easy: $f \in \text{Hom}_R(M, N)$.

Since M is torsion-free, $f(m) = sm \neq 0 \Rightarrow \ker f = 0$.

Thus $f: M \hookrightarrow N$.

Since R is a PID, M finitely generated & N free,

Thm 2.12 $\Rightarrow M$ is free. □

Thm 5.3: If M is a finitely generated R -module, then M has a free submodule F of finite rank such that $M = \text{Tor} \oplus F$. The rank of F is uniquely determined by M .

PF: Let $\eta: M \rightarrow M/\text{Tor}(M)$ be the natural quotient map.

By Prop 3.1, $M = F \oplus \ker \eta = \text{Tor}(M) \oplus F$ for some free R -module F , which is free by Prop 5.2.

By FHT for modules, $F \cong M/\text{Tor}(M)$.

If $M = \text{Tor}(M) \oplus F'$ for another free R -module F' , then

$F' \cong M/\text{Tor}(M) \cong F \Rightarrow F \subseteq F' \Rightarrow \text{rank } F = \text{rank } F'$. □

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Def: If M is a finitely generated R -module, then the rank of M is $\text{rank } F$, where $M = \text{Tor}(M) \oplus F$.

By Thm 5.3, this is well-defined.

Let M be a torsion module. Suppose $\exists r \in R^* \text{ with } rx=0$
for all $x \in M$.

The set $I = \{s \in R : sx=0 \ \forall x \in M\}$ is an ideal of R ,
and its generator $a \in I$ (i.e., $I = (a)$) is called the
exponent of M .

Clearly, $a|r$ and a is unique up to associates.

Remark: If M has an exponent $a \in M$, then $a = \text{lcm}\{x : x \in M\}$.

Not every torsion module has an exponent.

For example, $M = \bigoplus_{n=1}^{\infty} \mathbb{Z}_n$.

Finitely generated torsion modules have an exponent:

If $M = R\langle x_1, \dots, x_n \rangle$ and $r_i \in \text{Ann}(x_i)$, then

$r = r_1 r_2 \dots r_n \in \text{Ann}(x_i) \Rightarrow M$ has an exponent $a|r$.

Exercise: If M is an R -module and $x, y \in \text{Tor}(M)$ with

$|x| = r$, $|y| = s$, and $(x, y) = 1$ in R . Then $|x+y| = rs$.

Pf: Exercise.

Prop 5.4: If M is a torsion module with exponent $r \in R$, then M has an element of order r .

PF: Since R is a PID, it is a UFD.

Write $r = p_1^{d_1} p_2^{d_2} \dots p_k^{d_k}$, p_i 's distinct primes $d_i > 0$.

Put $r_i = r/p_i$.

Since $r \neq 0$, $\exists x_i \in M$ s.t. $r_i x_i \neq 0$ for each i .

Put $y_i = (r/p_i^{d_i}) x_i$.

Note: $p_i^{d_i} y_i = 0$ but $p_i^{d_i-1} y_i = r_i x_i \neq 0 \Rightarrow |y_i| = p_i^{d_i}$.

Put $x = y_1 + \dots + y_k$. By Exercise, $|x| = \prod_{i=1}^k |y_i| = r$. \square

Def: If M is an R -module and $s \in R$, define $M[s] = \{x \in M : sx = 0\}$ and let $sM = \{sx : x \in M\}$.

Exercise (easy): $M[s]$ & sM are submodules of M .

Notation: To avoid confusion, we'll write $(r, s) = \gcd(r, s)$ and $\langle r, s \rangle =$ ideal generated by r & s .

Prop 5.5: Suppose $M = R\langle y \rangle$ is cyclic of order r , and $s \in R$. Then

$$(i) \ M[s] = R\langle \frac{r}{(r,s)} y \rangle \cong R/\langle r, s \rangle$$

$$(ii) \ sM = R\langle sy \rangle \cong R/\langle \frac{r}{(r,s)} \rangle.$$

i.e., $M[s]$ is cyclic of order (r, s)

sM is cyclic of order $\frac{r}{(r, s)}$.

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Pf: (i) $M[s] = \{uy : u \in R, suy = 0\}$, and $suy = 0$ iff $\frac{r}{(r,s)} \mid u$.

$$\Rightarrow M[s] = R \left\langle \frac{r}{(r,s)} y \right\rangle$$

Define $\varphi: R \rightarrow M[s]$

$$v \mapsto \frac{vr}{(r,s)} y.$$

Clearly, φ is onto and $\ker \varphi = \langle r, s \rangle$. Apply FHT modules. ✓

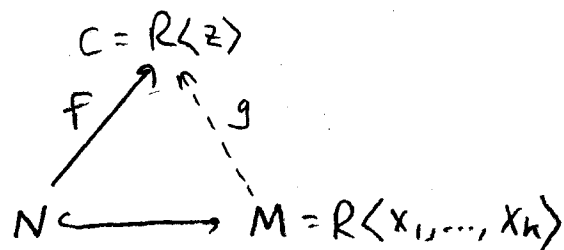
(ii) Similar (exercise). ✓

Cor: If $(r,s) = 1$, then $M[s] = 0$ and $sM = M$.

Prop 5.6: Suppose $C = R\langle z \rangle$ is cyclic of order r_0 , M is a finitely generated torsion R -module whose exponent

divides r_0 , N is a submodule of M , and $f \in \text{Hom}_R(N, C)$. Then

f extends to $g \in \text{Hom}_R(M, C)$.



Pf: Say $M = R\langle x_1, \dots, x_k \rangle$.

Put $N_1 = N + R\langle x_1 \rangle$, $N_2 = N_1 + R\langle x_2 \rangle$, ..., $N_k + R\langle x_k \rangle = M$.

By induction, it suffices to show that f extends to $g \in \text{Hom}_R(N_i, C)$.

Let $s = |x_i + N| \Rightarrow s(x_i + N) = N \Rightarrow sx_i \in N$

By def'n, $r_0 x_i = 0 \Rightarrow r_0 x_i + N = N \Rightarrow s \mid r_0$ (say $r_0 = st$).

Since $r_0 x_1 = 0$, $f(r_0 x_1) = f(t s x_1) = t f(s x_1) = 0 \Rightarrow |f(s, t)| \mid t$ in $C = R\langle z \rangle$.

Say that $f(s x_1) = u z$, $u \in R$.

Then $t f(s x_1) = t u z = 0 \Rightarrow r_0 \mid t u$

Recall that $r_0 = s t \Rightarrow s t \mid t u \Rightarrow s \mid u$ in R

So let $u = s v$, $v \in R$.

Now, $f(s x_1) = u z = (s v) z = s(v z)$. Let $z_0 = v z$.

Summary thus far: We don't have $x_1 \in N$, but $s x_1 \in N$,

and $f: s x_1 \mapsto s z_0$. (Recall: $f: N \rightarrow R\langle z \rangle$).

Define $f': N \oplus R \rightarrow R\langle z \rangle = C$

$$(y, r) \mapsto f(y) + r z_0$$

Define $h: N \oplus R \rightarrow N + R\langle z \rangle = N$,

$$(y, r) \mapsto y + r x_1$$

Claim: $\ker h \subseteq \ker f'$

PF: Take $(y, r) \in \ker h$.

$$\Rightarrow y + r x_1 = 0 \Rightarrow r x_1 = -y \in N$$

$$\Rightarrow r x_1 \in N$$

$$\Rightarrow r(x_1 + N) = N \Rightarrow s \mid r \text{ in } R$$

Since $s \mid r$, say $r = -s w$ for some $w \in R$.

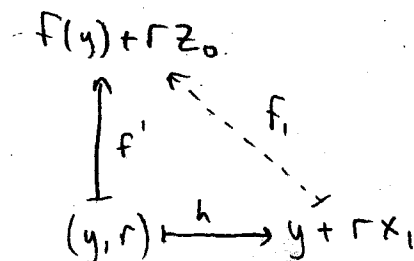
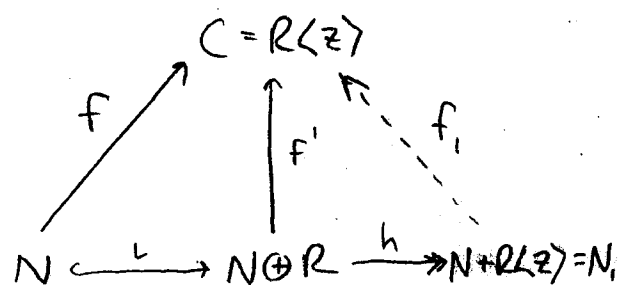
Then $y = -r x_1 = s w x_1 \Rightarrow (y, r) = (s w x_1, -s w)$.

$$\text{Thus, } f'(y, r) = f'(s w x_1, -s w) = f'(s w x_1) - f'(s w)$$

$$= f'(s w x_1) - s w z_0$$

$$= f'(s w x_1) - w s z_0 = f(s w x_1) - w f(s x_1) = 0 \checkmark$$

Since $\ker h \subseteq \ker f'$, $\exists!$ $f_1: N \rightarrow C$ s.t. $f' h = f_1$ □



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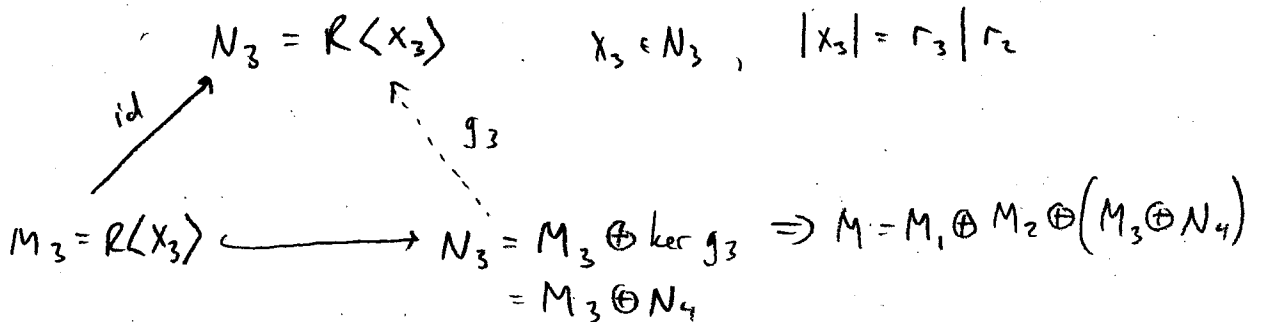
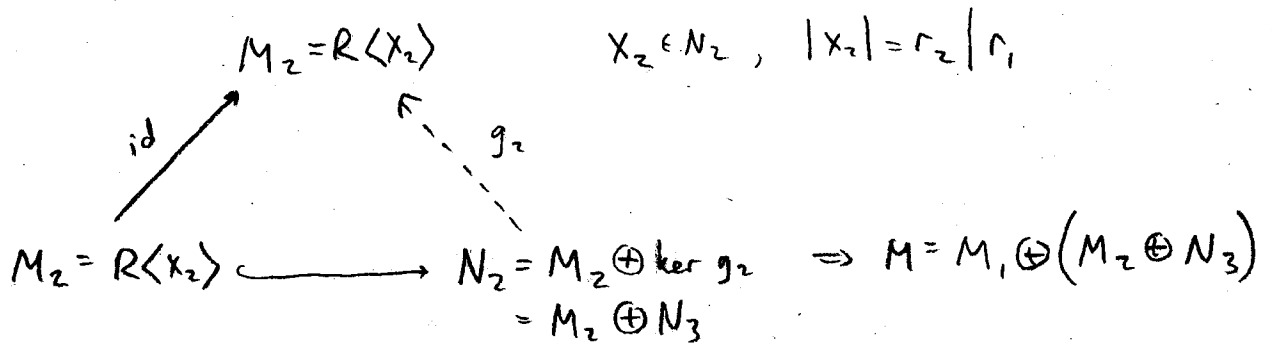
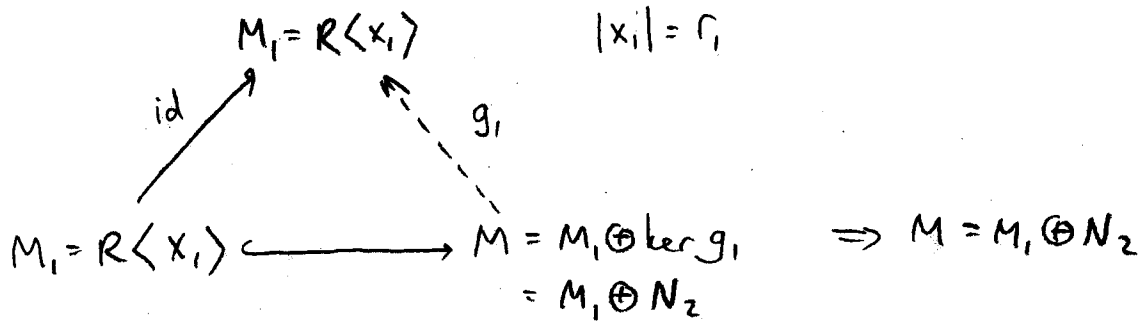
Thm 5.7: If M is a finitely generated torsion R -module, then
 $M = M_1 \oplus M_2 \oplus \dots \oplus M_k$, where each M_i is cyclic of order r_i ,
 with $r_i \mid r_{i-1}$, $2 \leq i \leq k$, and r_1 is the exponent of M .

Pf: By Prop 5.4, $\exists x_1 \in M$ with $|x_1| = r_1$. Set $M_1 = R\langle x_1 \rangle$.

By Prop 5.6, $\text{id}: M_1 \rightarrow M_1$ extends to $g_1: M \rightarrow M_1$.

Set $N_2 = \ker g_1$.

Outline of proof in diagrams (* we need to verify that this works):



⋮

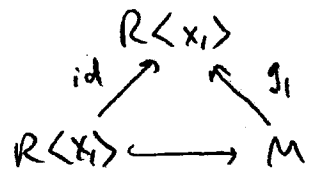
Eventually $N_{k+1} = 1$ (M Noetherian) $\Rightarrow M = M_1 \oplus M_2 \oplus \dots \oplus M_k$.

Why this works:

Prop 5.6 \Rightarrow $id: M_1 \rightarrow M_1$ extends to $g_1: M \rightarrow M_1$. Set $N_2 = \ker g_1$.

Claim: $M = M_1 \oplus N_2$.

Pf: Pick $x \in M$: Note: $g_1(x) \in R\langle x_1 \rangle$ and g_1 extends $id: R\langle x_1 \rangle \rightarrow R\langle x_1 \rangle$
 $\Rightarrow g_1(g_1(x)) = g_1(x)$.



Now, $g_1(x - g_1(x)) = g_1(x) - g_1(x) = 0$
 $\Rightarrow x - g_1(x) \in \ker g_1 = N_2$

If $x \in M \cap N_2$ then $x = g_1(x) = 0 \Rightarrow M = M_1 \oplus N_2$ (Thm 2.3). \checkmark

Continuing, if N_2 has exponent r_2 , then $(r_1) \subseteq (r_2)$ as ideals, so $r_2 | r_1$.

By Prop 5.4, $\exists x_2 \in N_2$ with $|x_2| = r_2$. Set $M_2 = R\langle x_2 \rangle$.

Continue as above to conclude $N_2 = M_2 \oplus \ker g_2 = M_2 \oplus N_3$.

Repeat this process, \ddots it terminates because M is Noetherian, i.e, at some point $N_{k+1} = 1$ and $M = M_1 \oplus \dots \oplus M_k$. \square

Cor: $M \cong R/(r_1) \oplus R/(r_2) \oplus \dots \oplus R/(r_k)$.

Pf: Take R and M_i as above, and consider the map

$$R \longrightarrow R\langle x_i \rangle = M_i$$

$$r \longmapsto r x_i.$$

By construction, the kernel is (r_i) , so $M_i \cong R/(r_i)$ by FHTM. \square

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Not surprisingly, this decomposition is "unique."

Thm 5.8: Suppose M is a finitely generated torsion R -module and

$$M = M_1 \oplus \dots \oplus M_k = N_1 \oplus \dots \oplus N_m$$

with M_i cyclic of order r_i , N_j cyclic of order s_j , $r_i | r_{i-1}$ and $s_j | s_{j-1}$, and $r_k, s_m \notin U(R)$. Then $k=m$, and $r_i \sim s_i$ (associates in R) and $M_i \cong N_i$ for each i .

Pf: Assume WLOG that $k \geq m$.

Choose a prime $p \in R$ s.t. $p | r_k$ (and hence $p | r_i$ $i=1, \dots, k$).

Since r_i, s_i are exponents of M , $r_i \sim s_i$ (Recall: an exponent is an LCM of the orders of the elements of M ; LCM's are unique up to assoc.)

Thus, $p | s_k \Rightarrow p | s_1$.

Claim: $p | s_i$ for all $i=1, \dots, k$.

Suppose not, i.e., that $p | s_j$ $j=1, \dots, n$ but $p \nmid s_{n+1}$.

Note: R is a PID and p prime $\Rightarrow R/\langle p \rangle = K$ is a field.

Plainly, $\langle r_i, p \rangle = \langle p \rangle$

By Prop 5.5 (i), $M_i[p] \cong R/\langle r_i, p \rangle = R/\langle p \rangle = K$

Also, since $p \nmid s_{n+1}$, $N_i[p] \cong R/\langle s_{n+1}, p \rangle = R/\langle 1 \rangle = 1$.

Therefore, $M[p] = M_1[p] \oplus \dots \oplus M_k[p] \cong \bigoplus_{i=1}^k K$ (k -dim'l vect. space)

Similarly, $M[p] = N_1[p] \oplus \dots \oplus N_n[p] \oplus \dots \oplus N_m[p]$
 $= N_1[p] \oplus \dots \oplus N_n[p] \cong \bigoplus_{j=1}^n K$ (n -dim'l vect. space)

Since $M[p] \cong \bigoplus_{i=1}^k K \cong \bigoplus_{i=1}^n K$, (a k -dim'l & n -dim'l vector space, respectively, $k=n$. Also, $n \leq m \leq k \Rightarrow m=k$. ✓

Now, suppose that $r_i \neq s_i$ for some i (and wlog, $r_i \sim s_i$ for $i < n$).

Also wlog, assume that $s_n \nmid r_n$

Put $M' = r_n M$. Then $r_n M_n = r_n M_{n+1} = \dots = 0$ but $r_n N_n \neq 0$.

$$\begin{aligned} \text{Thus, } M' &= r_n M_1 \oplus \dots \oplus r_n M_{n-1} \\ &= r_n N_1 \oplus \dots \oplus r_n N_{n-1} \oplus r_n N_n \end{aligned} \quad \downarrow$$

We conclude that $r_i \sim s_i$ for all i .

Finally, $M_i \cong R/(r_i) = R/(s_i) \cong N_i$ for each i . □

Def: If $M = M_1 \oplus \dots \oplus M_k$ as above (each M_i cyclic with order r_i), then r_1, \dots, r_k are the invariant factors of M .

Example: $R = \mathbb{Z}$. An finitely generated R -module is an abelian group G .

Take the orders $r_i = n_i$ to be positive integers.

By Thm 5.7, $|G| = n_1 n_2 \dots n_k \Rightarrow G$ is finite.

Thms 5.7 & 5.8 imply that finite abelian groups are determined up to isomorphism by their invariant factors.

Example 1: $|G| = 100 = 50 \cdot 2 = 20 \cdot 5 = 10 \cdot 10$

$\Rightarrow G \cong \mathbb{Z}_{100}$	$\xleftrightarrow{\cong}$	$\mathbb{Z}_{25} \oplus \mathbb{Z}_4$	}	Classification as described by Thm 7.11 (Groups)
$\mathbb{Z}_{50} \oplus \mathbb{Z}_2$	\longleftrightarrow	$(\mathbb{Z}_{25} \oplus \mathbb{Z}_2) \oplus \mathbb{Z}_2$		
$\mathbb{Z}_{20} \oplus \mathbb{Z}_5$	\longleftrightarrow	$(\mathbb{Z}_4 \oplus \mathbb{Z}_5) \oplus \mathbb{Z}_5$		
or $\mathbb{Z}_{10} \oplus \mathbb{Z}_{10}$	\longleftrightarrow	$(\mathbb{Z}_5 \oplus \mathbb{Z}_2) \oplus (\mathbb{Z}_5 \oplus \mathbb{Z}_2)$		

[2]

Example 2: $|G| = 48 = 24 \cdot 2 = 12 \cdot 4 = 12 \cdot 2 \cdot 2 = 6 \cdot 2 \cdot 2 \cdot 2$

$$\Rightarrow G \cong \mathbb{Z}_{48} \quad \xleftrightarrow{\cong} \quad \mathbb{Z}_3 \oplus \mathbb{Z}_{16}$$

$$\mathbb{Z}_{24} \oplus \mathbb{Z}_2, \quad \xleftrightarrow{\quad} \quad \mathbb{Z}_3 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2$$

$$\mathbb{Z}_{12} \oplus \mathbb{Z}_4, \quad \xleftrightarrow{\quad} \quad \mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$$

$$\mathbb{Z}_{12} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad \xleftrightarrow{\quad} \quad \mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

$$\text{or } \mathbb{Z}_6 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad \xleftrightarrow{\quad} \quad \mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

Classification by Thms 4.7 & 4.8
(by invariant factors)

Classification by Thm 7.11 Groups
(by cyclic p_i -subgroups)

Thms 5.3, 5.7, & 5.8 can be combined for a complete description of the structure of finitely generated modules over a PID. (i.e., not necessarily torsion modules).

Thm 5.9: If M is a finitely generated R -module, then

$\exists m \in \mathbb{Z}_{\geq 0}$ and $r_1, \dots, r_k \in R \setminus U(R)$ with $r_i \mid r_{i-1}$ ($i=2, \dots, k$)

such that $M = M_1 \oplus \dots \oplus M_k \oplus F$, where M_i is cyclic of order r_i , and F is free of rank m . Equivalently,

$$M \cong R/(r_1) \oplus \dots \oplus R/(r_k) \oplus R^m,$$

and this is uniquely determined (up to R -isomorphism) by m & r_1, \dots, r_k .

Cor: If G is a finitely generated abelian group, then $\exists m \in \mathbb{Z}_{\geq 0}$

and $1 \mid n_1 \mid n_2 \mid \dots \mid n_k$ such that $G \cong \mathbb{Z}_{n_1} \oplus \dots \oplus \mathbb{Z}_{n_k} \oplus \mathbb{Z}^m$.

G is determined up to isomorphism by the sequence n_1, \dots, n_k of invariant factors, and rank m . The order of the torsion subgroup of G is $n_1 n_2 \dots n_k$.

Def: A torsion R -module M is primary or p -primary, if its exponent is p^d for some prime $p \in R$.

Prop 5.10: Suppose M is a torsion module with exponent $r \in R$.

If $r = p_1^{d_1} p_2^{d_2} \dots p_k^{d_k}$ for distinct primes $p_i \in R$, then

$M = M[p_1^{d_1}] \oplus \dots \oplus M[p_k^{d_k}]$, a direct sum of primary submodules.

Pf: It suffices to show that if $r = st$, $(s, t) = 1$, then

$M = M[s] \oplus M[t]$ (the full result will then follow by induction).

Write $1 = as + bt$ for some $a, b \in R$ (See Prop 3.2 Rings).

For any $x \in M$, $x = t \cdot bx + s \cdot ax \in M[s] + M[t]$.

If $x \in M[s] \cap M[t]$, then $sx = 0$ and $tx = 0$, so

$x = asx + btx = 0$. By Thm 2.3, $M = M[s] \oplus M[t]$. □

Remark: The primary submodules $M[p_i^{d_i}]$ above are unique because $M[p_i^{d_i}]$ consists precisely of the elements of M having order a prime power $p_i^{d_i}$, where $p_i \mid r$ (the exponent).

The Invariant Factor Theorem (5.7) said that a finitely generated torsion R -module M can be decomposed into cyclic R -modules $M = M_1 \oplus \dots \oplus M_k$ of order $r_i \mid r_{i-1}$. Applying this to each cyclic R -module M_i yields the following theorem:

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Thm 5.11: Suppose M is a finitely generated torsion R -module of exponent $r = \prod_{i=1}^k p_i^{d_i}$ (p_i 's distinct primes, $d_i > 0$). Then

$$M = M_1 \oplus \dots \oplus M_k, \text{ where } M_i \text{ is } p_i\text{-primary, and}$$

$$M_i = M_{i1} \oplus \dots \oplus M_{ik_i}, \text{ where } M_{ij} \text{ is cyclic of order } p_i^{d_{ij}} \text{ with}$$

$$1 \leq e_{ij} \leq e_{i(j-1)} \leq e_i \quad \forall i, j.$$

Moreover, M is determined up to isomorphism by the set

$$\{p_i^{d_{ij}} : 1 \leq j \leq k_i, 1 \leq i \leq k\}, \text{ called the } \underline{\text{elementary divisors}} \text{ of } M.$$

Application: Classification of abelian groups of order $72 = 2^3 \cdot 3^2$.

Let $p_1 = 2, p_2 = 3$. Then $\sum_j e_{1j} = 3$ and $\sum_j e_{2j} = 2$.

We have 6 possible sets of elementary divisors:

Elementary divisors Invariant factors

$$\mathbb{Z}_8 \oplus \mathbb{Z}_9 \longleftrightarrow \{2^3, 3^2\} \longleftrightarrow 72 \longleftrightarrow \mathbb{Z}_{72}$$

$$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \longleftrightarrow \{2^3, 3, 3\} \longleftrightarrow 24 \cdot 3 \longleftrightarrow \mathbb{Z}_{24} \oplus \mathbb{Z}_3$$

$$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \longleftrightarrow \{2^2, 2, 3^2\} \longleftrightarrow 36 \cdot 2 \longleftrightarrow \mathbb{Z}_{36} \oplus \mathbb{Z}_2$$

$$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \longleftrightarrow \{2^2, 2, 3, 3\} \longleftrightarrow 12 \cdot 6 \longleftrightarrow \mathbb{Z}_{12} \oplus \mathbb{Z}_6$$

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \longleftrightarrow \{2, 2, 2, 3^2\} \longleftrightarrow 18 \cdot 2 \cdot 2 \longleftrightarrow \mathbb{Z}_{18} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \longleftrightarrow \{2, 2, 2, 3, 3\} \longleftrightarrow 6 \cdot 6 \cdot 2 \longleftrightarrow \mathbb{Z}_6 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_2$$