7. The Euclidean isometry group

Recall that the set $\mathrm{Isom}(\mathbb{E}^2)$ of Euclidean isometries is a group, under function composition:

* If $f, g \in \mathrm{Isom}(\mathbb{E}^2)$, then $f \circ g \in \mathrm{Isom}(\mathbb{E}^2)$.
* $Id \in \mathrm{Isom}(\mathbb{E}^2)$ such that $f \circ Id = Id \circ f = f$ for each f.
* For each $f \in \mathrm{Isom}(\mathbb{E}^2)$, $f^{-1} \in \mathrm{Isom}(\mathbb{E}^2)$.
* Associative law holds

Note that $\mathrm{Isom}(\mathbb{E}^2)$ is non-abelian, because in general, $f \circ g \neq g \circ f$.

We'll study the conjugacy classes of $\mathrm{Isom}(\mathbb{E}^2)$ next.

Def: Let G be a group. We say that $f, g \in G$ are conjugate if there is some $h \in G$ such that $hfh^{-1} = g$.

Remark: Conjugacy is an equivalence relation. Elements in the same conjugacy class "have the same structure."

Example: In linear algebra, conjugacy is usually called similarity.

Let $G = GL_n(\mathbb{R})$, the $n \times n$ invertible matrices.

$A \sim B$ iff A & B represent the same abstract linear transformation relative to different bases. [Change of basis theorem.]

In linear algebra, there are certain properties (eigenvalues, determinant, trace, Jordan canonical form, etc.) that are <u>invariant</u> under change of basis, i.e., conjugacy.

<u>Goal</u>: Do the same thing for $\mathrm{Isom}(\mathbb{E}^2)$, and then $\mathrm{Isom}(\mathbb{H}^2)$.

Throughout, let X be a geometry and $G = \mathrm{Isom}(X)$.

<u>Theorem</u> (Hsu 5.1.3): For $f, g \in G$, $x, y \in X$, g takes x to y iff fgf^{-1} takes $f(x)$ to $f(y)$

$$\begin{tikzcd} x \arrow[r, mapsto, "g"] \arrow[d, "f"'] & y \arrow[d, "f"] \\ f(x) \arrow[r, mapsto, dashed, "h"] & f(y) \end{tikzcd}$$

<u>Proof</u>: Exercise.

<u>Def</u>: For $g \in G$, the <u>fixed points</u> of g are the points $x \in X$ such that $g(x) = x$.

<u>Cor</u>: (Hsu 5.1.5) $x \in X$ is a fixed point of g iff $f(x)$ is a fixed point of fgf^{-1}.

<u>Remark</u>: This holds more generally for any group G that acts on a set X.

Let $f : \mathbb{E}^2 \to \mathbb{R}$ be a non-negative function. Recall that the <u>infimum</u> of f, denoted $\inf\limits_{x \in \mathbb{E}^2} f(x)$ is the greatest lower bound on $f(x)$. It exists, though $f(x)$ might not ever achieve this bound. (It might get arbitrarily close.)

<u>Def</u>: Let $g \in \mathrm{Isom}(\mathbb{E}^2)$. The <u>minimal motion</u> of g, is defined as $\mu(g) := \inf_{x \in \mathbb{E}^2} d(x, g(x))$. The <u>set of minimal motion</u> of g is $\{x \in \mathbb{E}^2 : d(x, g(x)) = \mu(g)\}$.

<u>Theorem</u> (Hsu 5.2.3): Let $f, g \in \mathrm{Isom}(\mathbb{E}^2)$, let S be the set of minimal motion of g, and S' the set of minimal motion of fgf^{-1}. Then $\mu(g) = \mu(fgf^{-1})$ and $f(S) = S'$.

<u>Proof</u>: By def'n,

$$\begin{aligned}
\mu(g) &= \inf_{x \in \mathbb{E}^2} d(x, g(x)) \\
&= \inf_{x \in \mathbb{E}^2} d(f^{-1}(x), g(f^{-1}(x))) \\
&= \inf_{x \in \mathbb{E}^2} d(f(f^{-1}(x)), f(g(f^{-1}(x)))) \\
&= \inf_{x \in \mathbb{E}^2} d(x, fgf^{-1}(x)) = \mu(fgf^{-1}) \checkmark
\end{aligned}$$

$$\begin{tikzcd}
S \arrow[r, mapsto] \arrow[d, mapsto] & g(S) \arrow[d, mapsto] \\
f(S) \arrow[r, mapsto] & f(g(S))
\end{tikzcd}$$

Now, let $\mu = \mu(g) = \mu(fgf^{-1})$.

Note that $x \in S \iff d(x, g(x)) = \mu \iff d(f(x), f(g(x))) = \mu$

$\iff d(f(x), fgf^{-1}(f(x))) = \mu \iff f(x) \in S'$. $\square$

<u>Big idea</u>: Theorem 5.2.3 says that the minimal motion of an isometry is an <u>invariant</u> of a conjugacy class.

<u>Next goal</u>: Determine the conjugacy classes of $\mathrm{Isom}(\mathbb{E}^2)$.

Approach:

1. Determine the minimal motion & set of minimal motion for each type of isometry
2. Given an arbitrary isometry of a certain type, conjugate it to an isometry of the same type in standard position.
3. Find the conjugacy classes of each type by considering the ones in standard position:

Theorem (Hsu 5.3.1): The 4 types of Euclidean isometries have minimal motions & sets of minimal motions as follows:

1. Reflection: $\boxed{\rho_m}$ has line m as its fixed point set (and so $\mu(\rho_m)=0$ and the set of min'l motion is m)
2. Rotation: $\boxed{R_{C,\alpha}}$ has one fixed point, C. (so $\mu(R_{C,\alpha})=0$ and the set of min'l motion is C.
3. Translation: $\boxed{\tau_v}$ has minimal motion $\mu(\tau_v)=|v|$ and the set of min'l motion is $\mathbb{E}^2$.
4. Glide reflection: $\boxed{\gamma_{AB}}$ has minimal motion $\mu(\gamma_{AB})=d(A,B)$ and the set of minimal motion is the line through AB.

Cor: Reflections are only conjugate to reflections, rotations are only conjugate to rotations, and so on.

Theorem (Hsu 5.3.2): All Euclidean reflections are conjugate.

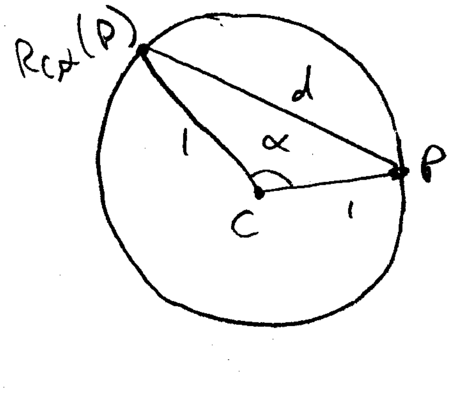
Proof: Let ρ_k, ρ_m be reflections and pick any $f \in Isom(\mathbb{E}^2)$ that sends $m \mapsto k$. By Cor 5.1.5, $f \rho_m f^{-1}$ fixes every point on k. But the only non-trivial isometry fixing all points on k is ρ_k, and so $f \rho_m f^{-1} = \rho_k$.

Classifying rotations by conjugacy takes more work.

Theorem: (Hsu 5.3.3) Two rotations $R_{C,\alpha}$ and $R_{D,\beta}$ with $-\pi \le \alpha, \beta \le \pi$ are conjugate $|\alpha| = |\beta|$.

Proof: Assume all angles are in $[-\pi, \pi]$.

Let $R_{C,\alpha}$ be a rotation and let $P \in \mathbb{E}^2$ be a point with $|C-P| = 1$ and let $d = d(P, R_{C,\alpha}(P))$.



By the law of cosines, $d = \sqrt{2 - 2\cos\alpha}$.

Since P is arbitrary, any rotation $R_{C,\alpha}$ moves all points at distance 1 from C exactly $\sqrt{2-2\cos\alpha}$.

Claim: Every conjugate of $R_{C,\alpha}$ is a rotation of angle $\pm\alpha$.

Pick $f \in Isom(\mathbb{E}^2)$ and let $D = f(C)$.

By Cor 5.1.5, $f \circ R_{C,\alpha} \circ f^{-1}$ fixes only D.

Thus, $f \circ R_{C,\alpha} \circ f^{-1}$ is a rotation around D; call it $R_{D,\beta}$. <u>Goal</u>: Show $\beta = \pm\alpha$.

$$\begin{tikzcd} \mathbb{E}^2 \arrow[r, "R_{C,\alpha}"] \arrow[d, "f"'] & \mathbb{E}^2 \arrow[d, "f"] \\ \mathbb{E}^2 \arrow[r, "R_{D,\beta}"] & \mathbb{E}^2 \end{tikzcd}$$

Let $x \in \mathbb{E}^2$ be a point with $|x - D| = 1$.

Now, $\sqrt{2-2\cos\beta} = d(x, R_{D,\beta}(x)) = d(x, f \circ R_{C,\alpha} \circ f^{-1}(x))$

$= d(f^{-1}(x), R_{C,\alpha} \circ f^{-1}(x)) = \sqrt{2-2\cos\alpha}$.

Therefore, $\sqrt{2-2\cos\beta} = \sqrt{2-2\cos\alpha} \Rightarrow \cos\alpha = \cos\beta \Rightarrow \alpha = \pm\beta$.

It now remains to show that $R_{C,\alpha}$ is conjugate to both $R_{O,\pm\alpha}$.

2 cases: (i) $R_{O,\alpha}$ (HW 1)

(ii) $R_{O,-\alpha}$ (HW 6)

$\square$

Now, we'll consider isometries without fixed points.

<u>Theorem</u> (Hsu 5.3.4): Two translations τ_v & τ_w are conjugate iff $|v| = |w|$.

<u>Proof</u>: $(\Rightarrow)$ If τ_v & τ_w are conjugate, then by Thm 5.2.3,

$|v| = \tau_v = \tau_w = |w|$.

$$\begin{tikzcd} \mathbb{E}^2 \arrow[r, "\tau_v"] \arrow[d, "R"'] & \mathbb{E}^2 \arrow[d, "R"] \\ \mathbb{E}^2 \arrow[r, "\tau_{R(v)}"] & \mathbb{E}^2 \end{tikzcd}$$

$(\Leftarrow)$ <u>Claim</u>: If R is a rotation, then

$R \tau_v R^{-1} = \tau_{R(v)}$. (Exercise).

Now, if $|v| = |w|$, then let R be a rotation about O such that $R(v) = w$.

$\square$

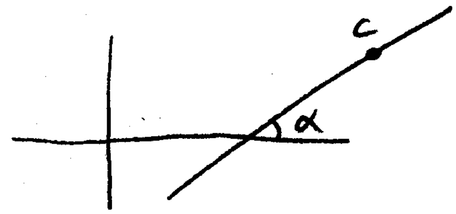
Theorem (Hsu 5.3.5): Two glide reflections γ_{AB}, γ_{CD} are conjugate iff $d(A,B) = d(C,D)$.

Proof: Exercise. (Hw 6) $\square$

Euclidean isometries, analytically

We can describe $f \in \mathrm{Isom}(\mathbb{E}^2)$ as a bijection $f : \mathbb{C} \to \mathbb{C}$:

* Translations: $\tau_c(z) = z + c$
* Rotations: $R_{c,\alpha}(z) = e^{i\alpha}(z-c) + c$
* Reflections: $\rho_m(z) = e^{2i\alpha}\,\overline{z-c} + c$



Summary (Stahl Thm 9.1.7): The isometries of $\mathbb{E}^2$ have the form $f(z) = e^{i\alpha} z + c$ (rotations, translations)

or $f(z) = e^{i\alpha}\bar{z} + c$ (reflections, glide-reflections).

Moreover, all maps of these forms are isometries.

Proof: Exercise (elementary verification).