4. Euclidean vs. Hyperbolic geometry

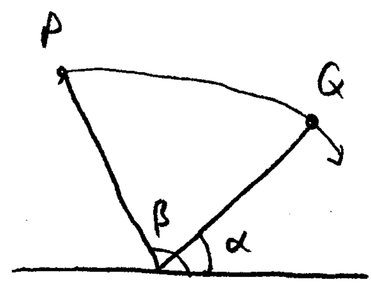
Euclid used 5 "postulates" in his book "The Elements." As we'll see, the first 4 hold for hyperbolic geometry.

- Postulate 1: Any two points can be connected by a straight line.

In $\mathbb{H}^2$: Clear. ✓

- Postulate 2: Any line segment can be extended indefinitely in a straight line.

In $\mathbb{H}^2$: Consider a bowed geodesic γ from P to Q.



$$\lim_{\alpha \to 0} h(P,Q) = \lim_{\alpha \to 0} \ln\left(\frac{\csc\beta - \cot\beta}{\csc\alpha - \cot\alpha}\right)$$

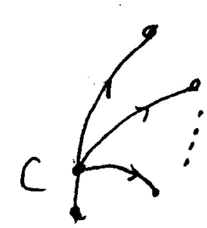
$$= \lim_{\alpha \to 0} \ln\Big[\underbrace{(\csc\beta - \cot\beta)}_{< \infty}\underbrace{(\csc\alpha + \cot\alpha)}_{= \infty}\Big]$$

Similarly, $\beta \to \pi$ can be checked.

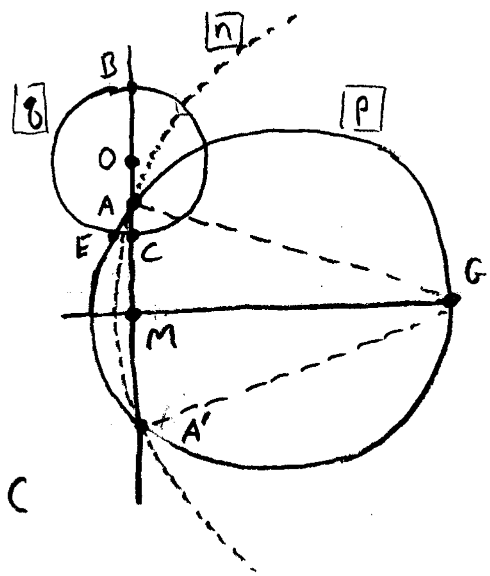
Also, the case when γ is a straight geodesic is an exercise.

- Postulate 3: Given any straight line segment, a circle can be drawn having the segment as radius & one endpoint as center.

In $\mathbb{H}^2$: Given any point $C \in \mathbb{H}^2$, fix r, and let γ be a ray (half-geodesic) from C. The set of all such points is the hyperbolic circle with center C & radius r.



<u>Thm</u> (Stahl 5.1.1): Every Euclidean circle in the upper half-plane is also a hyperbolic circle.



<u>Proof</u>: Let q be a Euclidean circle with center O, diameter BC perpendicular to the x-axis.

If q is in fact a hyperbolic circle, then the hyperbolic midpoint A of BC must be the hyperbolic center.

<u>Lemma</u>: The hyperbolic midpoint M of the points (a,p) & (a,q) is the point $M=(a,b)$, where b satisfies $\frac{b}{p}=\frac{q}{b}$, i.e, $M=(a,\sqrt{pq})$

<u>Proof</u>: Exercise. (easy)

Claim: Every geodesic p through A divides q into two hyperbolically congruent parts (via the hyperbolic reflection I_p).

<u>Note</u>: This only happens if $p \perp q$ (Thm 2.2.2 / Stahl 3.1.7), or equivalently, if $I_{O,k}(p)=p$. (where k is Euclidean radius of q)

To show this, Suppose $O=(\cdot, b)$, $B=(\cdot, b+k)$, $C=(\cdot, b-k)$.

By Lemma, $A=(\cdot, \sqrt{b^2-k^2})$, and let $A'=(\cdot, -\sqrt{b^2-k^2}) \in q$.

<u>Note</u>: $OA \cdot OA' = (b-\sqrt{b^2-k^2})(b+\sqrt{b^2-k^2}) = k^2 \Rightarrow A'$ is on $I_{O,k}(p)$.

Since $A' \in p$ and $I_{O,k}(p)$, $p=I_{O,k}(p)$. (Why?)

Thus, $p \perp q \Rightarrow q = I_p(q)$. (Claim proven ✓)

Remains to show: $h(A,E)$ is independent of the position of p, where $E := p \cap q$.

Approach: Product a hyperbolic reflection moving AE onto AC.

Let $G = p \cap x$-axis, and let n be the geodesic centered at G and passing through A & A'.

Note: $I_n(p)$ is the straight line through BC, because p goes through the center of n, and $A, A' \in I_n(p)$.

Since n is a geodesic through A, $n \perp q \Rightarrow I_n(q) = q$.

Now, consider $E = p \cap q$. $I_n(E) = I_n(p) \cap I_n(q) = BC \cap q = C$.

Thus, $h(A,E) = h(A,C)$, and q is a hyperbolic circle. $\square$

Prop (Stahl, 5.1.2): If a Euclidean circle has Euclidean center (h,k) and Euclidean radius $r<k$, then it has hyperbolic center (H,K) and radius R, where

$$H = h, \qquad K = \sqrt{k^2 - r^2}, \qquad R = \frac{1}{2}\ln\left(\frac{k+r}{k-r}\right).$$

Proof: $R = \frac{1}{2}\int_{k-r}^{k+r} \frac{dy}{y} = \frac{1}{2}\ln\left(\frac{k+r}{k-r}\right)$. ✓

To find K: $\int_{k-r}^{K} \frac{dy}{y} = \ln\left(\frac{K}{k-r}\right) = \frac{1}{2}\ln\left(\frac{k+r}{k-r}\right) = \ln\sqrt{\frac{k+r}{k-r}}$

$$\Rightarrow \frac{K}{k-r} = \sqrt{\frac{k+r}{k-r}} \Rightarrow K = \sqrt{k^2-r^2}$$

The fact that $H = h$ is clear. $\square$

<u>Remark</u>: We can "invert" this process, i.e., given a hyperbolic circle $\mathscr{C}$ with center (H, K) and radius R, we can find the Euclidean center & radius by solving $\begin{cases} K = \sqrt{k^2 - r^2} \\ R = \ln\sqrt{\frac{k+r}{k-r}} \end{cases}$ for k & r.

$$e^R = \sqrt{\frac{k+r}{k-r}} \Rightarrow e^{2R} = \frac{k+r}{k-r}.$$

Now,
$$\left.\begin{aligned} e^{2R}+1 &= \frac{k+r}{k-r} + \frac{k-r}{k-r} = \frac{2k}{k-r} \\ e^{2R}-1 &= \frac{k+r}{k-r} - \frac{k-r}{k-r} = \frac{2r}{k-r} \end{aligned}\right\} \Rightarrow \frac{e^{2R}+1}{e^{2R}-1} = \frac{k}{r} = \coth R$$

$$\Rightarrow \boxed{r^2 = k^2 \tanh^2 R}$$

<u>Plug in $K^2 = k^2 - r^2$:</u>

$$r^2 = (r^2 + K^2)\tanh^2 R$$
$$r^2(1 - \tanh^2 R) = K^2 \tanh^2 R$$
$$r^2 \operatorname{sech}^2 R = K^2 \tanh^2 R$$
$$r^2 = K^2 \sinh^2 R$$
$$\boxed{r = K \sinh R}$$

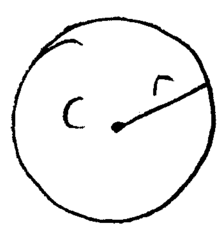
$$k^2 = r^2 \coth^2 R = (k^2 - K^2)\coth^2 R$$
$$k^2(\coth^2 R - 1) = K^2 \coth^2 R$$
$$k^2 \operatorname{csch}^2 R = K^2 \coth^2 R$$
$$k^2 = K^2 \cosh^2 R$$
$$\boxed{k = K \cosh R}$$

<u>Cor</u> (Stahl 5.1.3): Every hyperbolic circle is also a Euclidean circle. □

Let's compute the circumference of a hyperbolic circle $\mathscr{C}$.

Parametrize $\mathscr{C}$ as $x = h + r\cos t$, $y = k + r\sin t$

$dx = -r\sin t\, dt$ $\quad dy = r\cos t$



$$\int_q \frac{\sqrt{dx^2+dy^2}}{y} = 2\int_{-\pi/2}^{\pi/2} \frac{r\,dt}{k+r\sin t} = \left[2r\frac{2}{\sqrt{k^2-r^2}}\tan^{-1}\left(\frac{k\tan(\frac{t}{2})+r}{\sqrt{k^2-r^2}}\right)\right]_{-\pi/2}^{\pi/2}$$

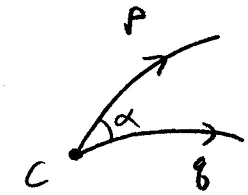
$$= \frac{4r}{\sqrt{k^2-r^2}}\left[\tan^{-1}\left(\frac{k+r}{\sqrt{k^2-r^2}}\right) - \tan^{-1}\left(\frac{-k+r}{\sqrt{k^2-r^2}}\right)\right]$$

$$= \frac{4}{\sqrt{k^2-r^2}}\left[\tan^{-1}\sqrt{\frac{k+r}{k-r}} + \tan^{-1}\sqrt{\frac{k-r}{k+r}}\right] = \frac{2\pi r}{\sqrt{k^2-r^2}} \qquad \left(\text{b/c } \tan^{-1}x + \tan^{-1}\frac{1}{x} = \frac{\pi}{2}\right)$$

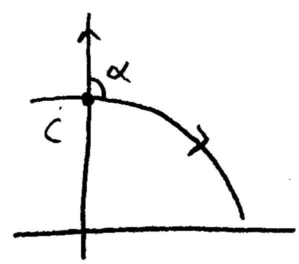
Compare:

$$\frac{\text{Euclidean circumference}}{\text{Euclidean diameter}} = \pi. \qquad \frac{\text{Hyperbolic circumference}}{\text{Hyperbolic diameter}} = \frac{2\pi r(k^2-r^2)^{-1/2}}{\ln\left(\frac{k+r}{k-r}\right)}.$$

- Postulate 4: All right angles are congruent.



By "angle," we mean a point C with an ordered pair (p, q) of rays from C.



Def: Say that a hyperbolic angle is in standard position if one of its sides is the ray $\{(0,y) \mid 1 \le y < \infty\}$ and the other is in the half-plane $\{(x,0) \mid x > 0\}$.

In $\mathbb{H}^2$: All right angles are congruent.

Proof: We'll show that every hyperbolic right angle is congruent to one in standard position with $C = (0,1)$.

First, consider two geodesics $p \perp q$.

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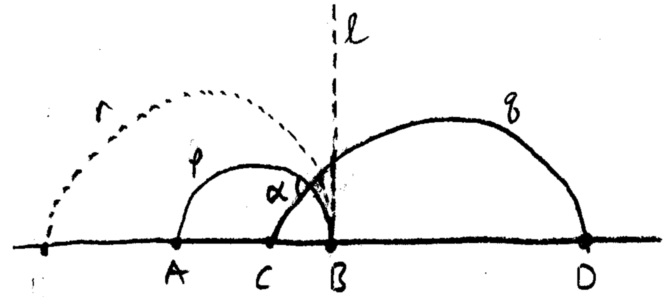
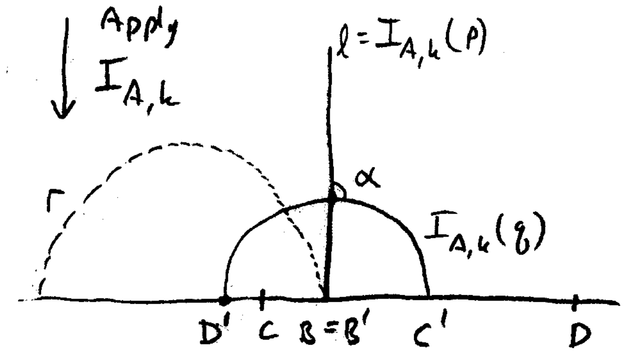


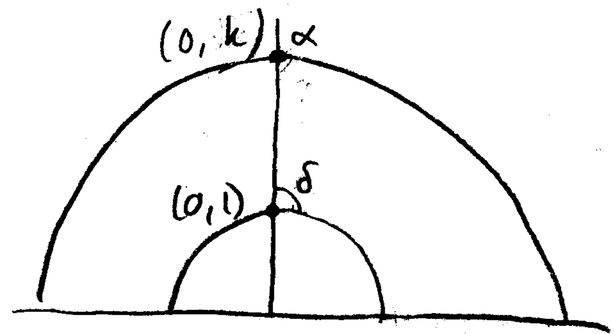
Assume p & q are Euclidean semi-circles with endpoints A, B and C, D on the x-axis.



Then the inversion $I_{A,k}$ $(k = AB)$ carries p to a vertical line.

If α is on the left side of ℓ, we can reflect it through ℓ to bring it to standard position.

We can also translate α so its center is on the line $x=0$.



Now, if α is at point $(0,k)$, the inversion $I_{0,\sqrt{k}}$ will carry it to $(0,1)$. Then apply $I_{0,1}$ and/or reflection across $x=0$ to move it to standard position.

□

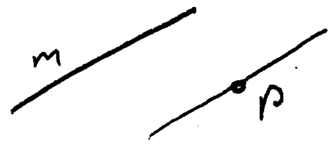
Remark: The above argument can be generalized to handle non-right angles.

Prop (Stahl 5.1.4): In $\mathbb{H}^2$, every angle is congruent to an angle in standard position.

Proof: Exercise.

□

- <u>Postulate 5</u> ("Parallel postulate"): Given a straight line m and a point $P \notin m$, there is a unique straight line parallel to m containing P.

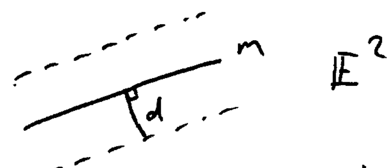


<u>In $\mathbb{H}^2$</u>: Fail.

But there are several ways to define "parallel" in $\mathbb{H}^2$:

(i) Non-intersecting geodesics

(ii) The set of all points a fixed distance d from m.

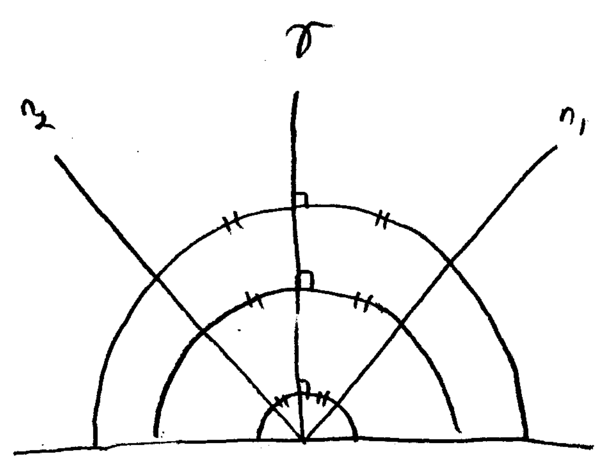


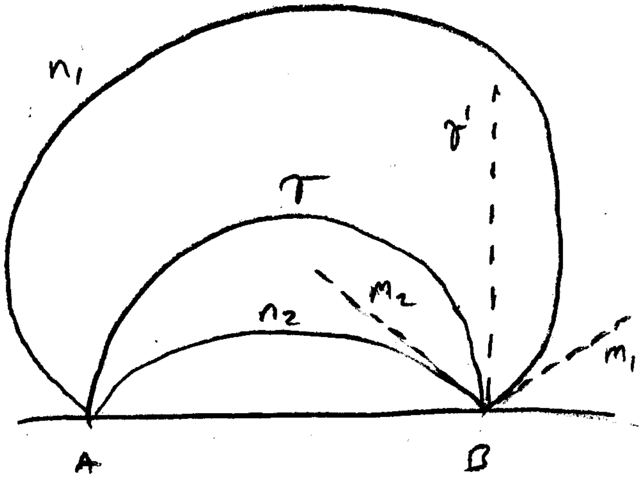
It still fails but it's worth analyzing what these type of "parallel" lines look like in $\mathbb{H}^2$.

Consider the set of points equidistant to a fixed geodesic γ.

If $P \notin \gamma$, then there exists a unique geodesic of minimal length from P to γ (orthogonal to γ). This can be proven from scratch, but it also follows from Euclid's Props 12, 17 & 27.

If γ is a <u>straight geodesic</u>, then the set of points a fixed distance from γ consists of 2 Euclidean straight lines n_1 & n_2 (see at right).

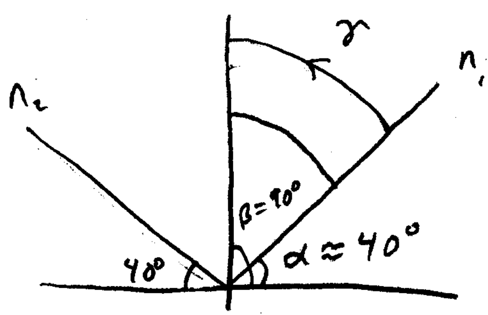




If γ is a bowed geodesic with endpoints A & B, then the inversion $O_{A,k}$ $(k = AB)$ maps γ to a straight line geodesic γ'. If m_1 & m_2 are the set of points shown at right, then their preimage is the set of points a fixed distance from γ. These are the hyperbolic straight lines n_1 & n_2.

Note that n_1 & n_2 are tangent (in $\mathbb{E}^2$) to m_1 & m_2

Example 1 Find the set of points that lie at a constant hyperbolic distance 1 from the y-axis.

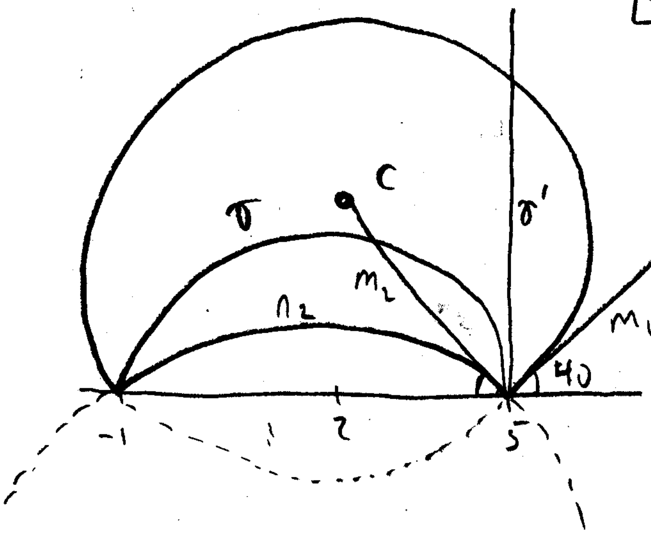


We need to solve

$$\ln\left(\frac{\csc(\pi/2) - \cot(\pi/2)}{\csc\alpha - \cot\alpha}\right) = 1 \quad \text{for } \alpha.$$

$$\Rightarrow \begin{cases} \csc\alpha - \cot\alpha = \frac{1}{e} \\ \csc\alpha + \csc\alpha = e \end{cases} \rightarrow \alpha = \sin^{-1}\left(\frac{2}{e+e^{-1}}\right) \approx 40^\circ$$

The desired set of points are the lines n_1, n_2 at an angle of $\approx 40^\circ$ to the x-axis.



Example 2: Find the set of points that lie distance 1 from the bowed geodesic with center (2,0) and radius 3.

Soln: Put $B=(5,0)$. Construct the vertical line γ' through B, and the Euclidean rays m_1 & m_2 through B at $40°$ to the x-axis. The curves n_1 & n_2 we're looking for are the 2 Euclidean circles through $(-1,0)$ and $(5,0)$ tangent to m_1 and m_2.

We can compute the centers of these circles to be

$$\left(2, \pm\sqrt{3}\tan\left(\tfrac{\pi}{2}-\alpha\right)\right) \approx (2, \pm 3.52)$$

Remark: Euclid's Propositions 1-28 depend only on Postulates 1-4, and so they hold for $\mathbb{H}^2$ as well.

On HW #1, you proved the following statement, which is an "absolute proposition" (follows from Euclid's Props 1-28.):

Prop (Stahl 2.1.2): If two isometries agree on two distinct points, then they agree everywhere on the geodesic through them. $\square$

The following results are almost immediate (exercises).

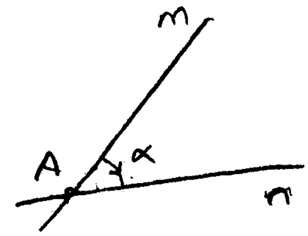
Theorem (Stahl 2.1.3): If two isometries agree on three noncolinear points, they agree everywhere. □

Cor (Stahl 2.1.4): If an isometry fixes three noncolinear points, it must be the identity. □

Since these are "absolute" results, they hold for $\mathbb{H}^2$ as well.

Several other absolute results apply to $\mathbb{H}^2$: (we skipped a few)

Prop (Stahl 2.2.6) Let m, n be two geodesics with $A = m \cap n$, and $\alpha = \measuredangle(m, n)$. Then



(ρ_n: reflection, R: rotation). [See HW #1]

Thm (Stahl 5.4.1): Every hyperbolic isometry is the composition of at most three hyperbolic reflections.

Proof: (outline). Follows from the analogous "absolute" statement; see Thm 2.4.2 in Stahl. □

Remark: Reflections (inversions) and rotations are simple to describe in $\mathbb{H}^2$, but translations are much more difficult, due to the lack of the parallel postulate. More on this later.