Read: Stahl, Chapters 2.2, 2.3, 2.4, 2.5, 3.1.

1. In this problem, use only the analytic definition of Euclidean geometry. Let $\mathbf{v}$ be a unit vector in $\mathbf{R}^2$, and define $r_{\mathbf{v}}(\mathbf{x}) = \mathbf{x} - 2(\mathbf{x}\cdot\mathbf{v})\mathbf{v}$.

   (a) Prove (analytically) that $r_{\mathbf{v}}$ is a Euclidean isometry.

   (b) Prove that $r_{\mathbf{v}}$ is the reflection in a line m through the origin, and find an equation for that line (both in $y = mx + b$, and in parametric form).

   (c) For an arbitrary $\alpha \in \mathbf{R}$, describe the isometry $f := \tau_{\alpha\mathbf{v}} \circ r_{\mathbf{v}} \circ \tau_{\alpha\mathbf{v}}^{-1}$ geometrically, and prove your description. This is the unique map that makes the following diagram commute:

$$\begin{array}{ccc} \mathbf{E}^2 & \xrightarrow{\;r_{\mathbf{v}}\;} & \mathbf{E}^2 \\ {\scriptstyle\tau_{\alpha\mathbf{v}}}\downarrow & & \downarrow{\scriptstyle\tau_{\alpha\mathbf{v}}} \\ \mathbf{E}^2 & \overset{f}{\dashrightarrow} & \mathbf{E}^2 \end{array}$$

   What do $r_{\mathbf{v}}$ and $\tau_{\alpha\mathbf{v}} \circ r_{\mathbf{v}} \circ \tau_{\alpha\mathbf{v}}^{-1}$ have in common?

In the following three problems, you will prove a series of results stated in Chapter 2 of Stahl. For each problem, you are free to use results in Stahl that appear before it, provided you clearly state and cite them. The notation ρ_m denotes the reflection of $\mathbf{E}^2$ across the line m, and τ_{AB} denotes the translation of $\mathbf{E}^2$ that carries A to B.

2. Prove Proposition 2.2.7 in Stahl: Let m and n be two parallel straight lines. Let AB be a line segment that first intersects m and then n, that is perpendicular to both, whose length is twice the distance between m and n. Prove that $\rho_n \circ \rho_m = \tau_{AB}$.

3. Prove Lemma 2.3.1 in Stahl: Let the line segment AB be perpendicular to the line m. Prove that $\rho_m \circ \tau_{AB} = \rho_n$, where n is a line parallel to m.

4. Prove the second parts of Propositions 2.3.2 and 2.3.3 in Stahl: Let ρ be any reflection, τ be any translation, and R be any rotation of $\mathbf{E}^2$. Prove that $\rho\circ\tau$ and $\rho\circ R$ are glide reflections. (Note: The facts that $\tau\circ\rho$ and $R\circ\rho$ also glide reflections constitute the first part of Propositions 2.3.2 and 2.3.3.)

5. In this problem, you may use the classification of Euclidean isometries as described in Theorem 2.4.3 of Stahl.

   (a) Let m be a line in $\mathbf{E}^2$. Describe all of the isometries of $\mathbf{E}^2$ that fix every point of m, and prove that your description is complete.

   (b) Let P be a point of $\mathbf{E}^2$. Describe all of the isometries of $\mathbf{E}^2$ that fix P, and prove that your description is complete.

6. Recall that the circle inversion in polar coordinates is the map $I_{O,k}\colon \mathbf{E}^2 \setminus O$ given by $I_{O,k}(r,\theta) = (k^2/r, \theta)$. Now, let $f(r,\theta)\colon (0,\infty)\times[0,2\pi)$ be an arbitrary function, and define the sets $\Gamma_0 := \{(r,\theta) \mid f(r,\theta) = 0\}$ and $\Gamma_1 := \{(r,\theta) \mid f(k^2/r,\theta) = 0\}$. Prove that $I_{O,k}(\Gamma_0) = \Gamma_1$.

7. Theorem 3.1.6 of Stahl says that $I_{C,k}$ is a conformal mapping of $\mathbf{E}^2 \setminus C$, that is, it preserves angles between paths. In this problem, we will prove this statement in a more rigorous and elegant fashion, using only the analytic methods we developed in class.

   (a) Prove that for any angle θ, the matrix

   $$B_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

   is orthogonal.

   (b) Prove that, at any given point $(x, y) \neq (0, 0)$, the inversion $I'_{O,k}(x, y)$ is a scalar multiple of the matrix B_θ for some value of θ.

   (c) Prove that $I_{O,k}$ preserves angles between paths in $\mathbf{E}^2 \setminus O$.

   (d) For a vector $\mathbf{v} \in \mathbf{R}^2$, describe the transformation $\tau_{\mathbf{v}} \circ I_{O,k} \circ \tau_{\mathbf{v}}^{-1}$ geometrically, and prove your descrpition. Draw a commutative diagram that shows how these maps are related. (Be careful: The domains of these maps are not quite all of $\mathbf{E}^2$!)

   (e) Prove that the transformation $I_{C,k}$ preserves angles between paths in $\mathbf{E}^2 \setminus C$.