2. Circle inversion

Circle inversion will be an application of Euclidean geometry, and an introduction to hyperbolic geometry.

Recall polar coordinates: $(x,y) = (r\cos\theta,\ r\sin\theta)$.

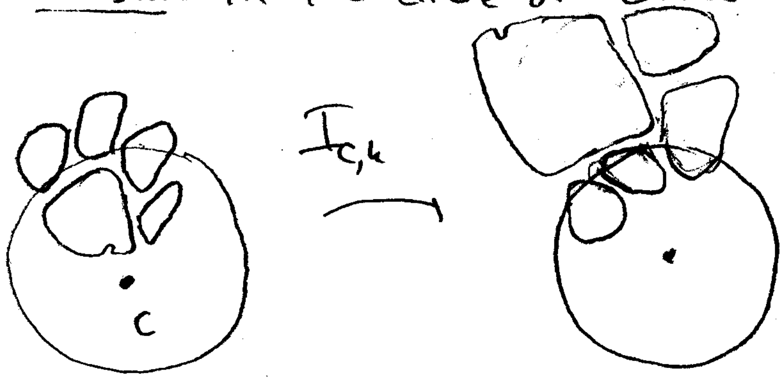
<u>Def 2.2.1</u>: Given $C \in \mathbb{E}^2$, $k > 0$, define a transformation

$$I_{C,k} : \mathbb{E}^2 \setminus C \longrightarrow \mathbb{E}^2 \setminus C$$

$$P \longmapsto P',$$ where P' is the unique point on $\overrightarrow{CP}$ such that $d(C,P)\, d(C,P') = k^2$.

$I_{C,k}$ is the <u>inversion</u> in the circle of center C, radius k.

<u>Example</u>:



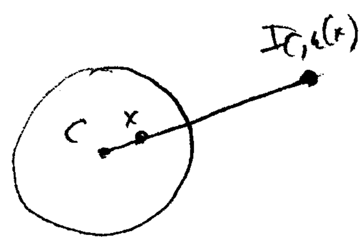
(approximately)

<u>Remark</u>: Points close to C get mapped far away

Points on the circle are fixed by $I_{C,k}$

The inside of the circle gets mapped to the outside, and vice versa.

We can think of C as getting mapped to ∞ (more on this later)



Thm 2.1.2: Inversion has the following properties:

1. $I_{c,k}$ is a bijection and $I_{c,k}^2 = 1$ (the identity map.)
2. In polar coordinates: $I_{0,k}(r,\theta) = \left(\frac{k^2}{r}, \theta\right)$
3. In rectangular coordinates: $I_{0,k}(x,y) = \left(\frac{k^2 x}{x^2+y^2}, \frac{k^2 y}{x^2+y^2}\right)$.

Proof: 1 & 2 are immediate from the def'n of $I_{c,k}$.

To show 3, observe that

$$I_{0,k}(x,y) = \left(\frac{k^2}{r}\cos\theta, \frac{k^2}{r}\sin\theta\right)$$

$$= \left(\frac{k^2 r\cos\theta}{r^2}, \frac{k^2 r\sin\theta}{r^2}\right) = \left(\frac{k^2 x}{x^2+y^2}, \frac{k^2 y}{x^2+y^2}\right). \quad \checkmark \qquad \square$$

Cor 2.1.3: Let $f:(0,\infty)\times[0,2\pi) \to \mathbb{E}^2$ be any function.

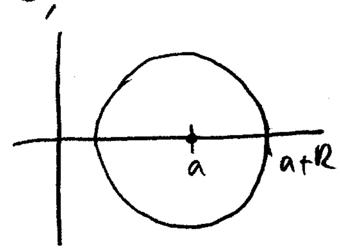
Define $\Gamma_0 = \{(r,\theta) \mid f(r,\theta) = 0\}$, $\Gamma_1 = \{(r,\theta) \mid f\left(\frac{k^2}{r},\theta\right) = 0\}$.

Then $I_{0,k}(\Gamma_0) = \Gamma_1$

Proof Exercise (HW 2)

Prop 2.1.4: (i) The circle with center $(a,0)$ and radius R is described by the equation $r^2 + br\cos\theta + c = 0$, where $b = -2a$ and $c = a^2 - R^2$.

(ii) Furthermore, for any b, c s.t. $b^2 - 4c > 0$, this equation is a circle



Proof: (i) This circle is $(x-a)^2+y^2=R^2$.

In polar: $(r\cos\theta - a)^2 + (r\sin\theta)^2 = R^2$

$$(r^2\cos^2\theta - 2ar\cos\theta + a^2) + r^2\sin^2\theta = R^2$$

$$r^2 - 2ar\cos\theta + (a^2-R^2) = 0. \quad ✓$$

(ii) Do steps in reverse. We need $\frac{\sqrt{b^2-4c}}{4} \in \mathbb{R}$. ✓ $\square$

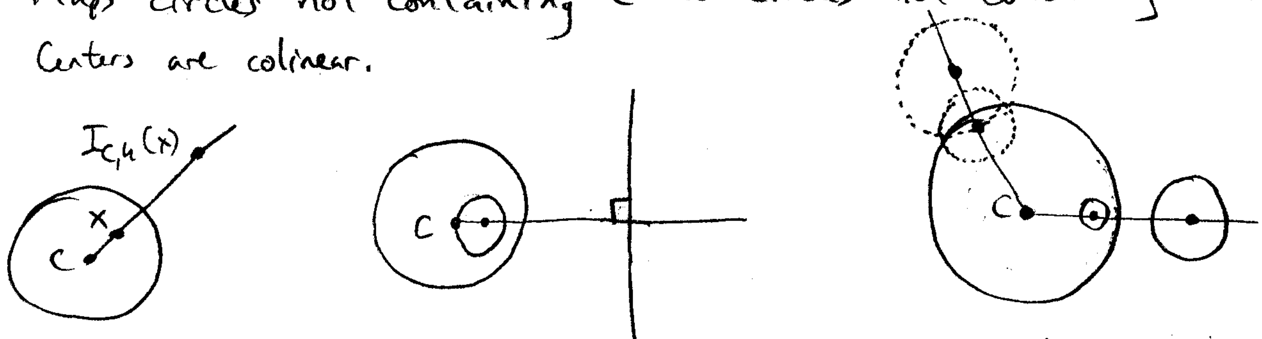
Thm 2.1.5 (Stahl 3.1.3): The inversion $I_{C,k}$:

(i) Maps straight lines containing C onto themselves

(ii) Exchanges straight lines not containing C with circles through the point C. Centers are orthogonal to the line.

(iii) Maps circles not containing C to circles not containing C. Centers are colinear.

Picture



(i) (ii) (iii)

Intrepretation: If straight lines are thought of as circles of infinite radius, then inversion $I_{C,k}$ preserves all circles.

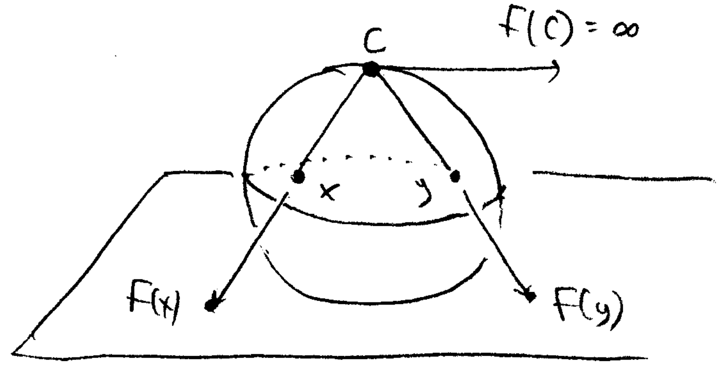
* Compare to stereographic projection:

There is a natural "homeomorphism"

$f: S^2 \setminus C \longrightarrow \mathbb{R}^2$, that extends to

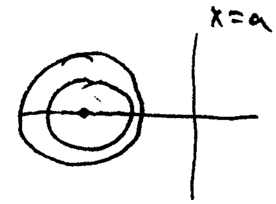
$\hat{f}: S^2 \longrightarrow \mathbb{R}^2 \cup \{\infty\}$.

circles through $C \longmapsto$ straight lines.



4

Proof: WLOG, we can use a "standard position" argument and assume $C=0$, the circles are centered on the positive x-axis, and the straight line is verticle. Also, (i) is trivial. ✓



(ii) Define $f(r,\theta) = r\cos\theta - a = 0$ $\quad a \neq 0$.

The horizontal line has equation $x=a$, or in polar: $f(r,\theta) = r\cos\theta - a = 0$.

Apply $I_{0,k}$. By Prop 2.1.3, this circle gets mapped to the set of points satisfying $f(\frac{k^2}{r}, \theta) = 0$

$$\Rightarrow \frac{k^2}{r}\cos\theta - a = 0.$$

$$\Rightarrow r^2 - \frac{k^2}{a} r\cos\theta = 0 \quad (\text{mult. by } -\tfrac{r^2}{a})$$

which is the circle with center $(\frac{k^2}{2a}, 0)$ through $(0,0)$.

Reverse the argument to get the other implication. ✓

(iii) Define $F(r,\theta) = r^2 + br\cos\theta + c$.

A circle centered on the x-axis not through $(0,0)$ consists of the set $\{(r,\theta) \mid F(r,\theta)\}$.

Apply $I_{0,k}$ to get $\{(r,\theta) \mid F(\frac{k^2}{r},\theta) = 0\}$, which is

$$\frac{k^4}{r^2} + \frac{bk^2\cos\theta}{r} + c = 0$$

$$\Rightarrow \frac{k^4}{c} + \frac{bk^2}{c} r\cos\theta + r^2 = 0$$

However, we must verify (by Prop 2.1.4): $\left(\frac{bk^2}{c}\right)^2 - \frac{4k^4}{c} > 0$.

Check: $\left(\frac{bk^2}{c}\right)^2 - \frac{4k^4}{c} = \frac{b^2k^4}{c^2} - \frac{4k^4}{c} = \frac{b^2k^4 - 4ck^4}{c^2}$

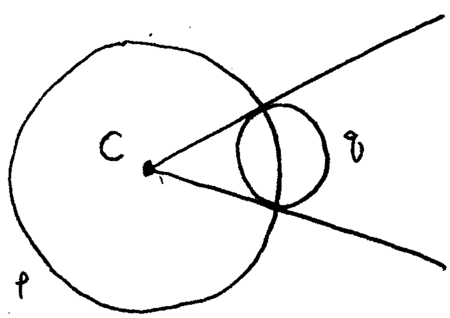
$$= \frac{k^4}{c^2}(b^2 - 4c) > 0. \qquad \square$$

Thm 2.2.1 (Stahl 3.1.6): The circle inversion $I_{C,k}$ preserves angles between paths in $\mathbb{E}^2 \setminus C$. (i.e, it is a conformal mapping.)

Proof: Exercise. (HW 2). $\square$

Def: Say that two circles are orthogonal iff they intersect and the tangent to each circle at their intersection points passes through the center of the other circle.

Picture:



Thm 2.2.2 (Stahl 3.1.7): Let p be the circle of center C, radius k, and let q be any other circle in $\mathbb{E}^2 \setminus C$.

Then $I_{C,k}(q) = q$ iff p and q are orthogonal.

Proof: Exercise (HW 3). $\square$