MTHSC 481: Topics in Geometry & Topology

1. What is a Geometry?

Def 1.2.1: A space is a subset $X \subseteq \mathbb{R}^n$.

Def 1.2.2: A path is a function $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ such that $\gamma'(t)$ is continuous and non-zero for all $t \in [a,b]$.

An intersection of two paths γ_1, γ_2 is a pair of times t_1, t_2 such that $\gamma_1(t_1), \gamma_2(t_2)$.

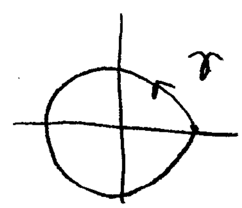
A change of parameter is a differentiable function $u : [c,d] \to [a,b]$ such that $u'(t) \neq 0$ for all $t \in [c,d]$.

Picture of this:

$$\begin{tikzcd}
[c,d] \arrow[r, "u"] \arrow[rr, bend right, "\gamma \circ u"'] & {[a,b]} \arrow[r, "\gamma"] & \mathbb{R}^n
\end{tikzcd}$$

Example: Let $\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2$,

$\gamma(t) = (\cos t, \sin t) \qquad 0 \leq t \leq 2\pi$



Define $u : [0, \pi] \longrightarrow [0, 2\pi]$, $u(t) = 2t$

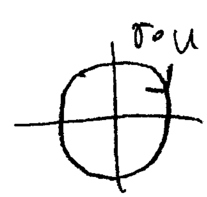
Then $(\gamma \circ u)(t) = (\cos 2t, \sin 2t)$, $0 \leq t \leq \pi$

traces out the same path, but under a change of parameter.

If $u' > 0$, say u is orientation-preserving, otherwise it is orientation-reversing.

Example (cont.) $u : [0, \pi] \to [0, 2\pi]$,

$u(t) = -2t$ is orientation-reversing.



<u>Def</u>: A <u>curve</u> is the image of a path in $\mathbb{R}^n$.

<u>Remark</u>: A curve is independent of choice of parameter.

<u>Def 1.2.3</u>: A <u>geometry</u> consists of the following:

(i) A space X

(ii) A definition of <u>length</u> of a path γ in X

(iii) A definition of <u>angle</u> between two intersecting paths γ_1, γ_2 in X.

Additionally:
- length should be invariant under change of parameter.
- angle should be invariant under orientation-preserving change of parameter.

Now, let X be a geometry.

<u>Def 1.2.4</u>: The <u>distance</u> between x & y is the minimum (technically, infimum) of the lengths of all paths from x to y, denoted $d(x,y)$.

If a path γ from x to y has length $d(x,y)$, say γ is a <u>geodesic segment</u> from x to y.

A <u>geodesic</u> is a path $\gamma : \mathbb{R} \to X$ such that every finite segment of γ is a geodesic segment.

Note: We may call a curve a geodesic too.

Def 1.2.5: An isometry of X is a bijective function $f: X \to X$ such that

(i) For any path γ in X, $|\gamma| = |f \circ \gamma|$.

(ii) For any paths γ_1, γ_2 in X, $\measuredangle(\gamma_1, \gamma_2) = \measuredangle(f \circ \gamma_1, f \circ \gamma_2)$.

Intuition: An isometry is a bijection $f: X \to X$ that preserves lengths and angles. (Also called symmetries, congruences, or rigid motions.)

Prop 1.2.6: If $f: X \to X$ is an isometry, then f is invertible and $f^{-1}: X \to X$ is an isometry.

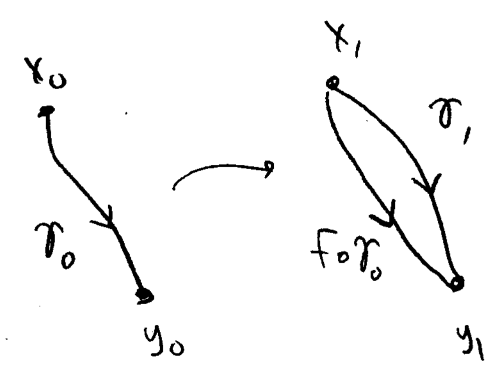
Prop 1.2.7: If $f, g: X \to X$ are isometries, then $f \circ g$ is an isometry and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Proofs: Exercise. (HW)

Remark: The isometries of X form a group.

Prop 1.2.8. If $f: X \to X$ is an isometry, then f sends geodesics to geodesics.

Proof: Suppose $f(x_0) = x_1$, $f(y_0) = y_1$, and γ_0 is a geodesic from x_0 to y_0



Suppose γ_1 is a path from x_1 to y_1, shorter than $f \circ \gamma_0$.
Then $f^{-1} \circ \gamma_1$ is a path from x_0 to y_0, shorter than γ_0. $\lightning$
Thus, $f \circ \gamma_0$ must be a geodesic from x_1 to y_1. $\square$

Cor 1.2.9: If $f: X \to X$ is an isometry, then $d(x,y) = d(f(x), f(y))$ for all $x, y \in X$. $\square$

Broad goals: Given a geometry X,

1. Classify the geodesics of X
2. Classify the isometries of X
3. Further study properties of X (triangles, area, etc.)

The Euclidean Plane.

Goal: Intrepret Euclidean geometry in this analytical setting.

Allowed: Algebra, calculus, linear algebra

Not allowed: Geometric intrepretations.

Example: OK: $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$

Not ok: $|\vec{v}|$ = "length of the vector $\vec{v}$", or distance from $\vec{v}$ to $\vec{0}$.

Not ok: "Parallel postulate." Given line ℓ, point $p \notin \ell$, $\exists!$ line through p parallel to ℓ.

<u>Def 1.3.2</u>: The <u>Euclidean plane</u> $\mathbb{E}^2$, is the following geometry:

1. The <u>space</u> $X = \mathbb{R}^2$
2. The <u>length function</u> (of a path $\gamma : [a,b] \to \mathbb{E}^2$)

$$|\gamma| = \int_a^b |\gamma'(t)|\, dt$$

3. The <u>angle function</u> $\measuredangle(\gamma_1(t_1), \gamma_2(t_2)) = \theta$, where

$$\theta = \frac{\gamma_1'(t_1) \cdot \gamma_2'(t_2)}{|\gamma_1'(t_1)|\,|\gamma_2'(t_2)|}$$

<u>Note</u>: If $\gamma(t) = (x(t), y(t))$, then $|\gamma| = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.

We must check:
- invariance of length — See Def 1.2.3
- invariance of angle — (HW)
- θ always exists (easy: $\gamma'(t) \neq 0$)

<u>Invariance of length</u>: Let $\gamma : [a,b] \to \mathbb{E}^2$ be a path, and $u : [c,d] \to [a,b]$ an orientation-preserving change of param.

Then
$$\int_a^b |\gamma'(t)|\, dt = \int_c^d |\gamma'(u(t))|\, u'(t)\, dt \qquad \text{(substitution)}$$
$$= \int_c^d |\gamma'(u(t))\, u'(t)|\, dt \qquad (u' \geq 0)$$
$$= \int_c^d |(\gamma \circ u)'(t)|\, dt \qquad \text{(chain rule)}$$

Orientation-reversing case: Exercise (HW)

<u>Invariance of angle</u>: (under orientation-preserving change of param. <u>only</u>)

Exercise (HW). □

<u>Def 1.3.5</u>: A <u>line</u> in $\mathbb{E}^2$ is a map $L: \mathbb{R} \to \mathbb{E}^2$ defined by $L(t) = \vec{v}t + \vec{b}$ for some fixed $\vec{v}, \vec{b} \in \mathbb{E}^2$. A <u>line segment</u> is a finite portion of a line.

<u>Next goal</u>: Show that geodesics of $\mathbb{E}^2$ are precisely the lines of $\mathbb{E}^2$.

<u>Approach</u>:
1. Move any pair of points to the x-axis via isometries, showing lines are preserved.
2. Prove the special case for 2 points on the x-axis.
3. Conclude the general theorem, for 2 points in $\mathbb{E}^2$.

This is a general technique for proving theorems about symmetries, called the "<u>method of standard position</u>."

0. State the theorem in a form invariant under isometries
1. Use isometries to simplify the problem (move the "general position" to the "standard position."
2. Prove the theorem in the standard position.
3. Conclude the theorem in general.

First, let's come up with some Euclidean isometries, and a way to describe them.

Def 1.3.6: An $n \times n$ matrix A is orthogonal if $A^T A = I$.

Remark: If $A = [\vec{c}_1 \dots \vec{c}_n]$, then $A^T A = \begin{bmatrix} c_1 \cdot c_1 & c_1 \cdot c_2 & \cdots & c_1 \cdot c_n \\ c_2 \cdot c_1 & c_2 \cdot c_2 & & \vdots \\ \vdots & & \ddots & \\ c_n \cdot c_1 & \cdots & & c_n \cdot c_n \end{bmatrix}$

Thus $A^T A = I$ iff the column vectors of A are an orthonormal set $(c_i \cdot c_j = \delta_{ij})$

Example: $A_\theta := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Prop 1.3.7: If A is an $n \times n$ orthogonal matrix, then for any $\vec{v}, \vec{w} \in \mathbb{R}^n$,

(i) $A\vec{v} \cdot A\vec{w} = \vec{v} \cdot \vec{w}$

(ii) $|A\vec{v}| = |\vec{v}|$.

Proof: Recall that $\vec{v} \cdot \vec{w} = \vec{v}^T \vec{w}$.

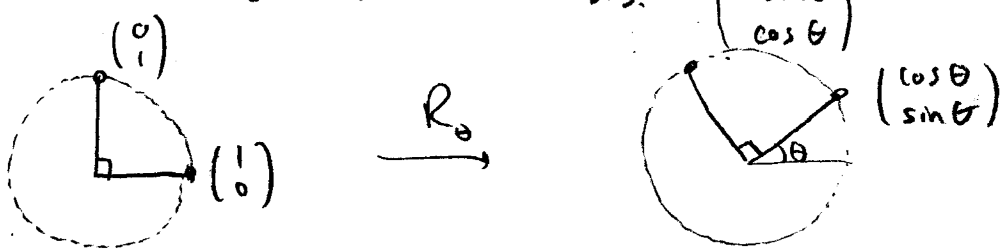
(i) $A\vec{v} \cdot A\vec{w} = (A\vec{v})^T (A\vec{w}) = \vec{v}^T (A^T A) \vec{w} = \vec{v}^T \vec{w} = \vec{v} \cdot \vec{w}$ ✓

(ii) Put $\vec{w} = \vec{v}$: $|A\vec{v}|^2 = A\vec{v} \cdot A\vec{v} = \vec{v} \cdot \vec{v} = |\vec{v}|^2$. ✓

$\square$

<u>Def 1.3.8</u>: The <u>rotation</u> $R_\theta : \mathbb{E}^2 \to \mathbb{E}^2$ is defined to be the transformation $R_\theta : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto A_\theta \begin{bmatrix} x \\ y \end{bmatrix}$.

<u>Note</u>: R_θ is the actual <u>map</u>, and A_θ is the <u>matrix</u> representation w.r.t. the standard basis.



<u>Thm 1.3.9</u>: R_θ is an isometry of $\mathbb{E}^2$ that preserves lines.

<u>Proof</u>: Note that R_θ preserves lines because:

$$R_\theta : (\vec{v}t + \vec{b}) \longmapsto (A_\theta \vec{v})t + (A_\theta \vec{b}). \quad ✓$$

Need to check that R_θ is an isometry.

Say $R_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$

$$R_\theta' = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = A_\theta$$

<u>Remark</u>. This is just a special case of the fact that the derivative (linear approximation) of a linear function is the function itself.

Check: R_θ preserves lengths: Let $\gamma : [a,b] \to \mathbb{E}^2$ be a path.

$$|R_\theta \circ \gamma| = \int_a^b |(R_\theta \circ \gamma)'(t)|\, dt = \int_a^b |R_\theta' \gamma'(t)|\, dt \qquad \text{(chain rule)}$$

$$= \int_a^b |A_\theta \gamma'(t)|\, dt$$

$$= \int_a^b |\gamma'(t)|\, dt \qquad (A_\theta \text{ is orthog.})$$

$$= |\gamma| \quad \checkmark \qquad \text{(Def'n)}$$

Check: R_θ preserves angles: Suppose $\gamma_1(t_1)$, $\gamma_2(t_2)$ intersect.

Let $\alpha = \measuredangle(\gamma_1(t_1), \gamma_2(t_2))$, $\quad \beta = \measuredangle(R_\theta \circ \gamma_1(t_1), R_\theta \circ \gamma_2(t_2))$

Need to show $\cos\beta = \cos\alpha$

$$\cos\beta = \frac{(R_\theta \circ \gamma_1)'(t_1) \cdot (R_\theta \circ \gamma_2)'(t_2)}{|(R_\theta \circ \gamma_1)'(t_1)| \cdot |(R_\theta \circ \gamma_2)'(t_2)|} = \frac{R_\theta' \gamma_1'(t_1) \cdot R_\theta' \gamma_2'(t_2)}{|R_\theta \gamma_1'(t_1)|\,|R_\theta \gamma_2'(t_2)|}$$

$$= \frac{A_\theta \gamma_1'(t_1) \cdot A_\theta \gamma_2'(t_2)}{|A_\theta \gamma_1'(t_1)|\,|A_\theta \gamma_2'(t_2)|} = \frac{\gamma'(t_1) \cdot \gamma_2'(t_2)}{|\gamma_1'(t_1)|\,|\gamma_2'(t_2)|} = \cos\alpha. \quad \checkmark$$

$\square$

Def 1.3.10: Let $\vec{v} \in \mathbb{E}^2$ be a vector. The translation $\tau_{\vec{v}}$ is defined to be $\tau_{\vec{v}}(\vec{x}) = \vec{x} + \vec{v}$ of $\mathbb{E}^2$.

Theorem 1.3.11: The translation $\tau_{\vec{v}}$ is an isometry of $\mathbb{E}^2$ that preserves lines.

Proof: Exercise (HW).

Thm 1.3.12: Let $\vec{x}_0, \vec{x}_1 \in \mathbb{E}^2$. The line segment between $\vec{x}_0$ and $\vec{x}_1$ is the unique geodesic segment between $\vec{x}_0$ and $\vec{x}_1$.

Proof: Assume "standard position:" $\vec{x}_0 = (x_0, 0)$, $\vec{x}_1 = (x_1, 0)$.

Let $\gamma(t) = (x(t), y(t))$ be a path $[0,1] \to \mathbb{E}^2$.

That is, $\gamma(0) = (x_0, 0)$, $\gamma(1) = (x_1, 0)$

$$|\gamma| = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \geq \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2}\, dt = \int_0^1 \left|\frac{dx}{dt}\right| dt$$

equality iff $\frac{dy}{dt} = 0$

$$\geq \left| \int_0^1 \frac{dx}{dt}\, dt \right| = |x(1) - x(0)|$$

$$= |x_1 - x_0|.$$

equality since $\frac{dx}{dt} \neq 0$.

Thus, $|\gamma| \geq |x_1 - x_0|$.

Since $\gamma(t) = t\vec{x}_0 + (1-t)\vec{x}_1$ is a path with $|\gamma| = |x_1 - x_0|$, $d(x_1, x_0) = |x_1 - x_0|$, i.e, the line segment from $\vec{x}_0$ to $\vec{x}_1$ is the (unique) geodesic segment from $\vec{x}_0$ to $\vec{x}_1$. $\square$

We could continue establishing all of Euclidean geometry analytically, but it would be cumbersome. Stahl (Chapter 2) uses less-rigorous "ruler/compass" methods. These work for Euclidean geometry, but not for hyperbolic geometric, which we'll study next.