# 3. The hyperbolic plane.

Def: The hyperbolic plane $\mathbb{H}^2$ consists of

(i) The space $X = \{(x,y) \mid y > 0\}$ (the "upper half-plane.")

(ii) The length function (of a path $\gamma : [a,b] \to \mathbb{H}^2$)

$$|\gamma| = \int_a^b \frac{|\gamma'(t)|}{y(t)}\,dt, \quad \text{where } \gamma(t) = (x(t), y(t))\ dt.$$

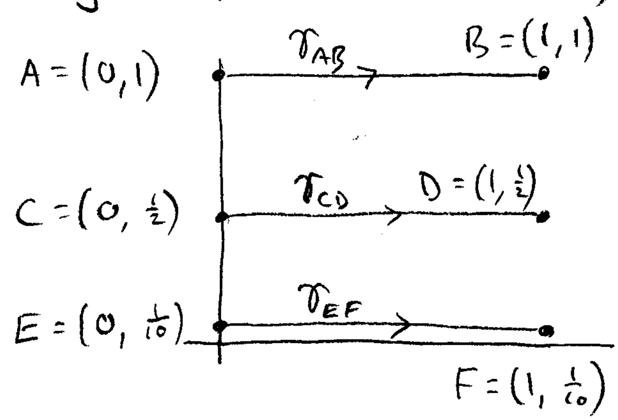
(iii) The angle function $\measuredangle(\gamma_1(t_1), \gamma_2(t_2)) = \Theta$, where

$$\cos\Theta = \frac{\gamma_1'(t_1)\cdot\gamma_2'(t_2)}{|\gamma_1'(t_1)|\,|\gamma_2'(t_2)|} \quad \text{(Same as Euclidean angle)}$$

Note: This means that $|\gamma| = \int_\gamma \frac{\sqrt{dx^2+dy^2}}{y}$.

Need to check: $|\gamma|$ is invariant under change in parameter. (Exercise.)

Example: Consider the points shown here:



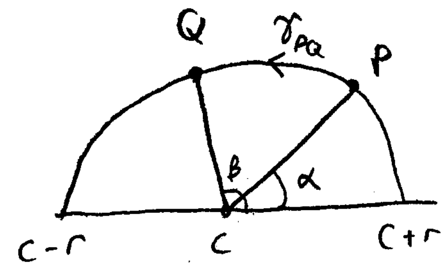

$$|\gamma_{AE}| = \int_{1/10}^{1} \frac{dy}{y} = \ln 1 - \ln \tfrac{1}{10} \approx 2.303.$$

$$|\gamma_{AB}| = \int_0^1 \frac{dx}{1} = 1$$

$$|\gamma_{CD}| = \int_0^1 \frac{dx}{1/2} = 2$$

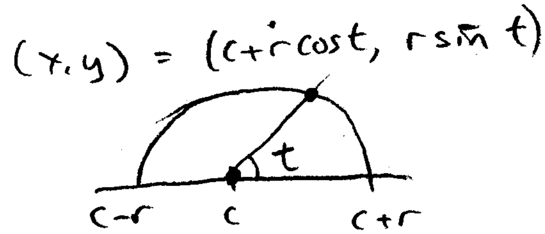
$$|\gamma_{EF}| = \int_0^1 \frac{dx}{1/10} = 10$$

2




**Prop** (Stahl 4.1.1): Let $g$ be a circle with center $C=(c,0)$ and radius $r$.

If $P$ & $Q$ are points of $g$ such that $CP$ & $CQ$ make angles $\alpha<\beta$ with the positive $x$-axis, then the length of the path $\gamma_{PQ}$ from $P$ to $Q$ along $g$ is
$$|\gamma_{PQ}| = \ln\left(\frac{\csc\beta-\cot\beta}{\csc\alpha-\cot\alpha}\right).$$



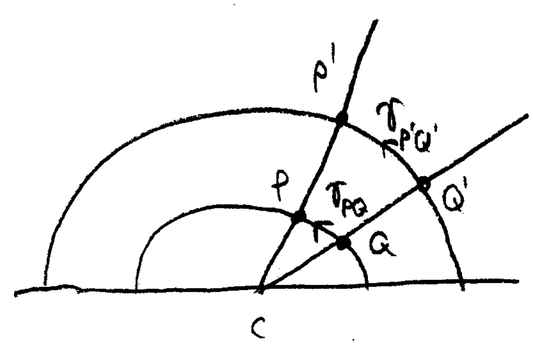

**Proof:** Let $t$ be the angle to a point $(x,y)$ on $g$. Then

$(x(t),y(t)) = (c+r\cos t,\ r\sin t)$

$\Rightarrow\quad dx = -r\sin t\,dt,\qquad dy = r\cos t\,dt$

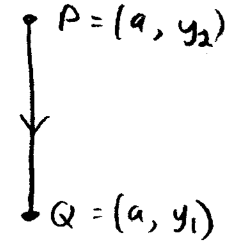
$$|\gamma_{PQ}| = \int_\alpha^\beta \frac{\sqrt{(-r\sin t\,dt)^2+(r\cos t\,dt)^2}}{r\sin t} = \int_\alpha^\beta \frac{r\,dt}{r\sin t}$$
$$= \int_\alpha^\beta \csc t\,dt = \ln\left(\frac{\csc\beta-\cot\beta}{\csc\alpha-\cot\alpha}\right).$$

$\square$




**Cor:** If $P$ & $Q$ lie on a circle with radius $r$, center $C$, and $P'$ & $Q'$ lie on a circle with radius $r'$ and center $C$, and, $C, P, P'$ are colinear, as are $C, Q, Q'$, then $|\gamma_{PQ}| = |\gamma_{P'Q'}|$.

<u>Prop</u> (Stahl 4.1.3): If $0 < y_1 \le y_2$, then the path $\tau$ from $(a, y_2)$ to $(a, y_1)$ along a Euclidean vertical line, has length $|\tau| = \ln\left(\frac{y_2}{y_1}\right)$.



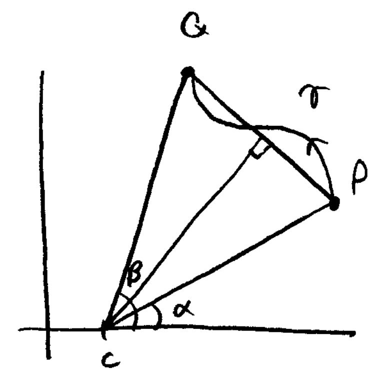

<u>Proof</u>. Exercise (HW 3).

## Geodesics.

<u>Thm</u> (Stahl 4.2.1): The geodesic segments of $\mathbb{H}^2$ are

(i) Arcs of Euclidean semicircles centered on the x-axis

(ii) Segments of Euclidean vertical straight lines.




<u>Proof</u>: Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be 2 points in $\mathbb{H}^2$, and let $\tau$ be a curve joining them.

<u>Case 1</u>: $x_1 \neq x_2$.

Let $C$ be the point on the x-axis perpendicular bisector to $PQ$.
Use polar coordinates, and let $r = f(\theta)$.

$$x = c + r\cos\theta$$
$$y = r\sin\theta$$

Calculus $\Rightarrow$ $\dfrac{dx}{d\theta} = \dfrac{dr}{d\theta}\cos\theta + r\dfrac{d\cos\theta}{d\theta} = r'\cos\theta - r\sin\theta$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\frac{d\sin\theta}{d\theta} = r'\sin\theta + r\cos\theta$$

$$\Rightarrow dx^2 + dy^2 = (r'\cos\theta - r\sin\theta)^2 d\theta^2 + (r'\sin\theta + r\cos\theta)^2 d\theta^2$$
$$= \left[(r')^2(\cos^2\theta + \sin^2\theta) - 2rr'(\cos\theta\sin\theta - \sin\theta\cos\theta) + r^2(\sin^2\theta + \cos^2\theta)\right] d\theta^2$$
$$= \left((r')^2 + r^2\right) d\theta^2.$$

Now, $|\gamma| = \int_\alpha^\beta \frac{\sqrt{(r')^2+r^2}}{r\sin\theta}\, d\theta \geq \int_\alpha^\beta \frac{\sqrt{r^2}}{r\sin\theta} = \int_\alpha^\beta \csc\theta\, d\theta$

$$= \ln\left(\frac{\csc\beta - \cot\beta}{\csc\alpha - \cot\alpha}\right) = |\gamma_{PQ}|,$$

where $\gamma_{PQ}$ is the path from $P$ to $Q$ along the arc of the circle with center $C$.

Since $|\gamma| \geq |\gamma_{PQ}|$ for any path $\gamma$, $\gamma_{PQ}$ is a geodesic segment. ✓


Case 2: $x_1 = x_2$.

Use rectangular coordinates, let $x = f(y)$, so $f' := \frac{dx}{dy}$.

$$|\gamma| = \int_{y_1}^{y_2} \frac{\sqrt{(f')^2 dy^2 + dy^2}}{y} = \int_{y_1}^{y_2} \frac{\sqrt{(f')^2+1}}{y}\, dy \geq \int_{y_1}^{y_2} \frac{dy}{y} = \ln\left(\frac{y_2}{y_1}\right) = |\gamma_{PQ}|,$$

where $\gamma_{PQ}$ is the straight-line path from $P$ to $Q$.

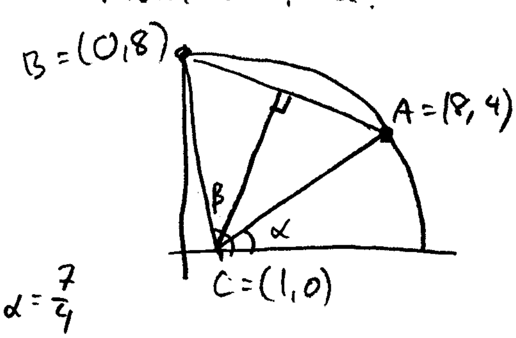
Since $|\gamma| \geq |\gamma_{PQ}|$ for any path $\gamma$, $\gamma_{PQ}$ is a geodesic segment. ✓ □

We now know that hyperbolic geodesics are either

(a) Euclidean circle arcs, called <u>bowed geodesics</u>, or

(b) Euclidean vertical straight lines, called <u>straight geodesics</u>.

<u>Notation</u>: Let $h(P,Q)$ denote the hyperbolic distance from $P$ & $Q$.




<u>Example</u>: Let $A=(8,4)$, $B=(0,8)$.

Then $C=(1,0)$ is the point on the x-axis equidistant to $A$ & $B$ (in $E^2$).

<u>Note</u>: $\csc\beta = \frac{\sqrt{65}}{8}$, $\cot\beta = \frac{-1}{8}$, $\csc\alpha = \frac{\sqrt{65}}{4}$, $\cot\alpha = \frac{7}{4}$

$$\Rightarrow h(A,B) = |\gamma_{AB}| = \ln\left(\frac{\sqrt{65}/8 - (-1/8)}{\sqrt{65}/4 - (7/4)}\right) \approx 1.450.$$

We didn't prove it rigorously (see Stahl, section 2.4), but Euclidean isometries have the following properties:

(i) They are generated by reflections

(ii) They are one of 3 types: (a) translation
(b) rotation
(c) glide-reflection.

It turns out that (i) still holds for $\mathbb{H}^2$, but (ii) does not.

For now, we'll primarily focus on hyperbolic reflections.

<u>Thm</u> (Stahl, 4.4.1): The following transformations of $\mathbb{H}^2$ preserve both hyperbolic length and angle.

(a) inversions $I_{C,k}$, where $C$ is on the x-axis.

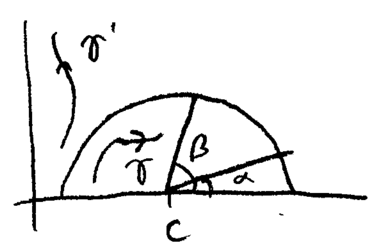
(b) reflections $\rho_m$, where $m$ is a line orthogonal to the x-axis.

(c) translations $\tau_{AB}$, where $AB$ is parallel to the x-axis.

<u>Proof</u>:

(a) Let $I_{C,k}$ be an inversion. By Thm 2.2.1, $I_{C,k}$ preserves angles. Consider the curve $\gamma : [a,b] \to \mathbb{H}^2$, where $r = f(\theta)$, $\alpha \le \theta \le \beta$.

Then $I_{C,k}$ maps $\gamma$ to the curve $\gamma'$ given by $r = F(\theta) := \dfrac{k^2}{f(\theta)}$, $\alpha \le \theta \le \beta$.

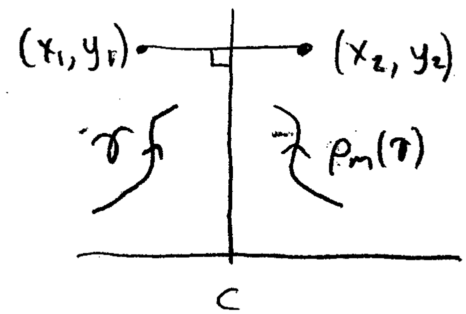


Now, $|\gamma'| = \int_{\gamma'} \frac{\sqrt{(r')^2 + r^2}}{r \sin\theta}\, d\theta = \int_{\alpha}^{\beta} \frac{\sqrt{(F')^2 + F^2}}{F \sin\theta}\, d\theta$

$$= \int_{\alpha}^{\beta} \frac{\sqrt{\left(\frac{-k^2 f'}{f^2}\right)^2 + \left(\frac{k^2}{f}\right)^2}}{k^2 \sin\theta / f} = \int_{\alpha}^{\beta} \frac{\sqrt{(f')^2 + f^2}}{f \sin\theta}\, d\theta$$

$$= \int_{\gamma} \frac{\sqrt{(r')^2 + r^2}}{r \sin\theta}\, d\theta = |\gamma|. \checkmark$$

(b) Let $m$ be the line $x = c$.




2 points are symmetrical w.r.t. $m$ iff

$c = \frac{x_1 + x_2}{2}$, $y_1 = y_2$.

Let $\gamma(t) = (u(t), v(t))$, $a < t < b$ be a path.

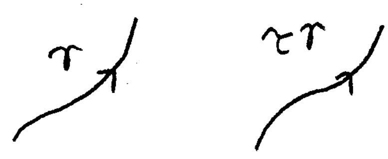
Then $\rho_m \gamma(t) = (2c - u(t), v(t))$, and

$\gamma'(t) = (u'(t), v'(t)) \Rightarrow dx = u'(t)\, dt$, $dy = v'(t)\, dt$

$(\rho_m \gamma)'(t) = (-u'(t), v'(t)) \Rightarrow dx = -u'(t)\, dt$, $dy = v'(t)\, dt$

Clearly, $|\rho_m \gamma| = \int_a^b \frac{\sqrt{u'(t)^2 + v'(t)^2}}{v(t)}\, dt = |\gamma|$ ✓

(c) Let $\tau(x, y) = (x + h, y)$, fixed $h$.




$\gamma(t) = (u(t), v(t))$, $a < t < b$

$\Rightarrow \tau\gamma(t) = (u(t) + h, v(t)) \Rightarrow dx = u'(t)\, dt$, $dy = v'(t)\, dt$

$\Rightarrow |\tau\gamma| = \int_a^b \frac{\sqrt{u'(t)^2 + v'(t)^2}}{v(t)}\, dt = |\gamma|$ ✓

$\square$

<u>Cor</u> (Stahl 4.4.2): All of the transformations in Thm 4.4.1 carry geodesics to geodesics.