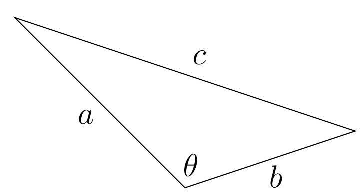
*Read*: Stahl, Chapters 4.1, 4.2, 4.3.

1. In this problem, we will prove Proposition 3.1.3 in Stahl in a more rigorous and elegant fashion, using only the analytic methods we developed in class.
   (a) Let $f$ be a Euclidean isometry, let $p_0$ be a circle, and let $f(p_0) = p_1$. Prove that if $I_{C,k}$ maps $p_0$ to a circle, then $f \circ I_{C,k} \circ f^{-1}$ maps $p_1$ to a circle.
   (b) Let $f$ be a Euclidean isometry, let $\ell_0$ be a line, and let $f(\ell_0) = \ell_1$. Prove the if $I_{C,k}$ maps $\ell_0$ to a line, then $f \circ I_{C,k} \circ f^{-1}$ maps $\ell_1$ to a line.
   (c) Prove that if $p$ is a circle and $q$ is either a circle or a line, then there exists $g \in \mathrm{Isom}(\mathbf{E}^2)$ such that $g(p)$ is centered at the origin, and $g(q)$ is orthogonal to the $x$-axis.
   (d) Prove that if $I_{O,k}$ maps circles and lines orthogonal to the $x$-axis to circles and lines, then for any $C \in \mathbf{E}^2$, $I_{C,k}$ maps circles and lines to circles and lines. Draw the appropriate commutative diagram illustrating your argument.
   (e) Let $p$ be a circle. Prove the following:
      i. If $p$ does not contain $C$, then $I_{C,k}(p)$ is a circle not containing $p$;
      ii. If $p$ contains $C$, then $I_{C,k}(p)$ is a straight line not containing $p$.

2. In this problem we will prove Proposition 3.1.7 analytically: Let $p$ be the circle of center $C$ and radius $k$, and let $q$ be any other circle in $\mathbf{E}^2 \setminus C$. Then $I_{C,k}$ sends $q$ to itself if and only if $q$ is orthogonal to $p$.
   (a) Prove that it is enough to prove that proposition for the case where the cetners of $p$ and $q$ are on the $x$-axis.
   (b) Prove that the proposition for the case where the centers of $p$ and $q$ are on the $x$-axis. You may find the following facts useful:
      i. Let $q$ be a circle with center $C$, and let $P$ be a point on $q$. Then the tangent line to $q$ at $P$ is orthogonal to $\overline{CP}$.
      ii. In the triangle below, $a^2 + b^2 = c^2$ if and only if $\theta = \pi/2$. (This follows from the law of cosines.)




3. Prove Proposition 4.1.2 in Stahl: The hyperbolic length of the Euclidean line segment joining the points $P = (a, y_1)$ and $Q = (a, y_2)$, where $0 < y_1 \leq y_2$, is $\ln(y_2/y_1)$. Do not use any later results that trivialize the problem.

4. Find the inversion $I_{C,k}$ that takes the bowed geodesic from $(-4, 0)$ to $(2, 0)$ to the bowed geodesic from $(10, 0)$ to $(20, 0)$.

5. Given a hyperbolic geodesic $\gamma$ and a point $P$ on $\gamma$, describe a Euclidean method for constructing a hyperbolic geodesic through $P$ that is orthogonal to $\gamma$.

6. Given a hyperbolic geodesic $\gamma$ and a point $P$ not on $\gamma$, describe a Euclidean method for constructing a hyperbolic geodesic through $P$ that is orthogonal to $\gamma$.