12. Simplicial complexes

The goal of this section is to formalize what we mean by saying that, e.g.,  is a torus.

Def: Let X, Y be subsets of $\mathbb{R}^n$. A surjective map $q: X \to Y$ is a quotient map if $V \subseteq Y$ is open iff $q^{-1}(V) \subseteq X$ is open.

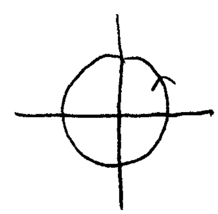
In this case, we call Y a quotient space of X.

Remark: $V \subseteq Y$ open $\Rightarrow$ $q^{-1}(V) \subseteq X$ open means q is continuous.
The converse, $U \subseteq X$ open $\Rightarrow$ $q(U) \subseteq Y$ open has further implications:

Since the open sets of X determine the open sets of Y (given q), we say Y has the quotient topology induced by X & q.

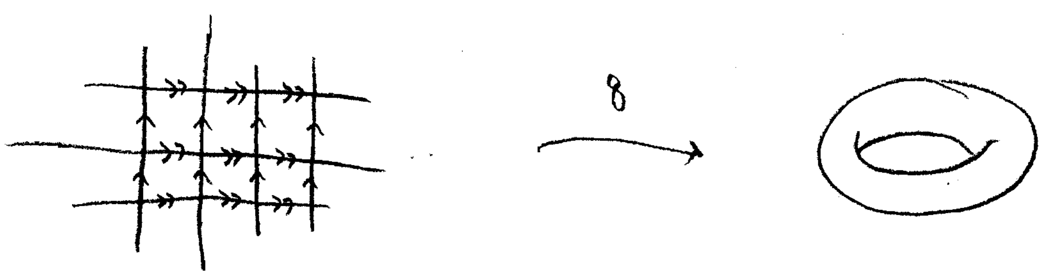
Examples:

(1). $q: \mathbb{R} \to S^1 \subseteq \mathbb{R}^2$, $\quad t \longmapsto (\cos t, \sin t)$



(2) $q: \mathbb{R}^2 \to T^2 \subseteq \mathbb{R}^3$,

$$(s,t) \longmapsto ((\cos s)(2+\cos t), (\sin s)(2+\cos t), \sin t)$$



The following is called the <u>universal property</u> of quotient maps.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "q"'] & & Z \\ & Y \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

<u>Theorem</u> (Hsu 8.12): Let $q: X \to Y$ be a quotient map, and $f: X \to Z$ any continuous map such that f is constant on $q^{-1}(y)$ for all $y \in Y$. Then the function $h: Y \to Z$, defined by $h(y) = f(q^{-1}(y))$ is continuous.

<u>Proof</u>: Since f is constant on $q^{-1}(y)$, it's well-defined and $f = h \circ q$.

To show h is continuous, take any open set $U \subseteq im(h) \subseteq Z$.

* We must show that $h^{-1}(U)$ is open in Y.

Now, $q^{-1}h^{-1}(U) = \{x \in X : q(x) \in h^{-1}(U)\}$ by def'n

$= \{x \in X : h(q(x)) \in U\}$ by applying h

$= \{x \in X : f(x) \in U\}$ because $f = h \circ q$

$= f^{-1}(U)$.

Since f is continuous, $f^{-1}(U) = q^{-1}h^{-1}(U)$ is open.

Since q is a quotient map, $q(q^{-1}h^{-1}(U)) = h^{-1}(U)$ is open. ✓

Thus, h is continuous. $\square$

In this case, we say that f factors through the quotient map

<u>Example</u>: Consider $q: \mathbb{R}^2 \to T^2$ as before.

$$\begin{tikzcd} \mathbb{R}^2 \arrow[rr, "f = \text{"color"}"] \arrow[dr, "q"'] & & \mathbb{R}^2 \\ & T^2 \arrow[ur, dashed, "q^{-1} \circ f = \text{"color"}"'] & \end{tikzcd}$$

Suppose we color each point of $\mathbb{R}^2$ via f in a continuous fashion, so that

$f(s + 2\pi k, t + 2\pi l) = f(s,t) \quad \forall k, l \in \mathbb{Z}$.

This will uniquely determine a way to color T^2 via $q^{-1} \circ f$.

Key reason: f is <u>constant</u> on $q^{-1}(z)$.

<u>Theorem</u> (Hsu 8.13): Let $X \subseteq \mathbb{R}^n$ be compact and $q: X \to Y \subseteq \mathbb{R}^m$ a continuous surjection. Then q is a quotient map.

<u>Proof</u>: It suffices to show that $V \subseteq Y$ is open $\Leftrightarrow$ $q^{-1}(V)$ is open.

($\Leftarrow$) Because q is continuous. ✓

($\Rightarrow$) Let $U \subseteq X$ be open, and $A := X \setminus U$ (which is closed).

Since X is compact, so is A.

Since q is continuous and A compact, $q(A)$ is compact.

Thus, $q(A)$ is closed $\Rightarrow$ $Y \setminus q(A) = q(U)$ is open. ✓ $\square$

<u>Remark</u>: This result holds more generally when X is <u>compact</u> and Y <u>Hausdorff</u>. (Exercise)

Next: Build up topological spaces using polygons & quotient maps.

Def: Define the <u>standard n-gon</u> to be a regular n-gon of side length 1 in $\mathbb{R}^2$.

Def: A <u>polygon complex</u> is a set $X \subseteq \mathbb{R}^n$ that is a finite disjoint union of polygons X_0, along with a continuous map $q : X_0 \to X$ such that

1. Restricted to any polygon $P \in X_0$, with interior P_0, $q : P_0 \to q(P_0)$ is a homeomorphism.
2. The images of the interiors of the polygons are pairwise disjoint.
3. For $P_1, P_2 \in X_0$, if E_i is an edge of P_i, then one of the following occurs:

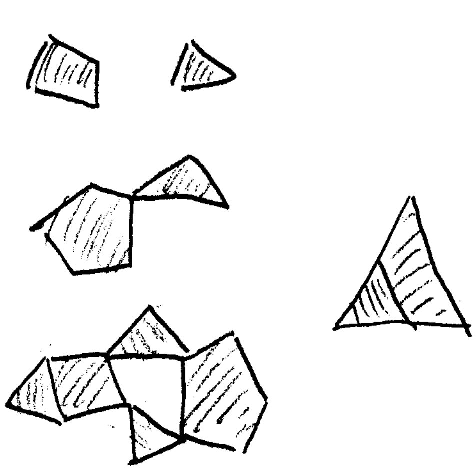
   (i) $q(E_1) = q(E_2) = \emptyset$

   (ii) $q(E_1) \cap q(E_2)$ is a point that is the image of an endpoint in E_1 & E_2.

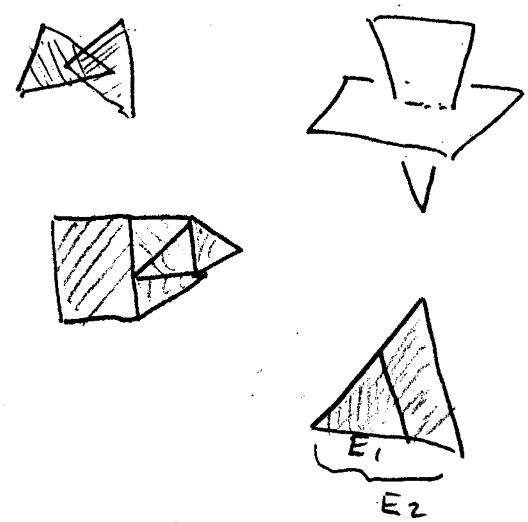
   (iii) $q(E_1) = q(E_2)$, and if $q_i = q|_{E_i}$, then $q_2^{-1} q_1$ is an <u>isometry</u> from $E_1 \to E_2$.

Remarks:

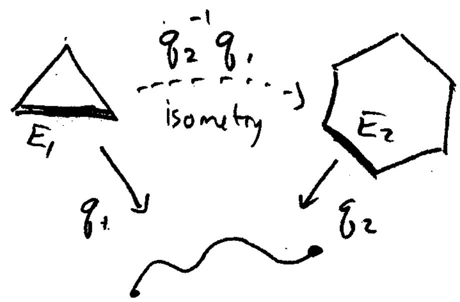
* Allowed



Not allowed



* Condition (iii) allows us to ignore self-homeomorphisms $[0,1] \to [0,1]$. (Think: Non-uniform stretching)

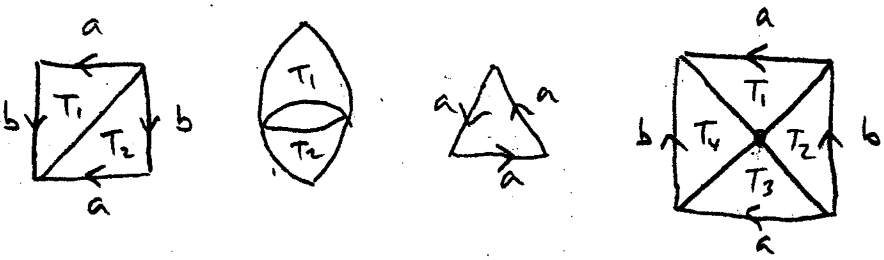


* Usually, we'll abuse notation and think of X as being "subdivided into polygons"

* By Theorem 8.1.3, q is a quotient map!

Def: A polygon complex made from triangles is called a triangle complex. A triangle complex in which any 2 triangles intersect in a single edge, vertex, or $\emptyset$, is called a simplicial 2-complex.

(6)

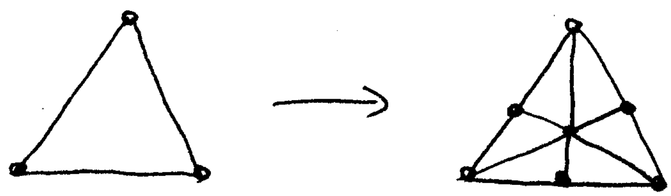
What this disallows: 

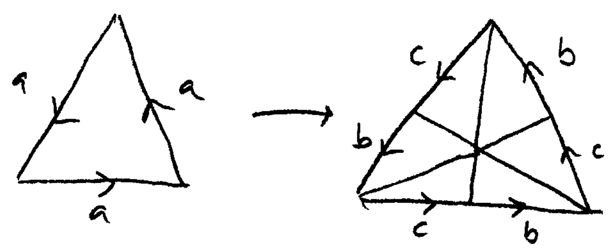
<u>Theorem</u> (Hsu 8.2.5): Any polygon complex is homeomorphic to a simplicial complex.

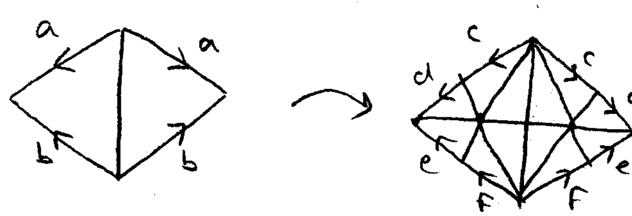
<u>Proof</u>: Subdivide! Let X be a polygon complex.

We may assume that X is a triangle complex (by subdivision).

Now, we will <u>barycentrically subdivide</u> each triangle.

Picture definition: 

We do this because it gets rid of the problem intersections, such as self-intersections, shown above: 

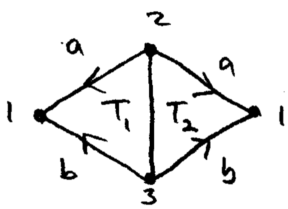
There is one potential problem case that we could be left with. But we can fix this by barycentrically subdividing again. 

(Note that the resulting complex need not be a surface!) $\square$

Working with simplicial complexes has the following main advantage:

* The <u>topology</u> of a simplicial 2-complex X can be <u>completely described combinatorially</u>: let V be the vertices. Then specifying which 3-subsets are triangles determines <u>everything</u>.

<u>Examples</u> (of non-simplicial 2-complexes):

* 

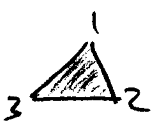
S^2

Vertices $V = \{1,2,3\}$

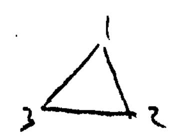
Triangles: $T_1 = \{1,2,3\}$, $T_2 = \{1,2,3\}$

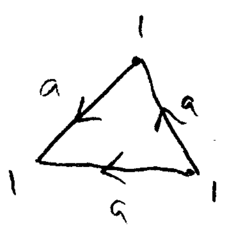
$T_1 \cap T_2$ is 3 edges.

What should happen: If $V = \{1,2,3\}$, $T = \{1,2,3\}$, then X should be a triangle (homeomorphic to a disk):



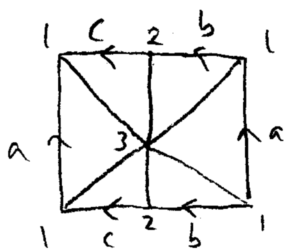
A similar example: $V = \{1,2,3\}$, $T = \emptyset$. Now, X is a triangle boundary, homeom. to S^1.



* 

Vertices $V = \{1,2,3\}$

Triangles: $T = \{1,1,1\}$

* 

Vertices $V = \{1,2,3\}$

Triangles: 6 copies of $T_i = \{1,2,3\}$, $i = 1, \ldots, 6$.

<u>Note</u>: $T_i \cap T_j$ = edge $\cup$ vertex $i \neq j$. Not allowed.

Def: An <u>abstract simplicial complex</u> is a finite set V and a collection of size-3 subsets (called triangles).

<u>Remarks:</u> * The open sets of a simplicial 2-complex are completely determined by the quotient map property.

* There is <u>at most</u> one actual simplicial 2-complex (up to homeomorphism) from an abstract one. In fact, there is <u>exactly</u> one:

<u>Theorem</u> (Hsn 8.2.7): Any abstract simplicial 2-complex is the combinatorial description of an actual simplicial 2-complex.

* We say that any abstract simplicial 2-complex can be <u>realized</u>.

<u>Proof:</u> Construct it!

First, note that the set

$\{(x,y,z) : x+y+z=1,\ x,y,z \geq 0\}$ is an equilaterial triangle in $\mathbb{R}^3$.

Let $V = \{1, 2, \ldots, n\}$ be the vertex set of an abstract simplicial 2-complex C. For each subset $\{i,j,k\} \subseteq V$ (i.e., "triangle"),

let $T_{ijk} := \{(x_1, \ldots, x_n) : x_i + x_j + x_k = 1,\ x_i, x_j, x_k \geq 0,\ x_\ell = 0\ \ \ell \notin \{i,j,k\}\}$.

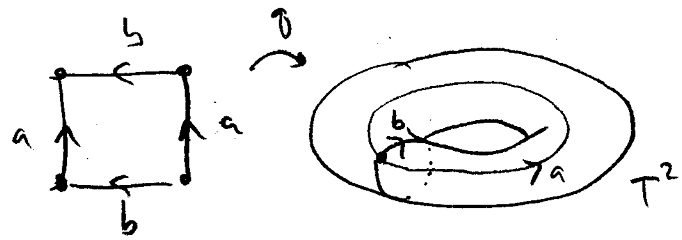
Now, define $X = \bigcup_{\{i,j,k\} \in C} T_{ijk}$. This is a simplicial complex whose combinatorial description is C (easy HW exercise). $\square$

We can now formally define what it means to glue edges.

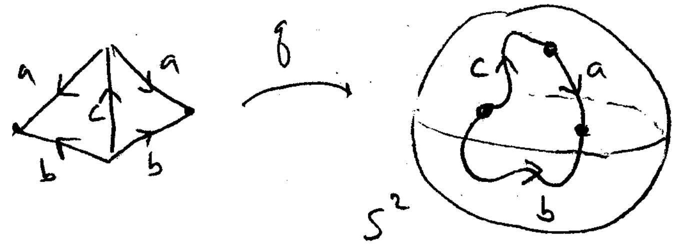
Recall that a polygon complex is a space X subdivided by polygons. Formally this is a disjoint union of polygons X_0 with a quotient map $q: X_0 \longrightarrow X$, with some natural restrictions on the images of edges.

We'll say that a vertex of a polygon complex X is a point $q(x_0) \in X$, where x_0 is a vertex of a polygon in X_0, and an edge of X is the image of an edge E_0 of some polygon.

Examples:



1 vertex
1 edge
1 face

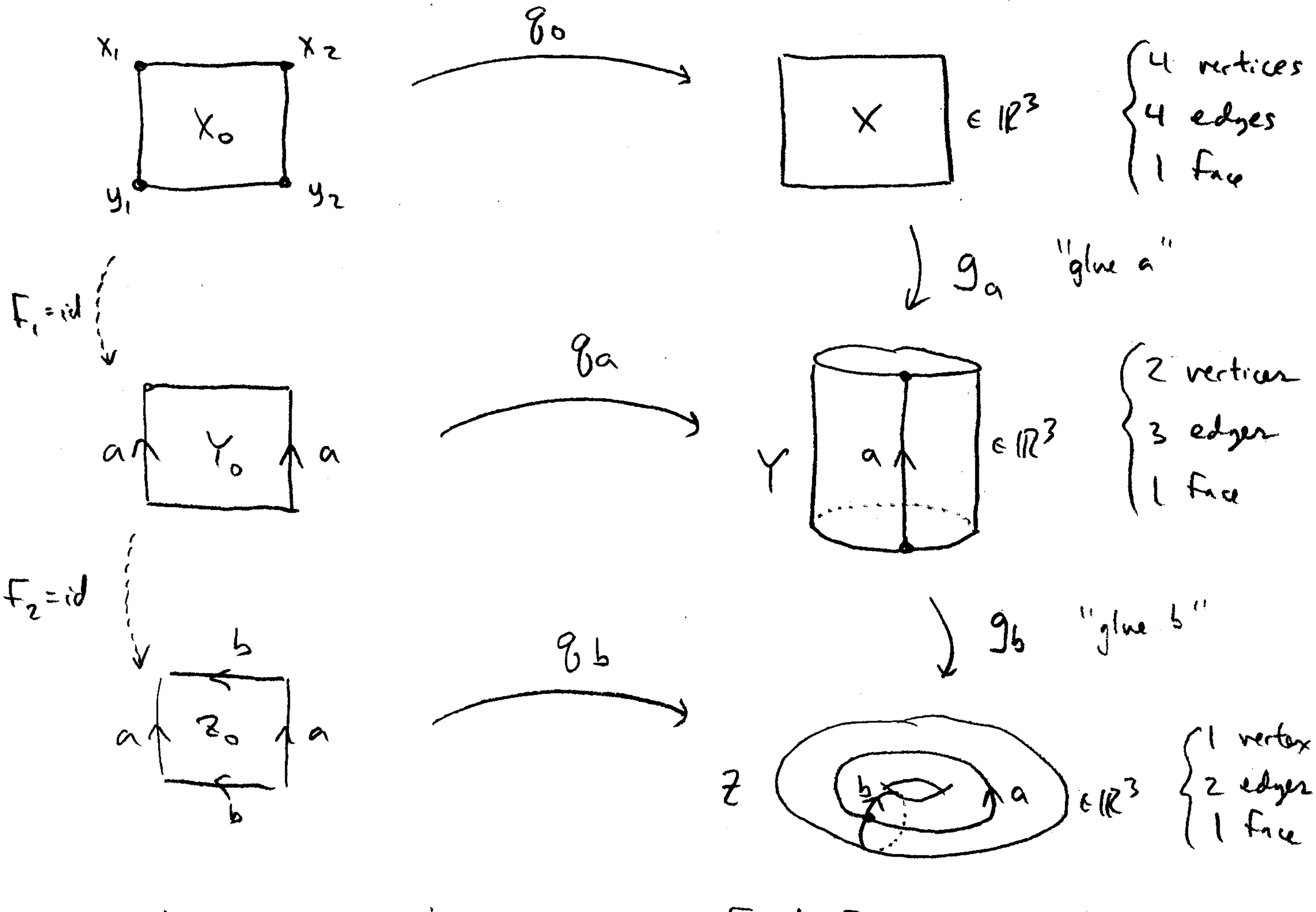


3 vertices
3 edges
2 faces

Def: Let X be a polygon complex. A gluing of X is another polygon complex Y, a subdivision of X, and a continuous surjective map $g: X \to Y$ (i.e., a quotient map) $g: X \to Y$ such that g maps:

1. Each vertex of X onto a vertex of Y
2. Each edge of X onto an edge of Y, isometrically.
3. The interior (i.e., face) of each polygon of X homeomorphically onto a polygon interior (i.e., face) of Y.

First, a motivating example: Start with a square, and glue opposite edges to get a torus.



Note that existence & uniqueness of f_1 & f_2 is guarenteed by the universal property of quotient maps.

<u>Remark</u>: The map g defines an <u>equivalence relation</u> $\sim$ on the edges of X. We say that g <u>glues</u> the edges of X according to $\sim$.

Also, note that by condition 2 (specifically, the "isometrically" part), there are only two ways to glue two edges together.

<u>Theorem</u> (Hsu 8.3.2): Let X be a polygon complex (technically, $q_X : X_0 \to X$) and let $\sim$ be any equivalence relation on the edges of X. Then there exists a subdivision of X, another polygon complex Y (technically, $q_Y : Y_0 \to Y$) and a continuous surjective (i.e, quotient) map $g : X \to Y$ such that g glues the edges of X according to $\sim$.

<u>Proof</u>: By subdividing into triangles, then barycentrically subdividing twice, we get a simplicial 2-complex for X, thus it suffices to consider this case.

After applying the equivalence relation $\sim$, we get (combinatorially) a triangle complex $q_Y : Y_0 \to Y$ realized in Euclidean space (by Theorem 8.2.7).

Note that $X_0 \to Y_0$ can be the identity map. We now have the following diagram:

$$\begin{tikzcd} X_0 \arrow[r, "q_x"] \arrow[d, "id"'] & X \arrow[d, dashed, "g"] \\ Y_0 \arrow[r, "q_y"'] & Y \end{tikzcd}$$

By the universal property of quotient maps, $\exists ! \, g : X \to Y$ that makes the diagram commute

(Note that this depends on our choice for q_y – there may be many of these such maps). $\square$

Cor: Any side-pairing of a polygon can be realized in $\mathbb{R}^n$.