5 Nonlinear models of interactions

Predator-prey model

Let P_t = size of prey, Q_t = size of predator.

$$\Delta P = F(P,Q), \qquad \Delta Q = G(P,Q).$$

Prey, without predators: $\Delta P = rP(1-P/k)$

Predators, without prey: $\Delta Q = -uQ$ $\qquad$ u = per capita death rate, so $0<u<1$.

One model: $\begin{cases} \Delta P = rP(1-P/K) - sPQ \\ \Delta Q = -uQ + vPQ \end{cases}$ "mass-action terms"

$$\Rightarrow \begin{cases} P_{t+1} = P_t\left(1+r(1-P_t/K)\right) - sP_tQ_t \\ Q_{t+1} = (1-u)Q_t + vP_tQ_t \end{cases} \qquad r,s,u,v,K>0, \ u<1.$$

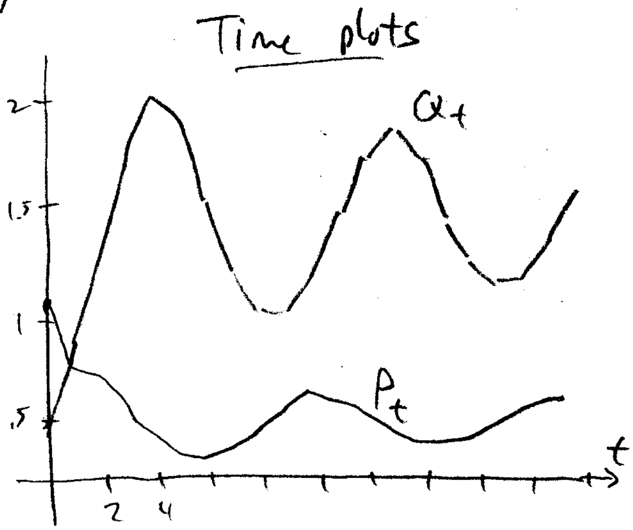
Hard: Estimate parameters.

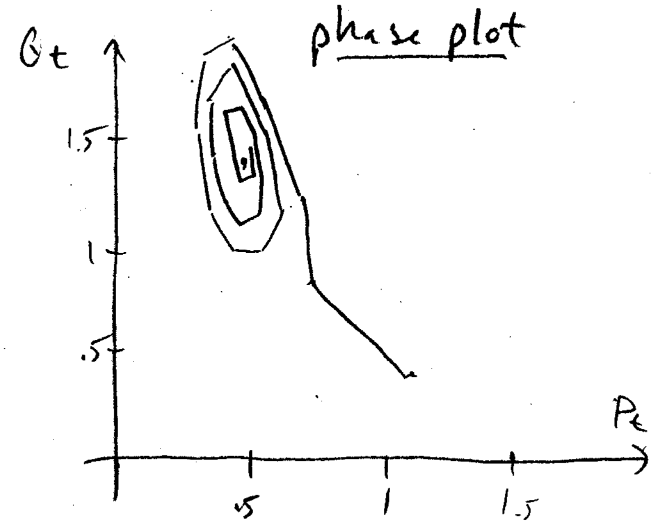
Qualitatively, larger $s, v \Rightarrow$ stronger predator-prey interaction.

We can plot the solutions several ways:

- Time plots: P_t, Q_t, vs. t.
- Phase plots: Q_t vs. P_t

$\boxed{2}$





Equilibria: Set $P_{t+1} = P_t = P^*$, $Q_{t+1} = Q_t = Q^*$.

Example: Same as above, but let $K=1$, $r=1.3$, $s=.5$
$u=.7$ $v=1.6$

$$P^* = P^*(1+1.3(1-P^*)) - .5P^*Q^*$$
$$Q^* = .3Q^* + 1.6P^*Q^*$$

$$\Rightarrow \begin{cases} 0 = P^*\,1.3(1-P^*) - .5P^*Q^* = P^*(1.3 - 1.3P^* - .5Q^*) \\ 0 = .7Q^* + 1.6P^*Q^* = Q^*(-.7+1.6P^*) \end{cases}$$

$\Rightarrow$ $Q^* = 0$ or $P^* = .7/1.6 = .4375$ (for Eq 2 to hold)

If $Q^* = 0$, then $P^* = 0$ or $P^* = 1$

If $P^* = .4375$, then $Q^* = 1.4625$.

* 3 equilibria: $(P^*, Q^*) = (0,0), (0,1), (.4375, 1.4625)$.

For *general* predator-prey: $\begin{cases} P_{t+1} = P_t\left(1 + r(1-P_t)\right) - sP_tQ_t \\ Q_{t+1} = (1-u)Q_t + vP_tQ_t \end{cases}$

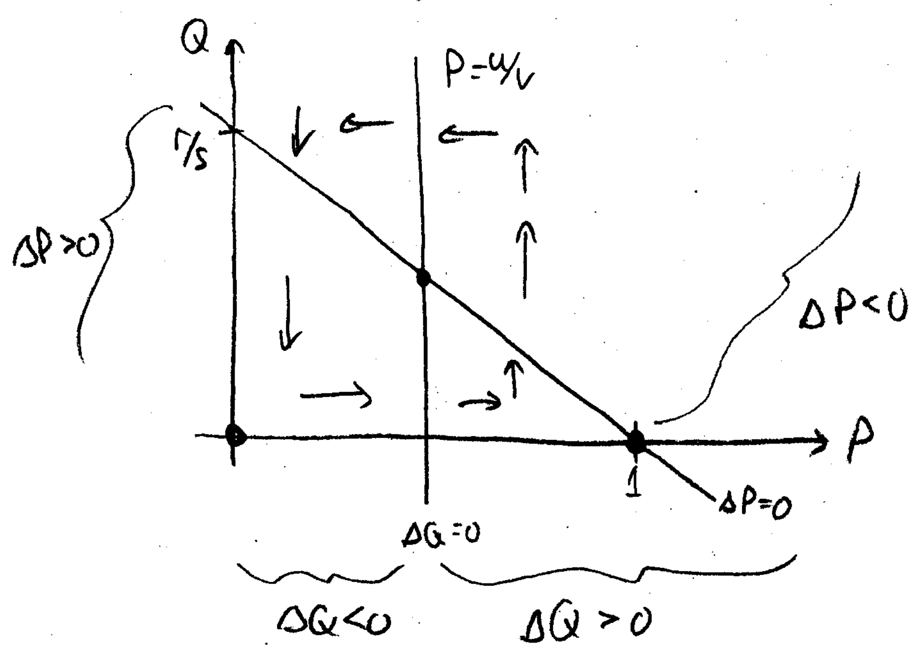
equilibrium equations are

$$0 = P^*(r(1-P^*)) - sP^*Q^* = P^*(r(1-P^*) - sQ^*)$$
$$0 = -uQ^* + vP^*Q^* = Q^*(-u+vP^*)$$

$\Rightarrow$ $Q^* = 0$ or $-u + vP^* = 0$ for Eq'n 2 to be satisfied

$P^* = 0$ or $r(1-P^*) - sQ^* = 0$ for Eq'n 1.

Graph these 4 lines: $P=0$, $Q = \frac{r}{s}(1-P)$, $Q=0$, $P = \frac{u}{v}$
(nullclines)



Steady-states

- $(0, 0)$
- $(1, 0)$
- $\left(\frac{u}{v}, \frac{r}{s}\left(1-\frac{u}{v}\right)\right)$

Remark: Changing r or s doesn't affect the Q-nullcline.

Example: Prey: Agricultural crop
Predator: Insect.

Idea: Introduce a new crop variety with higher r, to try to "outgrow" the predator.

Reality: P^* unchanged, Q^* increases.

Remark: ΔP & ΔQ don't change sign w/in regions.

We can deduce that the phase portrait spirals counterclockwise, but can't tell if inward or outward.

Linearization: (how to determine stability)

Plug in $P_t = P^* + p_t$, $\quad P_{t+1} = P^*_{t+1} + p_{t+1}$

$Q_t = Q^* + q_t$, $\quad Q_{t+1} = Q^*_{t+1} + q_{t+1}$

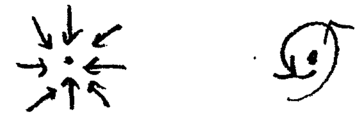
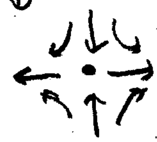
messy algebra $\longrightarrow$

$$p_{t+1} = .43125\,p_t - .21875\,q_t \underbrace{- \cancel{1.3\,p_t^2} - \cancel{.5\,p_t q_t}}_{\approx 0}$$
$$q_{t+} = 2.34\,p_t + q_t + \cancel{1.6\,p_t q_t}$$

$$\Rightarrow \begin{pmatrix} p_{t+1} \\ q_{t+1} \end{pmatrix} \approx \begin{pmatrix} .43125 & -.21875 \\ 2.34 & 1 \end{pmatrix} \begin{pmatrix} p_t \\ q_t \end{pmatrix}$$

$\lambda = .7156 \pm .6565\,i$, $\quad |\lambda| = \sqrt{(.7156)^2 + (.6565)^2} = .9711 < 1.$

* Types of equilibrium points

- $|\lambda_1| < 1$, $|\lambda_2| < 1$ stable 
- $|\lambda_1| > 1$, $|\lambda_2| > 1$ unstable 
- $|\lambda_1| < 1$, $|\lambda_2| > 1$ saddle 

*Other interaction models

Competition: 2 species fill the same niche in an environment.

$$\begin{cases} \Delta P = rP(1-(P+Q)/K) \\ \Delta Q = uQ(1-(P+Q)/K) \end{cases}$$

Question: Does one species "win"? Or can they co-exist?

More general: Now, suppose one preys on the other (e.g., wolf, deer).

$$\begin{cases} \Delta P = rP(1-(P+Q)/K) - sPQ \\ \Delta Q = uQ(1-(P+Q)/K) - vPQ. \end{cases}$$

Immune system vs. infective agent

P: immune cells

Q: level of infection

$$\begin{cases} \Delta P = rQ - sPQ \\ \Delta Q = uQ - vPQ \end{cases}$$

$-sPQ$: neg. effect on immune system from fighting

$-vPQ$: limited effect of immune system in fighting

rQ: Immune response is proportion to infection level.

Mutualism

P = sharks

Q = feeder fish

$$\begin{cases} \Delta P = rP(1-P/K) + sPQ \\ \Delta Q = -uQ + vPQ \end{cases}$$