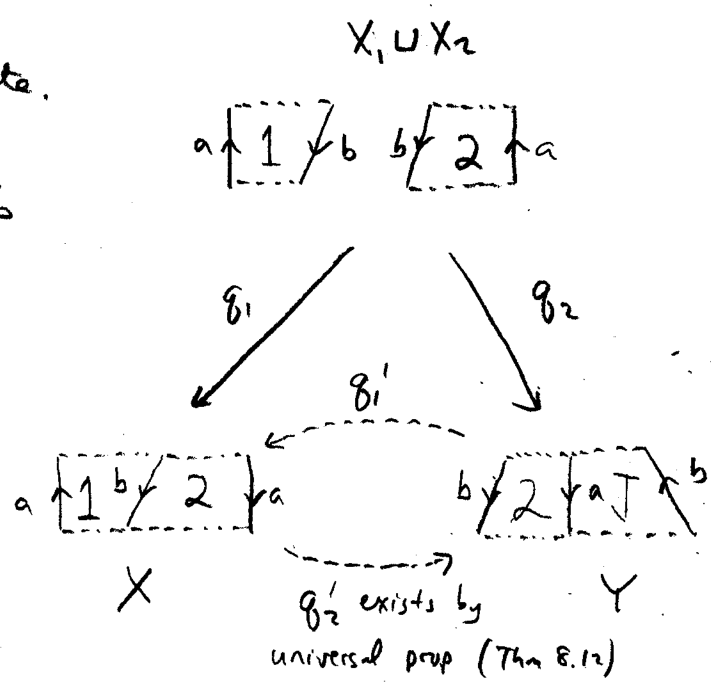
# 13. Classification of surfaces

The goal of this section is to prove that every surface is either the connected sum of tori or projective planes, and to use the Euler characteristic to classify them.

Along the way, we will need a few standard results that we don't have the tools to prove – algebraic topology is needed.

**Theorem** (Hsu 9.1.1): Let a space $X$ be obtained from a side-paired polygon $P$. If we cut-and-paste, cancel, or consolidate $P$, the resulting side-paired polygon yields a space homeomorphic to $X$.

Proof: First, consider cut-and-paste.




Suppose we cut $X$ along an edge $b$ to get two pieces, $X_1$ & $X_2$, then paste along edge $a$.

Let $Y$ be the end result.

Consider the two quotient maps

$q_1 : X_1 \sqcup X_2 \to X$

$q_2 : X_1 \sqcup X_2 \to Y$. We will construct a homeomorphism $X \to Y$.

By the universal property of quotient maps (Theorem 8.12), there exists a unique $q_2' : X \to Y$ s.t. $q_2 = q_1 \circ q_2'$ and a unique $q_1' : Y \to Y$ s.t. $q_1 = q_2 \circ q_1'$.

Claim: The (continuous) maps are inverses. [Note that by definition, this would imply that they are homeomorphisms, and so $X \simeq Y$.]

To show this, consider the following diagrams:

$$
\begin{tikzcd}
 & X_1 \sqcup X_2 \arrow[dl, "q_1"'] \arrow[d, "q_2"] \arrow[dr, "q_1"] & \\
X \arrow[r, "q_2'"] \arrow[rr, dashed, bend right, "\exists ! f"'] & Y \arrow[r, "q_1'"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X_1 \sqcup X_2 \arrow[dl, "q_2"'] \arrow[d, "q_1"] \arrow[dr, "q_2"] & \\
Y \arrow[r, "q_1'"] \arrow[rr, dashed, bend right, "\exists ! g"'] & X \arrow[r, "q_2'"] & Y
\end{tikzcd}
$$

By the universal property for quotient maps, $\exists ! \ f : X \to X$ such that $q_1 \circ f = q_1$ and a unique $g : Y \to Y$ such that $q_2 \circ g = q_2$.

Clearly, $f = id_X$ and $g = id_Y$. But $q_1' \circ q_2' : X \to X$ and $q_2' \circ q_1' : Y \to Y$ also satisfy this. Thus $q_1'$ and $q_2'$ are inverses of each other.

This shows that cut-and-paste preserves homeomorphism. Cancelation and consolidation are proven similarily (but easier – Exercise)

$\square$

**Theorem** (Hsu 9.1.2): Let $S$ be a space obtained from a side-paired polygon $P$. Then $S$ is a compact, connected surface.

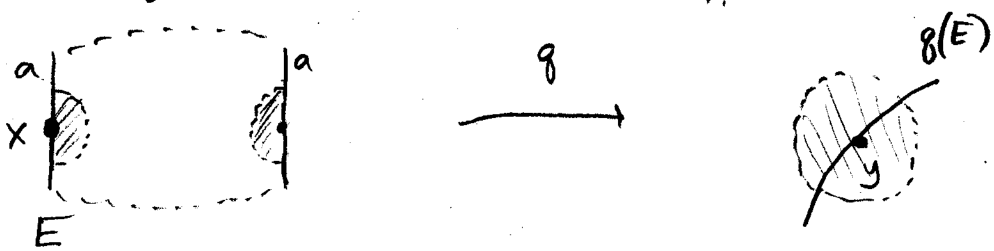
*Proof:* Since $P$ is compact and connected, and $q : P \to S$ (the quotient map) continuous, $q(P) = S$ is compact & connected.

We just need to show that $S$ is a surface.

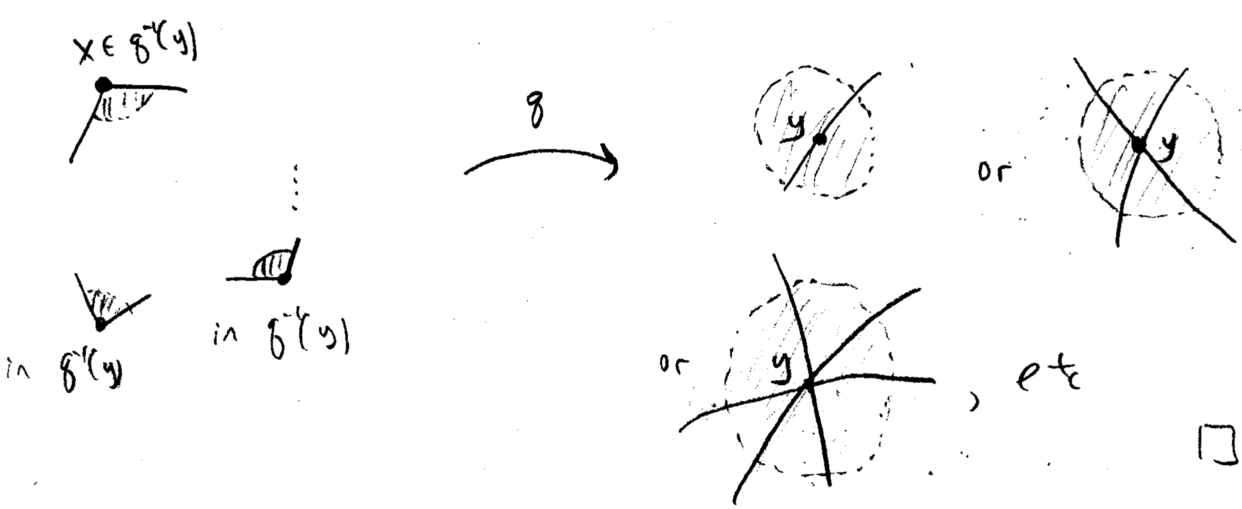
Take $y = q(x) \in S$. We need to show that $y$ contains a neighborhood homeomorphic to an open set in $\mathbb{R}^2$.

Case 1: $x \in$ interior$(P)$. This is clear: take any open $U \subseteq \text{int}(P)$ containing $x$. Since $q$ is a quotient map, $q(U)$ is open. ✓

Case 2: $x \in$ edge $E$ of $P$ (but not a vertex).




Case 3: $x \in$ vertex of $P$:




$\square$

The following theorem is well-known (but technical; best proven using algebraic topology).

<u>Theorem</u> (Hsu 9.1.3.): Every compact surface is homeomorphic to a simplicial 2-complex. $\square$

<u>Remarks</u>:

* This says that every compact surface can be triangulated.
* This seems obvious, but it <u>fails</u> in $\mathbb{R}^4$ (that is, for 4-manifolds). It does hold for 3-manifolds, though.

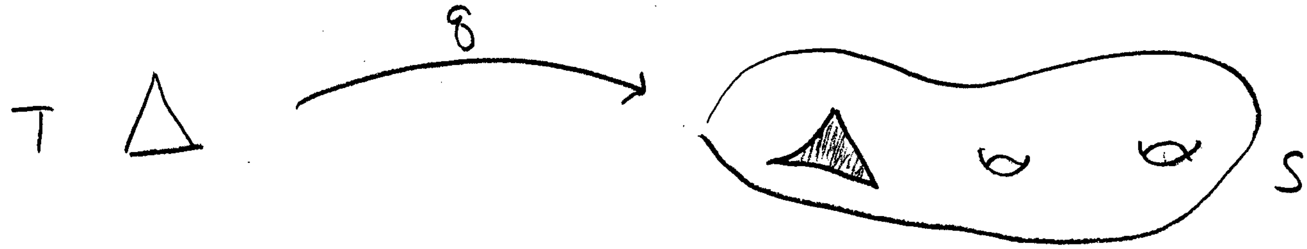
<u>Theorem</u> (Hsu 9.1.5): Let $S$ be a compact, connected surface that is also a simplicial 2-complex. Then there exists a polygon $P \subseteq \mathbb{R}^2$ such that $S$ is obtained from a side-pairing of $P$.

<u>Proof</u>: Given a surface $S$, we will construct a simplicial 2-complex $T$, called a <u>spanning tree of triangles</u>, with a gluing map $g: T \longrightarrow S$.

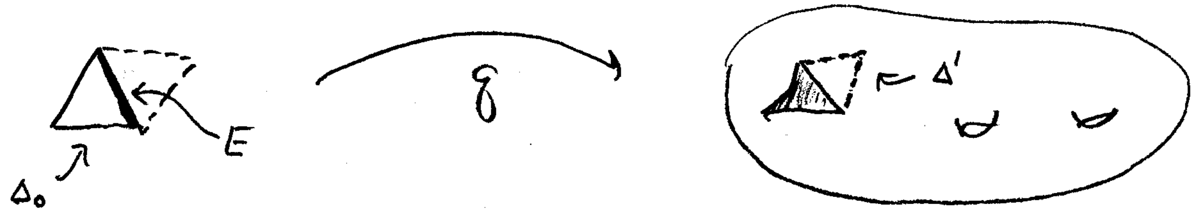
Recall that we can think of $S$ as a triangulized surface:

Step 1: Pick a triangle $\Delta' \subseteq S$, and let $T$ be a single triangle $\Delta$.

Define $q: T \to S$ by sending $q: \Delta \mapsto \Delta'$ homeomorphically (equivalently, as in the restriction of the polygon complex map to a single triangle.

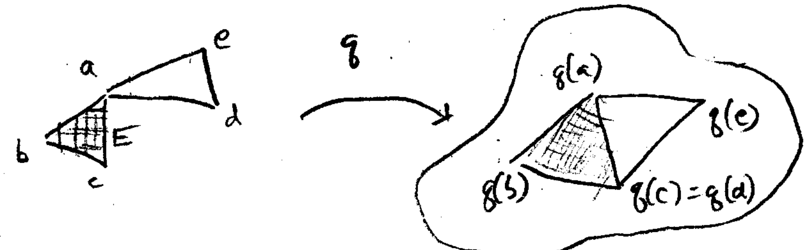



Step 2: While there are still triangles not in the image of $q: T \to S$, expand $T$ and $q$ as follows:




* Choose boundary edge $E$ of $T$. Say it's on the triangle $\Delta_0$. Let $\Delta'$ be the triangle of $S$ to which $q(\Delta_0)$ is attached, at the edge $q(E)$
* If $\Delta'$ is not in $q(T)$, then add a triangle $\Delta$ to $T$, attached to $\Delta_0$ at $E$.

Note: $\Delta'$ may be in $q(T)$:




Extend $q$ to $\Delta$, by sending $\Delta \to \Delta'$ if necessary.

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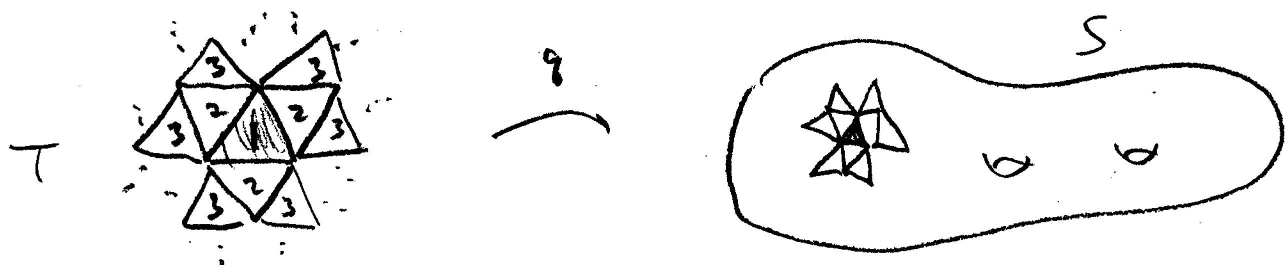
Remark: We can perform the process "layer-by-layer" if we wish, building outward from the initial triangle.



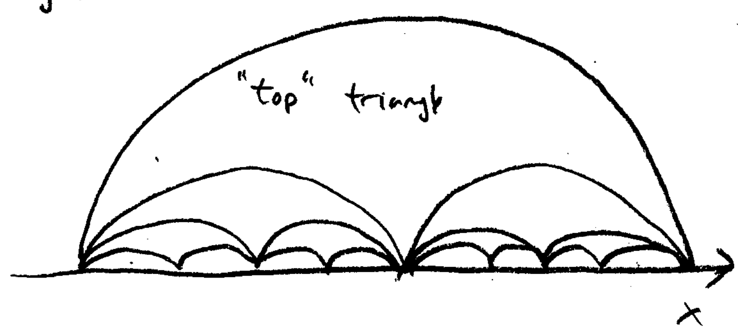

It is easy to see now that this construction will eventually terminate, and include all triangles in S.

Next: We must show that our polygon constructed by triangles can be embedded in $\mathbb{R}^2$.

Consider the following tiling of triangles in $\hat{\mathbb{H}}^2$:

Pick a "boundary" triangle $\bar{\Delta}$ in T

Map it to the "top" triangle shown at right.




Now, $\bar{\Delta}$ is adjacent to at most 2 triangles in T.

For each one of those, "hang" it off the corresponding edge.

Repeat. (Note: there may be several choices of edges to hang off of. Pick one.)

When this process stops, we have a connected region formed by "ideal triangles" in $\mathbb{R}^2$.

Consider the boundary of this region. It will have say, $m$ edges, and each edge is identified with another in $S$.

Thus, we can now construct a regular $m$-gon $P \in \mathbb{R}^2$ ($m$ is even) and we have a prescribed way to glue edges to yield the surface $S$. $\square$

Recall that we wish to prove the following:

Theorem (Hsu 9.2.1): Every compact, connected orientable surface is homeomorphic to $T^2 \# \cdots \# T^2$ ($g \geq 0$ copies). Every compact connected non-orientable surface is homeomorphic to $P^2 \# \cdots \# P^2$ ($n \geq 1$ copies).

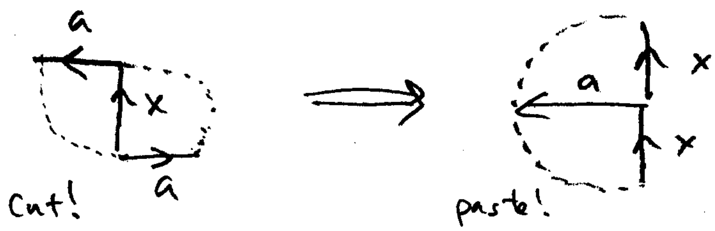
From what we've shown, it suffices to prove the following:

Theorem (Hsu 9.2.2): Let $S$ be the surface obtained from a side-paired polygon $P$. If $P$ has only orientation-preserving identifications (e.g., $a$ with $a^{-1}$), then $S$ is a sum of tori. Otherwise, it is a sum of projective planes.

**Proof:** We will cut-and-paste and cancel in 5 steps.

**Step 1.** Group $aa$ pairs (i.e., projective planes).

Picture:

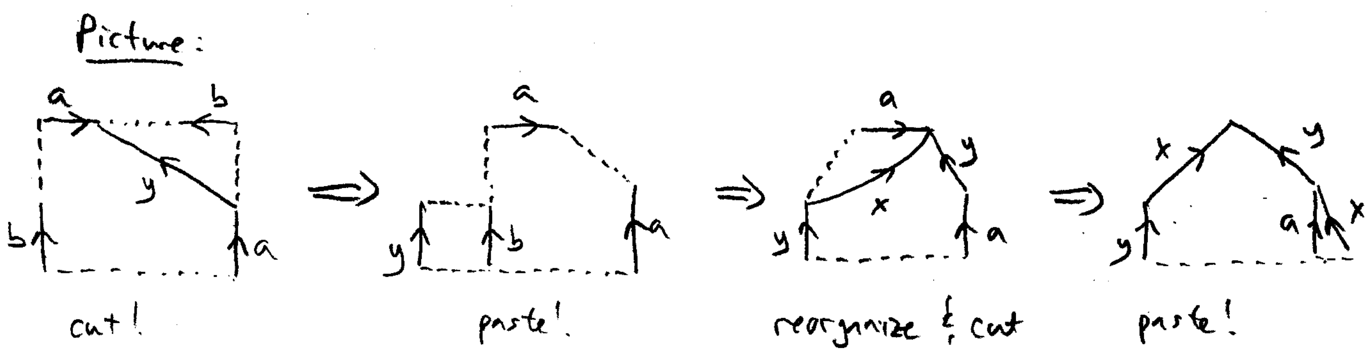



Combinatorially: $\ldots a \ldots\ldots a \ldots \Longrightarrow \ldots\ldots (aa) \ldots\ldots$

Remarks:
- This never destroys any other, e.g., $bb$ pairs
- Repeat this until there are no more separated $xx$ pairs remaining. Clearly, this will terminate.
- This could change $bb$ pairs to $b^{-1}b^{-1}$ pairs, but this is harmless. Also, we need to group $a^{-1}a^{-1}$ pairs too.

**Step 2.** Group $aba^{-1}b^{-1}$ subwords (i.e., tori).

Picture:




Combinatorially:

$\ldots a \ldots b \ldots a^{-1} \ldots b^{-1} \Longrightarrow$ substitute $y$ for $b$ $\Longrightarrow$ substitute $x$ for $a$ $\Longrightarrow \ldots\ldots xyx^{-1}y^{-1} \ldots$

<u>Remarks</u>: Repeat this until no such subwords remain ungrouped.

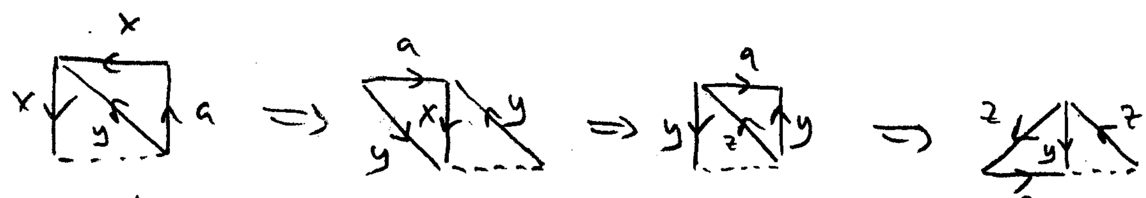
We now have, e.g., $\ldots aba^{-1}b^{-1} \ldots cc \ldots dd \ldots efe^{-1}fe \ldots gg \ldots$

<u>Step 3</u>: Put everything <u>not</u> in a $P^2$ or $T^2$ at the end.

Call such edges "loose edges."

Let $a$ be a loose edge. We can "pass it through" a $\boxed{P^2}$

<u>Picture</u>:

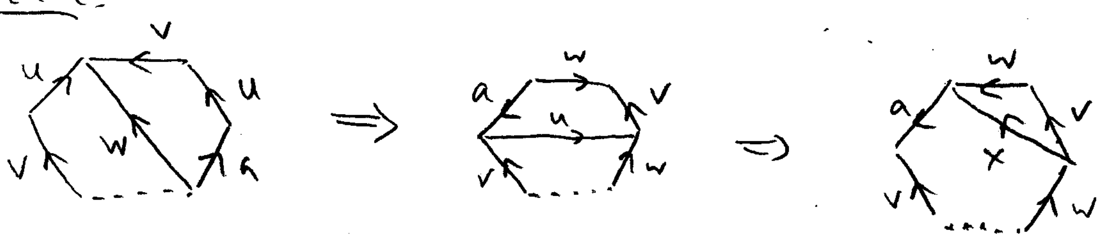



cut! paste rearrange & cut paste

<u>Combinatorially</u>: $\ldots axx \ldots \Rightarrow \ldots ya^{-1}y \ldots \Rightarrow \ldots zza \ldots$

Similarly, we can pass loose edges through a $\boxed{T^2}$:

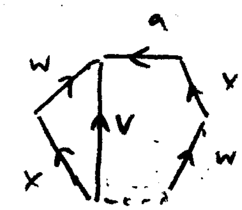
<u>Picture</u>:



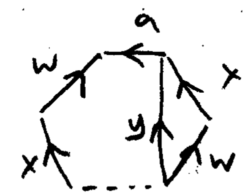

cut! paste! eliminate $u$, then cut again

$\ldots auvu^{-1}v^{-1} \ldots \Rightarrow \ldots wvw^{-1}av^{-1} \ldots$

(substitute $w$ for $u$)

$\Rightarrow$ 

paste to eliminate $v$

$\Rightarrow$ 

cut!

$\Rightarrow$ 

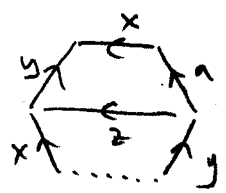
paste to eliminate $w$

$\cdots\; w x a w^{-1} x \cdots$

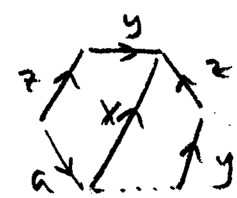
(substitute $x$ for $v$)

$\cdots\; y a x y^{-1} x^{-1} \cdots$

(substitute $y$ for $w$)

$\Rightarrow$ 

cut!

$\Rightarrow$ 

paste to eliminate $x$

$\cdots\; y z y^{-1} z^{-1} a \cdots$

substitute $z$ for $x$.

Remarks: Do this until all of the loose edges are at the end of the defining relation.

Step 4: Cancel the loose edges.

Note that if $a$ is a loose edge, so is $a^{-1}$ (by step 1).

Also, no two pairs of loose ends are intertwined, i.e,

$a \cdots b \cdots b^{-1} \cdots a^{-1}$ can happen but not $a \cdots b \cdots a^{-1} \cdots b^{-1}$

This is by step 2.

Thus, the string of loose edges must have some $xx^{-1}$ substring. Cancel this string, and then repeat.

Eventually, we will eliminate all loose edges in this manner.

Step 5: Eliminate tori.

We now have a defining relation built up of "blocks" of either projective planes, $(aa)$ or tori $(bcb^{-1}c^{-1})$.

For example: $(aa)(bcb^{-1}c^{-1})(ded^{-1}e^{-1})(ff)(gg)$.

$$P^2 \# T^2 \# T^2 \# P^2 \# P^2.$$

We can read off the surface as a connected sum of $P^2$ and $T^2$.

Recall (HW exercise): $P^2 \# T^2 \simeq P^2 \# P^2 \# P^2 \simeq T^2 \# P^2$.

Thus, if there is a $P^2$ in our sum, we can the replace every instance of $T^2$ with $P^2 \# P^2$.

We now have written our surface as either

$$\underbrace{T^2 \# \cdots \# T^2}_{g \geq 0} \quad \text{or as} \quad \underbrace{P^2 \# \cdots \# P^2}_{n \geq 1}. \qquad \square$$

## The Euler characteristic.

Now that we have classified all surfaces, we will assign them an *invariant*.

*Def.* A common refinement of two triangulations, $T_1$ & $T_2$ of $S$ is a third triangulation $T_3$ that can be obtained by repeated barycentric subdivision of the original triangulations.

$$\begin{tikzcd} (S, T_1) \arrow[dr] & & (S, T_2) \arrow[dl] \\ & (S, T_3) & \end{tikzcd}$$

*Theorem 10.1.2:* Any two triangulations have a common refinement. $\square$

This is one of those "hard" results for which we need algebraic topology.

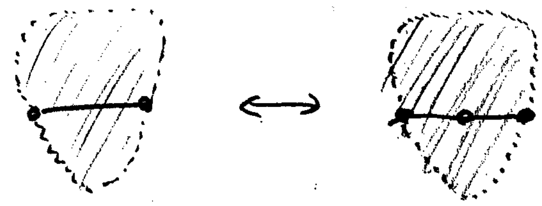
Additionally, we may need to "wiggle" the actual triangulations.

A more "formal" statement can be made on the level of *abstract* simplicial 2-complexes.

*Def.* The three *fundamental moves* on a polygon complex are:

1. Subdivide an edge

(Reverse: smooth a vertex)



2. Add a hanging edge inside a polygon

(Reverse: remove a hanging edge)

3. Subdivide a polygon into 2 polygons

(Reverse: Merge 2 polygons along a common edge)

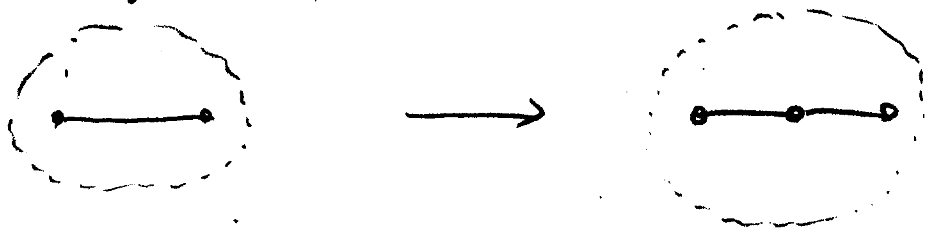
Theorem (Hsu 10.1.4): Barycentric subdivision can be achieved by performing only fundamental moves.

Proof: Exercise (easy).

Theorem (Hsu 10.1.5): Let S be a triangulated surface. Then changing the triangulation does not change the Euler characteristic of the surface.

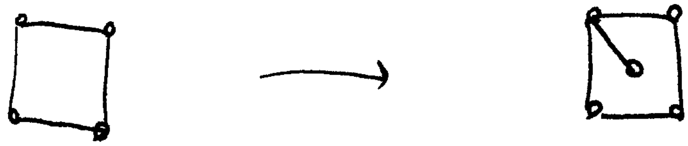
Proof: It suffices to check our 3 fundamental moves:

1. Subdivide edge



$\chi(S)$ $\quad V - E + F \longmapsto (V+1) - (E+1) + F$ ✓

2. Add edge



$\chi(S)$ $\quad V - E + F \longmapsto (V+1) - (E+1) + F$ ✓

3. Split a polygon

$\chi(S)$ $\quad V - E + F \longmapsto V - (E+1) + (F+1)$ ✓

$\square$

Since any 2 triangulations have a common refinement, their Euler characteristic must be the same.

Cor: If $S_1$ & $S_2$ are homemorphic triangulated surfaces, then $\chi(S_1) = \chi(S_2)$ $\square$

# Classification of surfaces (summary)

Thus far, we've shown the following:

Theorem (Hsu 10.2.1): Every compact, connected orientable surface is homeomorphic to $T^2 \# \cdots \# T^2$ ($g \geq 0$ copies).

Every compact, connected, non-orientable surface is homeomorphic to $P^2 \# \cdots \# P^2$.

What's left: Show that $T^2 \# \cdots \# T^2 \not\cong P^2 \# \cdots \# P^2$

and $\underbrace{T^2 \# \cdots \# T^2}_{k \text{ copies}} \not\cong \underbrace{T^2 \# \cdots \# T^2}_{\ell \text{ copies}}$ for $k \neq \ell$.

This is relatively easy using algebraic topology, but it takes some machinery to build up.

Theorem 10.2.2: No orientable surface is homeomorphic to a nonorientable surface.

Proof: Algebraic topology. $\square$