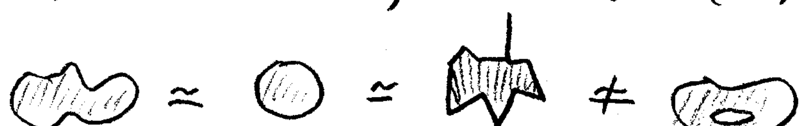
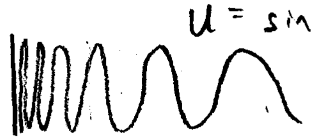
II. Some basic combinatorial topology

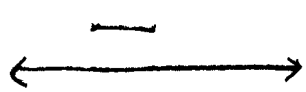
Topology is the study of properties preserved under continuous deformations of objects (e.g., stretching, tearing, twisting, but no tearing or gluing).

Def: Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be spaces. A homeomorphism $f: U \to V$ is a bijection such that f and f^{-1} are continuous. We say that U and V are homeomorphic, written $U \simeq V$.

Examples: $[0,1] \simeq [0,8] \not\simeq [0,8)$. , $(0,1) \simeq (-\infty, \infty) \simeq (0,\infty)$.

In $\mathbb{R}^2$, 

Sometimes things are less clear:  $U = \sin\frac{1}{x} \cup \{(0,y) : |y| < 1\}$

vs. $V = x\text{-axis} \cup \{(x,1) : |x| < 1\}$ 

or, Möbius strip  vs. annulus .

Field:	Objects	Structure-preserving maps	Equivalence
Algebra	Groups, rings, etc	Homomorphisms (Isomorphisms)	Isomorphic
Linear algebra	Vector spaces	Linear maps (bijective lin maps)	Isomorphic
Topology	Topological spaces	Continuous maps (homeomorphisms)	Homeomorphic
Diff Geo.	Manifolds	Smooth maps (diffeomorphism)	Diffeomorphic.

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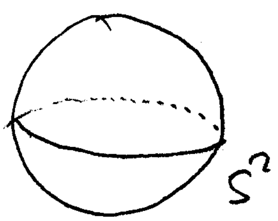
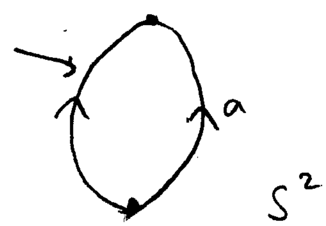
Def: A <u>surface</u> is a space $S \subseteq \mathbb{R}^n$ such that every $x \in S$ is contained in a neighborhood U that is homeomorphic to an open set in $\mathbb{R}^2$ (i.e, a 2-dimensional manifold).

There are several ways to describe a surface:

- As a subset of $\mathbb{R}^n$

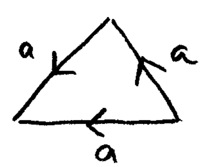


- As a "side-paired polygon."

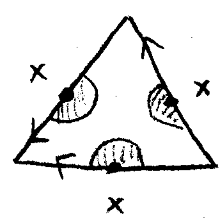


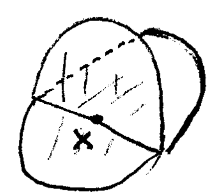
Remark: Not every such polygon leads to a valid surface.

Consider the "dunce cap":



Take a point x on the edge a, and a neighborhood of x:



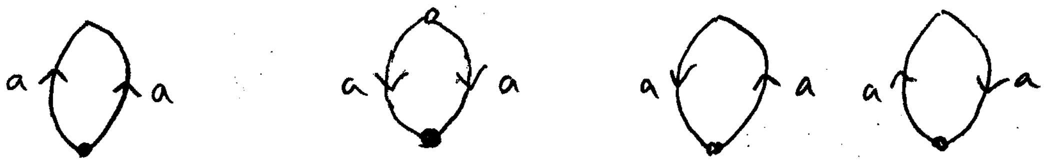


Not: homeomorphic to an open set in $\mathbb{R}^2$!

Def: A <u>side-paired polygon</u> consists of:

1. A polygon P with $2n$ sides
2. A partition of the sides of P into n (disjoint) pairs.
3. An identification of how to "glue" each pair.

Examples Side-paired 2-gons

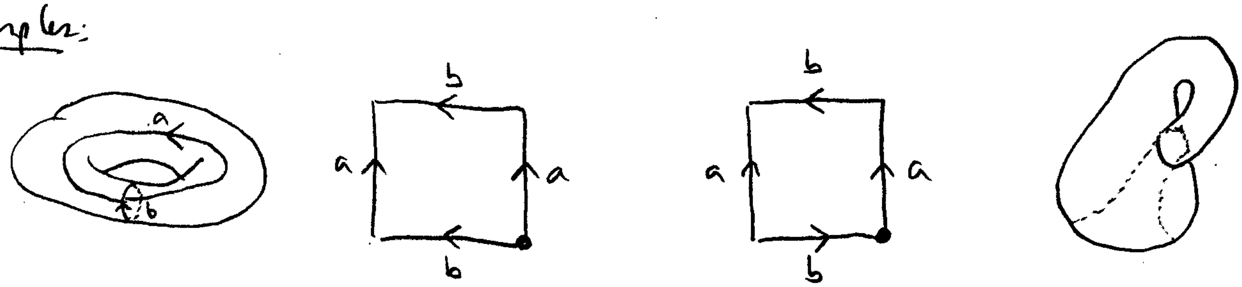


aa^{-1} $a^{-1}a$ aa $a^{-1}a^{-1}$

The first 2 are the same (homeomorphic), as are the second two.

Notation: Given a side-paired polygon with labels $a, b, c, \ldots$, choose a starting vertex and traverse the polygon counterclockwise. List the $2n$ edges in order, with an "inverse" for each edge that points clockwise.

Examples:



Torus: $T^2 \cong S^1 \times S^1$ $aba^{-1}b^{-1}$ Klein bottle: K, $aba^{-1}b$

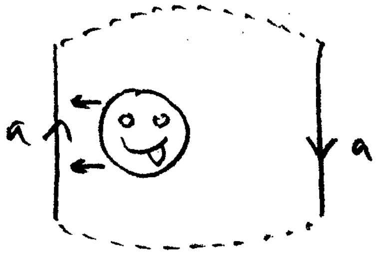
Such a label is called a defining relation for the surface.

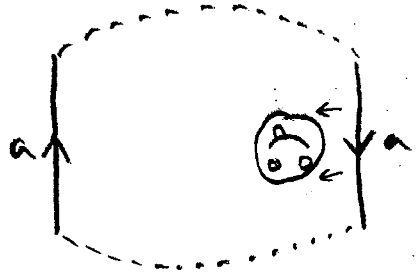
Def: A surface S is orientable if for any defining relation, each label a appears with its inverse.

Example: T^2 is orientable: $aba^{-1}b^{-1}$

K is nonorientable: $aba^{-1}b$ (no b^{-1}).

<u>Motivation</u>: Let S be nonorientable.

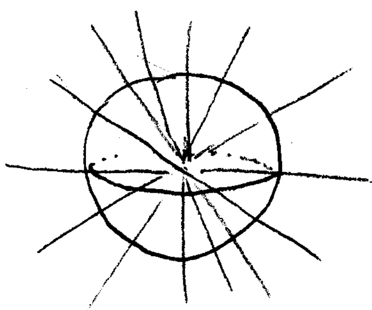


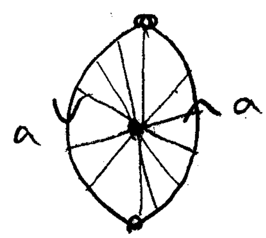


<u>Example</u>: The (real) projective plane $\mathbb{RP}^2$, or P^2, is the surface given by the side-paired polygon aa (i.e., ). This surface <u>cannot</u> be embedded in $\mathbb{R}^3$, but it <u>can</u> be embedded in $\mathbb{R}^4$!

It can be thought of as the sphere S^2 with antipodal points identified. Or equivalently, the set of all lines through $\vec{0}$ in $\mathbb{R}^3$.

P^2



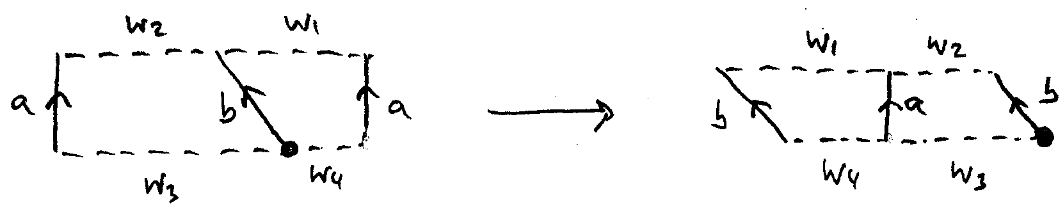


<u>Next goal</u>: Determine ways to alter a side-paired polygon without changing the surface (up to homeomorphism.)

We'll consider 3 <u>operations</u>. Both have natural <u>geometric</u> and <u>combinatorial</u> intrepretations.

Operations on side-paired polygons

1. Cut-and-paste:



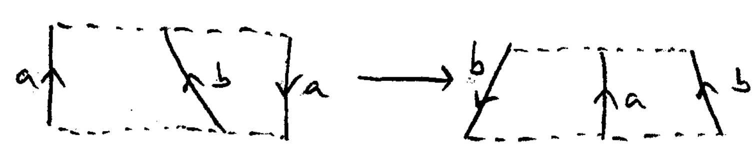
Combinatorially: $a w_1 w_2 a^{-1} w_3 w_4$

let $b = w_4 a w_1$

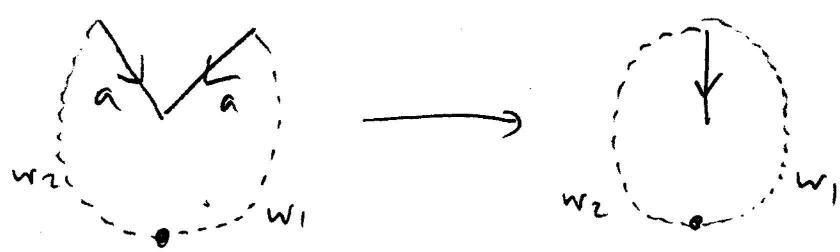
$\Rightarrow a = w_4^{-1} b w_1^{-1} \Rightarrow a^{-1} = w_1 b^{-1} w_4$

$$(w_4 a w_1) w_2 (a^{-1}) w_3 \xrightarrow{\text{eliminate } a} b\, w_2 (w_1 b w_4) w_3$$

Similarly, we can transform

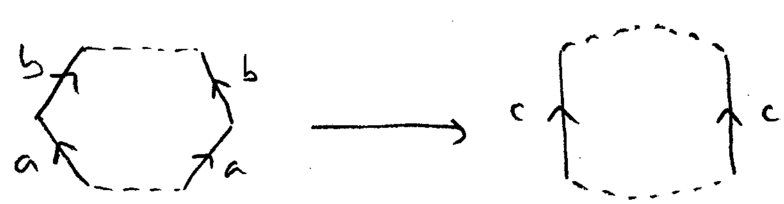


2. Cancel:



Combinatorially: $w_1 a a^{-1} w_2 \rightsquigarrow w_1 w_2$

3. Consolidate:



Combinatorially: $\cdots ab \cdots ab \cdots \rightsquigarrow \cdots c \cdots c \cdots$

<u>or</u> $\cdots ab \cdots b^{-1}a^{-1} \cdots \rightsquigarrow \cdots c \cdots c^{-1} \cdots$

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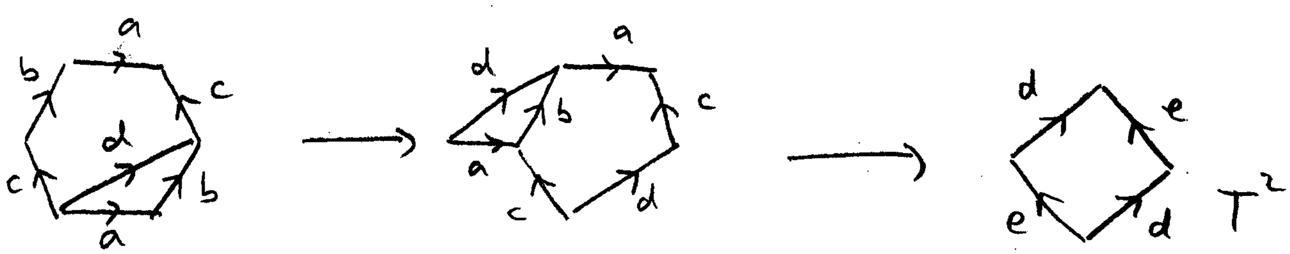
Theorem: (Hsu 7.2.4) Let S be a surface from a side-paired polygon P. Applying "cut & paste," "cancel," and "consolidate" to P yields a homeomorphic surface.

Proof: Later (we need some more tools, more rigorous definitions)

Example:



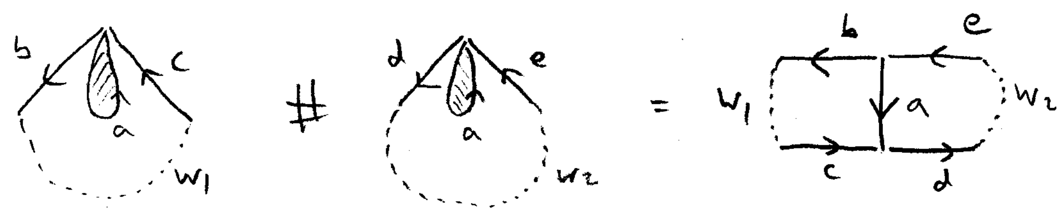
$$(ab)ca^{-1}(b^{-1})c^{-1} \xrightarrow{\text{Eliminate } b} (d)ca^{-1}(d^{-1}a)c^{-1}$$

Let $d = ab \Rightarrow b = a^{-1}d$ $\quad = d(ca^{-1})d^{-1}(ac^{-1}) \longrightarrow ded^{-1}e^{-1}$

$\Rightarrow b^{-1} = d^{-1}a$ $\quad$ set $e = ca^{-1}$ $\quad$ (Torus!)

Remark: We can do this visually, or entirely combinatorially.

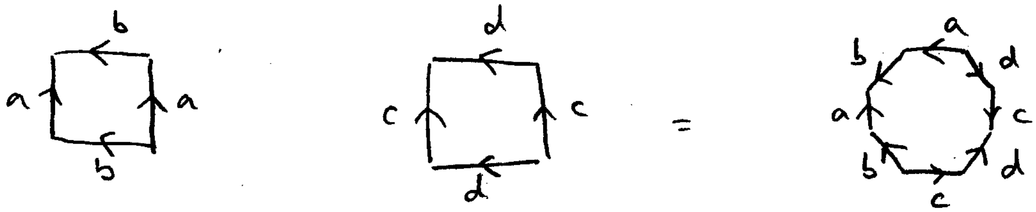
Def: Let S_1, S_2 be surfaces. The connected sum of S_1 & S_2, denoted $S_1 \# S_2$, is the surface obtained by removing a disk from each and identifying the boundary circle.



Combinatorially: $\quad bw_1c \quad \# \quad dw_2e \quad = (bw_1c)(dw_2e)$

Note: $a = bw_1c$, $\quad a^{-1} = dw_2e$

Example: $T^2 \# T^2$



$$aba^{-1}b^{-1} \quad \# \quad cdc^{-1}d^{-1} \quad = \quad aba^{-1}b^{-1}cdc^{-1}d^{-1}$$

Example: $P \# P^2 \cong K$ (Here, $P = \mathbb{RP}^2$, K = Klein bottle).

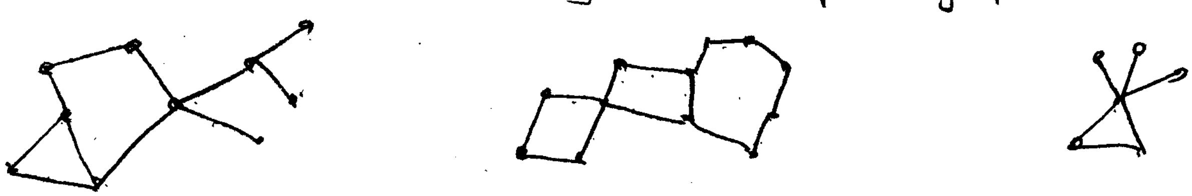
Proof: HW exercise.

Next goal: Find an invariant for surfaces up to homeomorphism.

Def: Let S be a surface obtained from side-paired polygons, with V vertices, E edges, and F faces. Then the Euler characteristic of S, denoted $\chi(S)$ is defined as

$$\chi(S) = V - E + F.$$

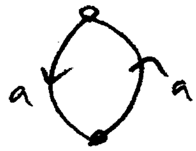
Remark: It is well-known in the field of graph theory that this quantity is 2, for any connected planar graph (on S^2).



$\chi(G) = 10 - 11 + 3 = 2$ $\qquad$ $\chi(G) = 12 - 14 + 4 = 2$ $\qquad$ $\chi(G) = 6 - 6 + 2 = 2$

Examples:

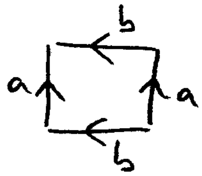
- S^2



2 vertices, 1 edge, 1 face.

$\chi(S^2) = V - E + F = 2 - 1 + 1 = 2$

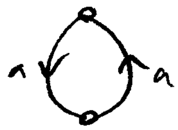
- T^2



1 vertex, 2 edges, 1 face

$\chi(T^2) = 1 - 2 + 1 = 0.$

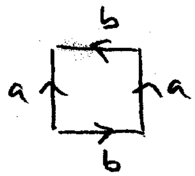
- P^2



1 vertex, 1 edge, 1 face

$\chi(P^2) = 1 - 1 + 1 = 1$

- K



1 vertex, 2 edges, 1 face

$\chi(P^2) = 1 - 2 + 1 = 0$

The Euler characteristic can be used to classify surfaces, regular polytopes, and much more. Details to follow, but here's a summary.

Theorem (Hsu 7.5.1): Every compact (closed & bounded) connected orientable surface is homeomorphic to the connected sum of tori. Every compact, connected non-orientable surface is homeomorphic to the connected sum of projected planes.

$$S_{or} \simeq T^2 \# T^2 \# \dots \# T^2$$

$$S_{non} \simeq P^2 \# P^2 \# \dots \# P^2$$

<u>Def</u>: A connected sum of g tori is a surface of <u>genus</u> g.

<u>Theorem</u> (Hsu 7.5.3): The Euler characteristic of a genus g surface is $2-2g$. The Euler characteristic of a connected sum of n projective planes is $2-n$.

<u>Remark</u>: It is well-founded to say $\chi(S^2)=2$.

<u>Theorem</u> (Hsu 7.5.4) If $S_1 \simeq S_2$, then $\chi(S_1)=\chi(S_2)$.

<u>Cor</u> (Hsu 7.5.5): Two compact connected surfaces S_1 & S_2 are homeomorphic iff.

- $\chi(S_1)=\chi(S_2)$
- They are either both orientable or both nonorientable.