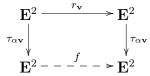
*Read*: Stahl, Chapters 2.2, 2.3, 2.4, 2.5, 3.1.

- 1. In this problem, use only the analytic definition of Euclidean geometry. Let  $\mathbf{v}$  be a unit vector in  $\mathbf{R}^2$ , and define  $r_{\mathbf{v}}(\mathbf{x}) = \mathbf{x} 2(\mathbf{x} \cdot \mathbf{v})\mathbf{v}$ .
  - (a) Prove (analytically) that  $r_{\mathbf{v}}$  is a Euclidean isometry.
  - (b) Prove that  $r_{\mathbf{v}}$  is the reflection in a line *m* through the origin, and find an equation for that line (both in y = mx + b, and in parametric form).
  - (c) For an arbitrary  $\alpha \in \mathbf{R}$ , describe the isometry  $f := \tau_{\alpha \mathbf{v}} \circ r_{\mathbf{v}} \circ \tau_{\alpha \mathbf{v}}^{-1}$  geometrically, and prove your description. This is the unique map that makes the following diagram commute:



What do  $r_{\mathbf{v}}$  and  $\tau_{\alpha \mathbf{v}} \circ r_{\mathbf{v}} \circ \tau_{\alpha \mathbf{v}}^{-1}$  have in common?

In the following three problems, you will prove a series of results stated in Chapter 2 of Stahl. For each problem, you are free to use results in Stahl that appear before it, provided you clearly state and cite them. The notation  $\rho_m$  denotes the reflection of  $\mathbf{E}^2$  across the line m, and  $\tau_{AB}$  denotes the translation of  $\mathbf{E}^2$  that carries A to B.

- 2. Prove Proposition 2.2.7 in Stahl: Let m and n be two parallel straight lines. Let AB be a line segment that first intersects m and then n, that is perpendicular to both, whose length is twice the distance between m and n. Prove that  $\rho_n \circ \rho_m = \tau_{AB}$ .
- 3. Prove Lemma 2.3.1 in Stahl: Let the line segment AB be perpendicular to the line m. Prove that  $\rho_m \circ \tau_{AB} = \rho_n$ , where n is a line parallel to m.
- 4. Prove the second parts of Propositions 2.3.2 and 2.3.3 in Stahl: Let  $\rho$  be any reflection,  $\tau$  be any translation, and R be any rotation of  $\mathbf{E}^2$ . Prove that  $\rho \circ \tau$  and  $\rho \circ R$  are glide reflections. (Note: The facts that  $\tau \circ \rho$  and  $R \circ \rho$  also glide reflections constitute the first part of Propositions 2.3.2 and 2.3.3.)
- 5. In this problem, you may use the classification of Euclidean isometries as described in Theorem 2.4.3 of Stahl.
  - (a) Let m be a line in  $\mathbf{E}^2$ . Describe all of the isometries of  $\mathbf{E}^2$  that fix every point of m, and prove that your description is complete.
  - (b) Let P be a point of  $\mathbf{E}^2$ . Describe all of the isometries of  $\mathbf{E}^2$  that fix P, and prove that your description is complete.
- 6. Recall that the circle inversion in polar coordinates is the map  $I_{O,k}$ :  $\mathbf{E}^2 \setminus O$  given by  $I_{O,k}(r,\theta) = (k^2/r,\theta)$ . Now, let  $f(r,\theta)$ :  $(0,\infty) \times [0,2\pi)$  be an arbitrary function, and define the sets  $\Gamma_0 := \{(r,\theta) \mid f(r,\theta) = 0\}$  and  $\Gamma_1 := \{(r,\theta) \mid f(k^2/r,\theta) = 0\}$ . Prove that  $I_{O,k}(\Gamma_0) = \Gamma_1$ .

- 7. Theorem 3.1.6 of Stahl says that  $I_{C,k}$  is a conformal mapping of  $\mathbf{E}^2 \setminus C$ , that is, it preserves angles between paths. In this problem, we will prove this statement in a more rigorous and elegant fashion, using only the analytic methods we developed in class.
  - (a) Prove that for any angle  $\theta$ , the matrix

$$B_{\theta} = \begin{pmatrix} \cos\theta & \sin\theta\\ \sin\theta & -\cos\theta \end{pmatrix}$$

is orthogonal.

- (b) Prove that, at any given point  $(x, y) \neq (0, 0)$ , the inversion  $I'_{O,k}(x, y)$  is a scalar multiple of the matrix  $B_{\theta}$  for some value of  $\theta$ .
- (c) Prove that  $I_{O,k}$  preserves angles between paths in  $\mathbf{E}^2 \setminus O$ .
- (d) For a vector  $\mathbf{v} \in \mathbf{R}^2$ , describe the transformation  $\tau_{\mathbf{v}} \circ I_{O,k} \circ \tau_{\mathbf{v}}^{-1}$  geometrically, and prove your description. Draw a commutative diagram that shows how these maps are related. (Be careful: The domains of these maps are not quite all of  $\mathbf{E}^2$ !)
- (e) Prove that the transformation  $I_{C,k}$  preserves angles between paths in  $\mathbf{E}^2 \setminus C$ .