9. The boundary of the hyperbolic plane

Throughout, we'll make the convention that $\frac{a\infty+b}{c\infty+d} = \frac{a}{c}$ and $\frac{a}{0} = \infty$.

The big idea of the last section can be summarized by the following:

<u>Theorem</u> (Hsu 6.3.1) The map $\mu : PGL_2(\mathbb{R}) \longrightarrow \mathrm{Isom}(\mathbb{H}^2)$,

$$A \longmapsto \left(\mu_A : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\right)$$

where
$$\mu_A(z) = \begin{cases} \dfrac{az+b}{cz+d} & \det A > 0 \\[2ex] \dfrac{a\bar{z}+b}{c\bar{z}+d} & \det A < 0 \end{cases}$$

is an isomorphism.

Now, consider the restriction $\mu_A\big|_{\hat{\mathbb{R}}}$. (Note that $\hat{\mathbb{R}} \subset \hat{\mathbb{C}}$.)

<u>Cor 6.3.2</u>: The Möbius map μ defines an action of $PGL_2(\mathbb{R})$ on $\hat{\mathbb{R}}$. For $x \in \hat{\mathbb{R}}$, $A \in PGL_2(\mathbb{R})$, this is given by

$$\mu_A(x) = \frac{ax+b}{cx+d}. \qquad \square$$

We can think of $\hat{\mathbb{R}}$ as the "boundary" of $\mathbb{H}^2$, because for any $x \in \mathbb{R}$, $\exists$ sequence $x_1, x_2, \ldots, x_n \longrightarrow x$ (in the Euclidean sense)

Def: View $\mathbb{H}^2 \subset \mathbb{C}$, and define the <u>ideal points</u> of $\mathbb{H}^2$ to be the set $\hat{\mathbb{R}}$. Denote $\overline{\mathbb{H}^2} = \mathbb{H}^2 \cup \hat{\mathbb{R}}$ (the <u>closure</u> of $\mathbb{H}^2$).

Def: Let G be a group acting on a set X.

G acts <u>transitively</u> on X if for any $x, y \in X$, $\exists\, g \in G$ such that $gx = y$.

G acts <u>n-transitively</u> on X if for any two n-tuples $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$, $x_i \neq x_j$, $y_i \neq y_j$, $i \neq j$, $\exists\, g \in G$ s.t. $(gx_1, \dots, gx_n) = (y_1, \dots, y_n)$.

Remark: If G acts n-transitively, it acts $(n-1)$-transitively.

Prop (Ilsn 6.3.5): Let G act on X. If $\exists$ an n-tuple $(a_1, \dots, a_n)$, $a_i \in X$ s.t. for <u>every</u> n-tuple $(x_1, \dots, x_n)$ of distinct elements, $\exists\, g \in G$ s.t. $(ga_1, \dots, ga_n) = (x_1, \dots, x_n)$, then G acts n-transitively on X.

Proof: Suppose $(ga_1, \dots, ga_n) = (x_1, \dots, x_n)$ and $(ha_1, \dots, ha_n) = (y_1, \dots, y_n)$.

Then
$$(hg^{-1}x_1, \dots, hg^{-1}x_n) = (ha_1, \dots, ha_n) = (y_1, \dots, y_n).$$

$$\begin{tikzcd}
(x_1, \dots, x_n) \arrow[rr, dashed, "hg^{-1}"] & & (y_1, \dots, y_n) \\
& (a_1, \dots, a_n) \arrow[ul, "g"] \arrow[ur, "h"'] &
\end{tikzcd}$$

$\square$

<u>Theorem</u> (Hsu 6.3.7): $PGL_2(\mathbb{R})$ acts 3-transitively on ideal points.

Furthermore, an element of $PGL_2(\mathbb{R})$ is completely determined by where it sends 0, 1, and ∞. (i.e, it's "sharply 3-transitive")

<u>Proof</u>: We'll prove that $PGL_2(\mathbb{R})$ acts 2-transitively. The 3-transitivity case will be HW.

For 2-transitively, it suffices to show (by Prop 6.3.5) that for any ideal points $a \neq b \in \hat{\mathbb{R}}$, $\exists A \in GL_2(\mathbb{R})$ such that

$$\mu_A(\infty) = a \quad \text{and} \quad \mu_A(0) = b.$$

<u>Claim</u>: $A = \begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}$ works:

$$\mu_A(\infty) = \frac{a\infty + b}{1\infty + 1} = \frac{a}{1} = a$$

(Note: $\det A = a - b \neq 0$)

$$\mu_A(0) = \frac{a \cdot 0 + b}{1 \cdot 0 + 1} = \frac{b}{1} = b. \qquad \square$$

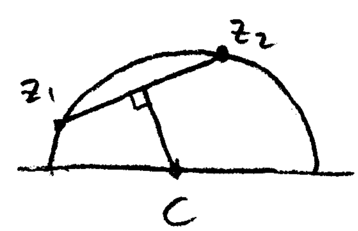
<u>The geometry of ideal points</u>:

Let's say that a geodesic "goes through ∞" iff it's vertical.

<u>Theorem</u> (Hsu 6.4.1): There is a unique geodesic between any two points z_1, z_2 of $\mathbb{H}^2$.

<u>Proof</u>: If $z_1 = \infty$, it's easy (vertical line through z_2)

Otherwise, consider the circle whose center is the intersection of the x-axis with the perpendicular bisector of $z_1 z_2$. $\square$



Ideal points can be thought of as "points on the horizon."

Theorem 6.4.1 says there's a 1-1 correspondence between ideal points & rays, or "lines of sight" from A.

<u>Theorem</u> (Hsu 6.4.2) Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a Möbius transformation. Then f is <u>continuous</u>. In particular, the restriction $f|_{\overline{\mathbb{H}^2}}$ is a continuous function $\overline{\mathbb{H}^2} \to \overline{\mathbb{H}^2}$.

<u>Proof</u> (sketch).

<u>Case</u> 1: $f(z) = \frac{az+b}{cz+d}$.

This is clearly continuous at z_0 if $z_0 \neq \infty$ and $f(z_0) \neq \infty$.

* If $f(z_0) = \infty$, then $cz_0 + d = 0$ & $N := az_0 + b \neq 0$.

If $z \approx z_0$ then $az+b \approx N$ and $cz+d \approx 0$, so $f(z) \to \infty$. ✓

* If $z_0 = \infty$, then as $z \to z_0$, $f(z) = \frac{az+b}{cz+d} \approx \frac{az}{cz} = \frac{a}{c} = \frac{a\infty + b}{c\infty + d}$

so $f(z) \to f(\infty)$. ✓

<u>Case</u> 2: $f(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$ is completely analogous. $\square$

<u>Def</u>: The <u>ends</u> of a geodesic m in $\mathbb{H}^2$ is $m \cap \hat{\mathbb{R}}$.

If m is vertical, then ∞ is one end.

Theorem 6.4.1 says that a hyperbolic geodesic is uniquely determined by its ends.

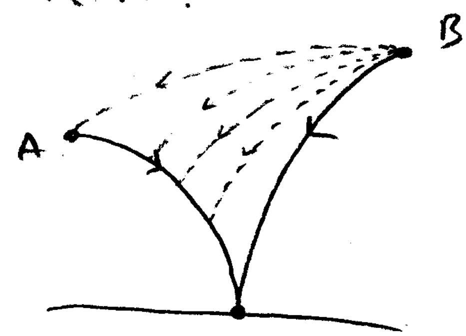
This, with Theorem 6.4.2 says:

<u>Cor</u>: (Hsu 6.4.4). Let $f \in \mathrm{Isom}(\mathbb{H}^2)$. Then f moves the geodesic m to n iff it moves the ends of m to the ends of n.

<u>Def</u>: Two geodesics <u>meet at infinity</u> if they intersect at an ideal point. We say that they meet <u>at an angle of 0</u>.

<u>Remark</u>: Since 2 points of $\overline{\mathbb{H}^2}$ determine a unique geodesic, if m & n meet at infinity, they do not meet elsewhere in $\overline{\mathbb{H}^2}$.

Two geodesics that meet at infinity are in a sense the "hyperbolic analog" of parallel lines.



"Flashlight tracing"

<u>Def</u>: An <u>ideal triangle</u> is a triangle whose vertices are all ideal points, or equivalently, all angles are 0.

As a corollary of Thm 6.3.7 & Cor 6.4.4, we have the following:

<u>Theorem</u> (Hsu 6.4.7): $PGL_2(\mathbb{R})$ acts transitively on ideal triangles, i.e., any two ideal triangles are congruent.

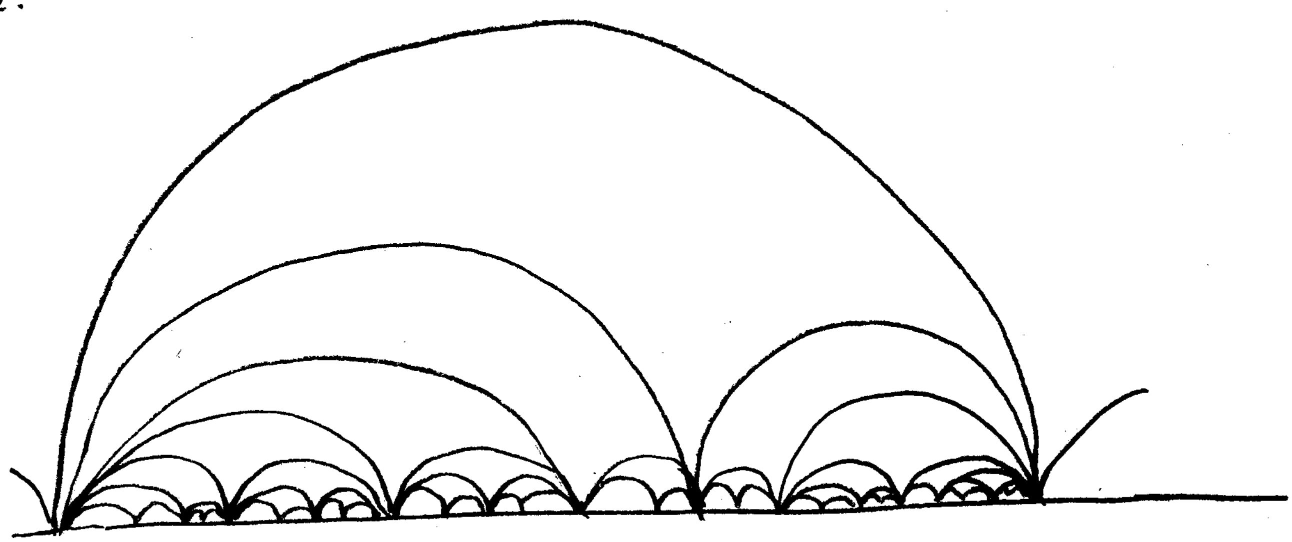
<u>Prop</u> (Hsu 6.4.8): Every ideal triangle has area π.

<u>Proof</u>: Exercise (HW).

$\boxed{6}$

By Thm 6.4.7, we can tile $\mathbb{H}^2$ with ideal triangles that are all congruent.

Example:



Note that the following is an ideal triangle ABC, where $A=0$, $B=1$, $C=\infty$: