10. Conjugacy classes of the Möbius group

Throughout, we'll consider $\mathrm{Isom}(\mathbb{H}^2)$, $PGL_2(\mathbb{R})$, and the set of Möbius transformations as the same.

Recall: Defining $\tau(A) = \dfrac{(\mathrm{tr}\, A)^2}{\det A}$, for any conjugate $f, g \in PGL_2(\mathbb{R})$:

(i) $\tau(f) = \tau(g)$

(ii) $\mathrm{sign}(\det f) = \mathrm{sign}(\det g)$

Def: If $\det f > 0$, then f is orientation-preserving.

If $\det f < 0$, then f is orientation-reversing.

Goal: Classify isometries by their fixed points and minimal motion sets in $\overline{\mathbb{H}^2}$, much like we did for $\mathrm{Isom}(\mathbb{E}^2)$.

Let $\mathrm{Fix}(f) = \{ z \in \overline{\mathbb{H}^2} : f(z) = z \}$.

First, we'll analyze orientation-preserving isometries. Denote this set as $\mathrm{Isom}^+(\mathbb{H}^2)$, or by $PGL_2^+(\mathbb{R})$. Here's a special case.

Prop (Hsu 6.5.1) Let $f \in \mathrm{Isom}^+(\mathbb{H}^2)$ with $f(\infty) = \infty$, $f \neq 1$. Choose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ s.t. $f = \mu_A$. Then $c = 0$, and we can choose some kA such that $d = 1$ and one of the following holds:

1. $a = 1$, $\tau(f) = 4$, $\mathrm{Fix}(f) = \{\infty\}$
2. $a \neq 1$, $\tau(f) > 4$, $\mathrm{Fix}(f) = \{\infty, x\}$ for some $x \in \mathbb{R}$.

Proof: Since $f(\infty) = \frac{a}{c}$, $f(\infty) = \infty \Rightarrow c = 0 \Rightarrow d \neq 0$. ✓

Clearly, we can scale A to kA s.t. $d = 1$, $f(z) = az + b$, $a \neq 0$.

Case 1: $a = 1$. $f(z) = z + b$, f fixes only ∞. $A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$

Now, $\tau(f) = \dfrac{(a+d)^2}{ad-bc} = \dfrac{(1+1)^2}{1 \cdot 1 - b \cdot 0} = 4$.

Case 2: $a \neq 1$: $A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, $f(z) = az + b$.

Fixed points: $f(z) = z \Rightarrow az + b = z \Rightarrow z = \dfrac{b}{1-a} \in \mathbb{R}$.

Thus, $\mathrm{Fix}(f) = \left\{ \infty, \dfrac{b}{1-a} \right\}$

Also, $\tau(f) = \dfrac{(a+d)^2}{ad-bc} = \dfrac{(a+1)^2}{a} > 4$ since $(a-1)^2 > 0$,
and $(a+1)^2 > 4a$. □

Now, we can drop the $f(\infty) = \infty$ condition and state the main theorem (orientation-preserving case)

Theorem (Hsn 6.5.2): Let $1 \neq f \in \mathrm{Isom}^+(\mathbb{H}^2)$. There are 3 cases:

1. $0 \leq \tau(f) < 4$, f fixes 1 point in $\mathbb{H}^2$. (f is of "elliptic type.")
2. $\tau(f) = 4$, f fixes one ideal point (f is of "parabolic type")
3. $\tau(f) > 4$, f fixes 2 ideal points (f is of "hyperbolic type")

Proof: Let $kA = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix form of f.

Note that Prop 6.5.1 establishes the theorem when $f(\infty) = \infty$.

Thus, assume that $f(\infty) \neq \infty$, i.e, that $c \neq 0$.

If $f(z) = z$, then

$$\frac{az+b}{cz+d} = z \implies az + b = cz^2 + dz \implies cz^2 + (d-a)z - b = 0$$

Let $D = (a-d)^2 + 4bc$.

By quadratic formula, there are 3 cases:

* $\underline{D<0}$: f fixes a point $z \in \mathbb{H}^2$ (note: $\bar{z} \notin \mathbb{H}^2$)
* $\underline{D=0}$: f fixes one point in $\mathbb{R}$ (ideal point)
* $\underline{D>0}$: f fixes two ideal points

Note: $D = (a-d)^2 + 4bc = (a+d)^2 - 4ad + 4bc = tr\,A - 4 \det A$.

So $D < 0 \iff \tau(f) < 4$

$D = 0 \iff \tau(f) = 4$

$D > 0 \iff \tau(f) > 4$.

Next: Classify isometries by type:

We'll see: Elliptic $\longleftrightarrow$ rotation

Parabolic $\longleftrightarrow$ "ideal" rotation

Hyperbolic $\longleftrightarrow$ "translation"

<u>Elliptic</u>: $\tau(f) < 4$, f fixes 1 point in $\mathbb{H}^2$.

<u>Theorem</u> (Hsu 6.5.3): Any isometry $f \in \mathrm{Isom}^+(\mathbb{H}^2)$ of elliptic type is conjugate to a unique transformation of the form

$$z \longmapsto \frac{(\cos\alpha)z + \sin\alpha}{(-\sin\alpha)z + \cos\alpha} \quad \text{for some } \alpha \in (0, \tfrac{\pi}{2}].$$

(i.e., $k\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \in PGL_2(\mathbb{R})$.

Moreover, two isometries f, g are conjugate iff $\tau(f) = \tau(g)$.

<u>Proof</u>: Suppose $f(z_0) = z_0 \in \mathbb{H}^2$.

Note that if $g \in \mathrm{Isom}(\mathbb{H}^2)$ satisfies $g(z_0) = i$, then $gfg^{-1}(i) = i$, so we may assume WLOG that $f(i) = i$.

Pick $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ s.t. $f = \mu_A$.

We may assume WLOG that
- $\det A = 1$
- $a > 0$, or $a = 0$ & $b > 0$.

Since $f(i) = i$, we have $\frac{ai+b}{ci+d} = i \Rightarrow ai + b = di - c$

$\Rightarrow a = d,\ b = -c \Rightarrow A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and $a^2 + b^2 = 1$.

Since $f \neq id$ and $\alpha \in (0, \frac{\pi}{2}]$, $\exists!\ \alpha \neq 0$ in $(-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $a = \cos\alpha$, $b = \sin\alpha$.

<u>Next</u>: We need to show that f is conjugate to one with $0 < \alpha \leq \frac{\pi}{2}$. (We just have $0 < |\alpha| \leq \frac{\pi}{2}$)

$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $\tau(F) = \dfrac{(\mathrm{tr}\, A)^2}{\det A} = 4a = 4\cos^2\alpha$

Observe that $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(-\alpha) & \sin(-\alpha) \\ -\sin(-\alpha) & \cos(-\alpha) \end{pmatrix}$. $\square$

<u>Parabolic</u> $\tau(f) = 4$, f fixes 1 ideal point

<u>Theorem</u> (Hsu 6.5.4): All isometries of parabolic type are conjugate.

<u>Proof</u>: Let $f \in \mathrm{Isom}^+(\mathbb{H}^2)$ be of parabolic type fixing $x \in \hat{\mathbb{R}}$.

Since $\mathrm{Isom}(\mathbb{H}^2)$ is transitive on $\hat{\mathbb{R}}$, $\exists\, g \in \mathrm{Isom}(\mathbb{H}^2)$ fixing ∞,

so assume WLOG that $f(\infty) = \infty$.

By Prop 6.5.1, $f = \mu_A$ for some $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $a \neq 0$.

Observe: $\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Thus, every isometry of parabolic type is conjugate to $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. $\square$

<u>Hyperbolic</u>: $\tau(f) = 4$, f fixes 2 ideal points.

<u>Theorem</u> (Hsu 6.5.5): Any isometry of hyperbolic type is conjugate to a transformation $z \mapsto az$, (i.e., $A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$), $a > 1$.

Also, f & g are conjugate iff $\tau(f) = \tau(g)$.

<u>Proof</u>: Let $f \in \mathrm{Isom}^+(\mathbb{H}^2)$ be of hyperbolic type, fixing $x_1, x_2 \in \hat{\mathbb{R}}$.

By 2-transitivity, we may assume WLOG that $\mathrm{Fix}(f) = \{0, \infty\}$.

[Reason: consider $g \in \mathrm{Isom}(\mathbb{H}^2)$ such that $g(x_1) = \infty$, $g(x_2) = 0$.]

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By Prop 6.5.1, $f = \mu_A$ for some $A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, $a \neq 1$.

Since $f(0) = 0$, $b = 0 \Rightarrow A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$.

Also, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} = a\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}$, so we may

assume $a > 1$. ✓

Remains to show that τ is a complete invariant.

Take $a, a' \in \mathbb{R}$ with $a > a' > 1$.

Consider the function $t(a) = \tau\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \frac{(a+1)^2}{a} = a + 2 + a^{-1}$.

This is an <u>increasing</u> function for $a > 1$ because $t'(a) = 1 - a^{-2} > 0$.

Thus, $t(a) > t(a')$. $\Rightarrow$ $\tau\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} > \tau\begin{pmatrix} a' & 0 \\ 0 & 1 \end{pmatrix}$. $\square$

<u>Orientation-reversing isometries</u>: $\det f < 0 \Rightarrow \tau(f) \leq 0$.

<u>Theorem</u> (Hsu 6.5.6-8:) Let $f \in \mathrm{Isom}(\mathbb{H}^2)$ be orientation-reversing. Then f fixes 2 points in $\hat{\mathbb{R}}$. Moreover, there are two cases:

1. $\tau(f) = 0$: f is a <u>hyperbolic reflection</u>, and all such reflections are conjugate.
2. $\tau(f) < 0$: f is conjugate to a transformation $z \mapsto -a\bar{z}$, $a > 1$ (i.e, $A = \begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}$). Moreover, f & g are conjugate iff $\tau(f) = \tau(g)$. This is a <u>hyperbolic glide reflection</u>.

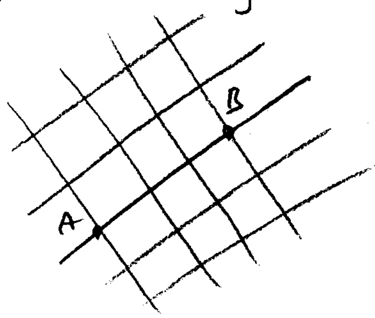
<u>Proof</u>: Series of exercises (HW)

Like the Euclidean case, a glide reflection is the product of a hyperbolic type isometry, & a commuting reflection, e.g., $z \mapsto az$ and $z \mapsto -\bar{z}$, or $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Visualizing hyperbolic isometries

We can describe Euclidean isometries by considering a coordinate system that is preserved by each type.

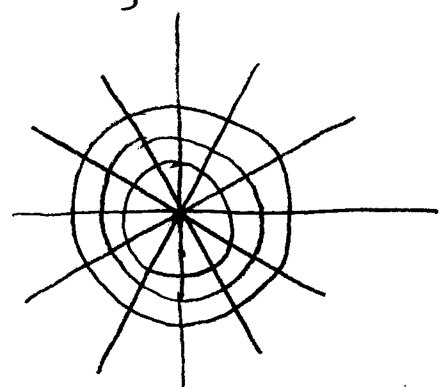
<u>Example</u>:



translation τ_{AB}

reflection ρ_{AB}

glide-reflection γ_{AB}



rotation $R_{C,\alpha}$

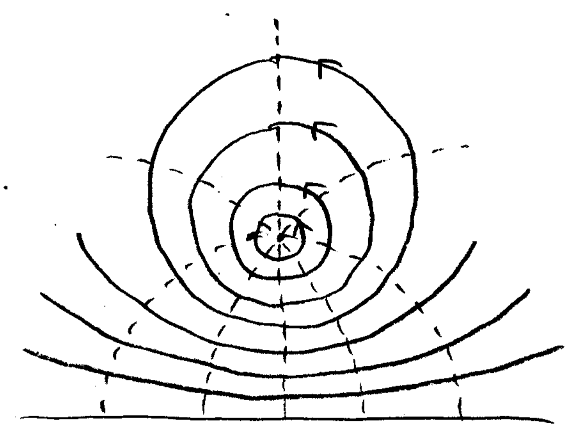
We'll do the same for hyperbolic isometries.

<u>Elliptic</u>: f is conjugate to $z \xmapsto{r_\alpha} \dfrac{(\cos\alpha)z + \sin\alpha}{-\sin\alpha\, z + \cos\alpha}$

$r_\alpha(i) = i$, so r_α preserves:

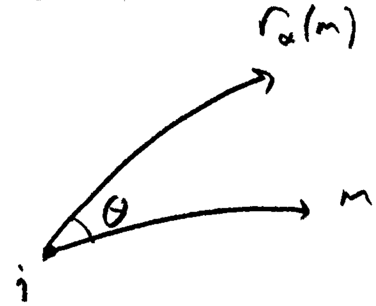
- all hyperbolic circles centered at i.
- all geodesics through i.

<u>Note</u>: These are orthogonal!



<u>Theorem</u> (Hsu 6.6.1): Let $\theta = 2\alpha$, and m a ray from i.

Then $r_\alpha(m)$ is a ray from i, at angle θ with m, i.e., r_α is a hyperbolic rotation of angle θ around i.



<u>Proof</u> (sketch). Need to show that $r_\alpha'(0,1) = \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}$,

(because if L is linear, then $L' = L$.)

or equivalently (easier!) $r_\alpha'(i) = e^{i\theta}$, where $r_\alpha : \mathbb{C} \to \mathbb{C}$.

By the quotient rule:

$$r_\alpha'(z) = \frac{-z\sin\alpha\cos\alpha + \cos^2\alpha + z\sin\alpha\cos\alpha + \sin^2\alpha}{(-z\sin\alpha + \cos\alpha)^2}$$

$$= (-z\sin\alpha + \cos\alpha)^{-2}$$

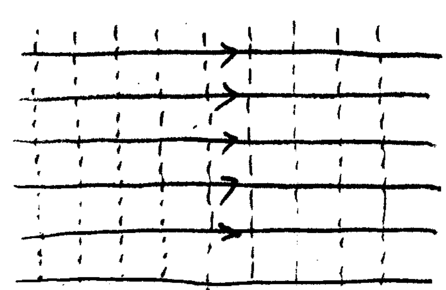
So $r_\alpha'(i) = (\cos\alpha - i\sin\alpha)^{-2} = (e^{-i\alpha})^{-2} = e^{i\theta}$. $\square$

Since $\tau(r_\alpha) = 4\cos^2\alpha = 2 + 2\cos\theta$, define the <u>rotation angle</u> of $f \in Isom(\mathbb{H}^2)$ of elliptic type to be the $\theta \in (-\pi, \pi]$ s.t. $\tau(f) = 2 + 2\theta$.

<u>Parabolic</u>: If $p \in Isom(\mathbb{H}^2)$ is parabolic with $p(z) = z+1$, the p preserves:

- all vertical geodesics
- curves of constant pos. imaginary part

These form an orthogonal coordinate system.



Note that the minimal motion is $\mu(p) = 0$

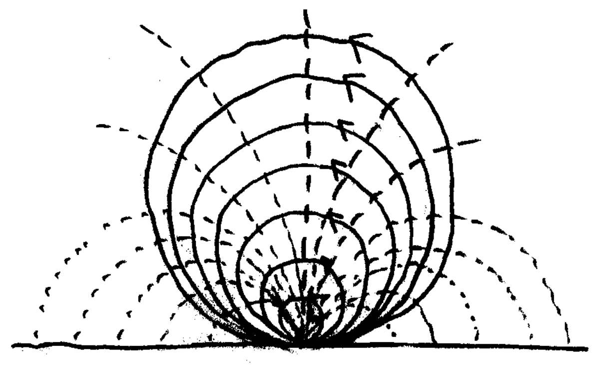
Thus, this holds for <u>any</u> parabolic isometry.

Now, consider a general isometry of parabolic type.

<u>Def</u>: A <u>horocycle</u> is a circle that intersects $\hat{\mathbb{R}}$ in one point.

Note that "horizontal" lines are horocycles; they go through ∞.

Consider the parabolic isometry fixing 0: $p(z) = \dfrac{z}{z+1}$.



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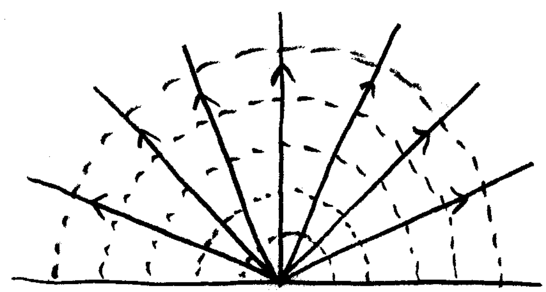
<u>Hyperbolic</u>: A dilation $t_a(z) = az$, $a > 1$ is hyperbolic.

The following curves are preserved by t_a:



- Geodesics centered at 0
- Euclidean rays through 0 (i.e., the locus of points a fixed distance from the y-axis.

Note that the only geodesic preserved by t_a is the y-axis.

<u>Theorem</u> (Hsu 6.6.4): Let $f \in \mathrm{Isom}(\mathbb{H}^2)$ be of hyperbolic type. The only geodesic preserved (setwise) by f is the geodesic connecting the two fixed (ideal) points of f. $\square$

Call this fixed geodesic the <u>axis</u> of f.

Revisiting $t_a \in \mathrm{Isom}(\mathbb{H}^2)$, consider a point $yi \in y$-axis.

Since $t_a(yi) = ayi$,

$$h(yi, t_a(yi)) = \ln \frac{ay}{y} = \ln a.$$

This means that any point on the <u>axis</u> of t_a is moved the same distance by t_a. Call this distance d the <u>translation distance</u> of t_a.

Note that $\tau(t_a) = \dfrac{(a+1)^2}{a} = a + a^{-1} + 2 = 2 + 2\cosh d$.

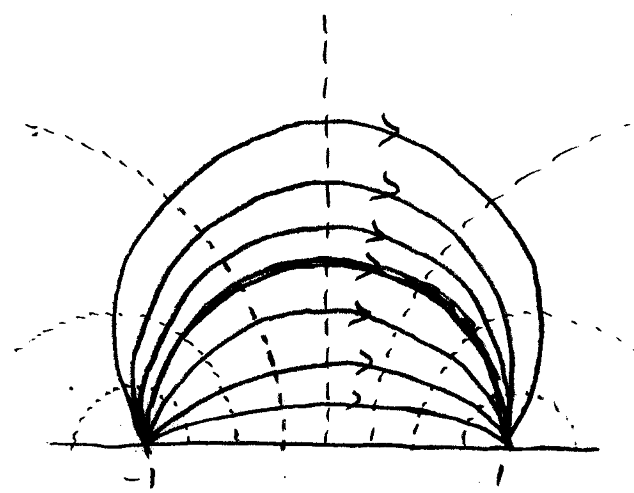
(Compare to $\tau(r_\alpha) = 2 + 2\cos\theta$).

Now, by Theorem 6.5.5, we have:

<u>Corollary</u> (Hsu 6.6.6): Two isometries of hyperbolic type are conjugate iff they have the same translation distance. $\square$

Now, consider a general isometry of hyperbolic type.

<u>Example</u>: Suppose that $\mathrm{Fix}(f) = \{-1, 1\} \subseteq \hat{\mathbb{R}}$.



<u>Note</u>: min'l motion of f is the translation distance, and the set of minimal motion is the axis.