Read: Stahl Chapters 6.1, 6.2, 7.1, 7.2, 8.1, 8.2, 8.3.

1. Prove that if $0 < \alpha < 2\pi$ then there is a hyperbolic quadrilateral whose angles sum to α.

2. For $a, b > 0$, let $R(a, b)$ be the Euclidean rectangle with corners $(0, 2)$, $(a, 2)$, $(0, 2 + b)$, and $(a, 2 + b)$.

   (a) Give a qualitative description of what R looks like to a resident of the hyperbolic plane. Specifically, which of its sides are hyperbolically straight or hyperbolically curved. If curved, in which direction?

   (b) For a given $a, b > 0$, let $\text{ha}(R(a, b))$ denote the hyperbolic area of $R(a, b)$, let $L(a, b)$ be the hyperbolic length of the left side of R, and let $B(a, b)$ be the hyperbolic length of the bottom side of R. Prove that

$$\lim_{a,b\to 0^+} \frac{L(a, b)B(a, b)}{\text{ha}(R(a, b))} = 1\,.$$

For the next two problems, consider a hyperbolic triangle ABC with right angle at C, let α and β denote the angles at A and B, respectively, and let a, b, and c be the hyperbolic lengths of the sides opposite A, B, and C, respectively.

3. Prove Parts (i)–(iii) of Proposition 8.2.2 in Stahl: If γ is a right angle, then the following identities hold.

$$\begin{array}{rlcl}
\text{i.} & \tanh a = \sinh b \tan \alpha & \text{and} & \tanh b = \sinh a \tan \beta \\
\text{ii.} & \sinh a = \sinh c \sin \alpha & \text{and} & \sinh b = \sinh c \sin \beta \\
\text{iii.} & \tanh b = \tanh c \cos \alpha & \text{and} & \tanh a = \tanh c \cos \beta.
\end{array}$$

   Additionally, find the Euclidean analogues of these formulas, and prove that the hyperbolic versions approach the Euclidean versions in the limit as $a, b, c \to 0$.

4. Theorem 8.3.2(i) in Stahl is the formula $\cos\alpha = \dfrac{\cosh b \cosh c - \cosh a}{\sinh b \sinh c}$. The proof of this began by considering d, the hyperbolic altitude from vertex A to side BC. The case when d falls inside the triangle was done in the book. Finish the proof of this statement when d falls outside the triangle.

5. Recall that hyperbolic triangles are "thin" in the sense that the sum of their angles must be less than π. In this problem, we establish another "thinness" property of hyperbolic triangles: If ABC is a hyperbolic triangle, then every point of AC is at most distance 1 from some point on AB or some point on BC.

   (a) Prove the theorem in the case where either one of the angles at A or C is non-acute.

   (b) Prove the theorem in the case where the angles at A and C are both acute.

6. Let G be a group. Prove that conjugacy is an equivalence relation.

7. Let X be a geometry and $G = \text{Isom}(X)$.

(a) Let $x, y \in X$ and $f, g \in G$, and define $h = fgf^{-1}$. Prove that $g\colon x \mapsto y$ iff $h\colon f(x) \mapsto f(y)$. Draw a commutative diagram relating f, g, and h.

(b) Prove that if x is a fixed point of g, then $f(x)$ is a fixed point of h.

8. Let $\tau_{\mathbf{v}}$ be the traslation by the vector $\mathbf{v} \in \mathbf{E}^2$.

   (a) Prove that the minimal motion of $\tau_{\mathbf{v}}$ is $|\mathbf{v}|$, and the set of minimal motion of $\tau_{\mathbf{v}}$ is all of $\mathbf{E}^2$.

   (b) Prove the if R is a rotation around the origin, then $R\tau_{\mathbf{v}}R^{-1} = \tau_{R(\mathbf{v})}$. Draw a commutative diagram showing this relation.