4 Linear models of structured populations

Motivation: Consider a population divided into several groups, e.g., children vs. adults, or egg, larva, pupa, adult.

Example: Insect population: Egg ⟶ Larva ⟶ Adult ⟶ Dead

E_t = # eggs at time t

L_t = # larvae at time t

A_t = # adults at time t

Data: 4% of eggs survive become larvae

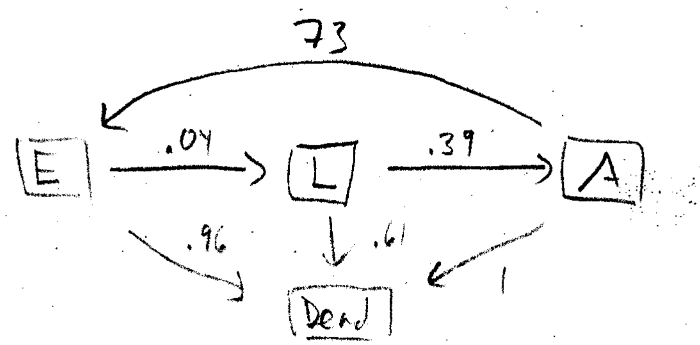
3% of larvae make it to adulthood

Average adult produces 73 eggs each.

$E_{t+1} = 73 A_t$

$L_{t+1} = .04 E_t$

$A_{t+1} = .39 L_t$



This is easy to solve: $A_{t+3} = (.39)(.04)(73) A_t = 1.1388 A_t$

Exponential growth

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Now, suppose instead of dying, 65% of adults survive another day:

$$\begin{cases} E_{t+1} = 73 A_t \\ L_{t+1} = .04 E_t \\ A_{t+1} = .39 L_t + .65 A_t \end{cases}$$

How to solve?

What's the growth rate?

Example: Forest has 2 species of trees, A & B.

A_t, B_t denotes # trees in year t.

Tree dies ⇒ new tree grows in its place (either species)

Each year: 1% of tree A's die

5% of tree B's die.

75% of vacant spots go to species A

25% of vacant spots go to species B.

$$A_{t+1} = .99 A_t + (.25)(.01) A_t + (.25)(.05) B_t$$

$$B_{t+1} = .95 B_t + (.75)(.05) B_t + (.75)(.01) A_t$$

Simplifying...
$$\begin{cases} A_{t+1} = .9925 A_t + .0125 B_t \\ B_{t+1} = .0075 A_t + .9875 B_t \end{cases}$$

<u>Questions</u>: What's the long-term behavior?

How does this depend on initial conditions?

<u>Approach</u>: Use matrices: $\begin{pmatrix} A_{t+1} \\ B_{t+1} \end{pmatrix} = \begin{pmatrix} .9925 & .0125 \\ .0075 & .9875 \end{pmatrix} \begin{pmatrix} A_t \\ B_t \end{pmatrix}$

Solve $\vec{X}_{t+1} = P\vec{X}_t$

<u>One sol'n</u>:
$$\vec{X}_1 = P\vec{X}_0$$
$$\vec{X}_2 = P\vec{X}_1 = P(P\vec{X}_0) = P^2\vec{X}_0$$
$$\vec{X}_3 = P\vec{X}_2 = P^3\vec{X}_0$$
$$\vdots$$

<u>Better</u>: Find the <u>eigenvalues</u> and <u>eigenvectors</u> of P.

Write the initial vector $\vec{X}_0$ as $\vec{X}_0 = c_1\vec{v}_1 + c_2\vec{v}_2$

Now, $\vec{X}_1 = A\vec{X}_0 = A(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1\lambda_1\vec{v}_1 + c_2\lambda_2 v_2$

$$\vec{X}_2 = A\vec{X}_1 = A^2\vec{X}_0 = A(c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2)$$
$$= c_1\lambda_1^2\vec{v}_1 + c_2\lambda_2^2\vec{v}_2$$
$$\vdots$$
$$X_t = A^t\vec{X}_0 = c_1\lambda_1^t\vec{v}_1 + c_2\lambda_2^t\vec{v}_2.$$

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Ex (cont.) $P = \begin{pmatrix} .9925 & .0125 \\ .0075 & .9875 \end{pmatrix}$

Eigenvalues/vectors: $P\begin{pmatrix}5\\3\end{pmatrix} = 1\begin{pmatrix}5\\3\end{pmatrix}$, $P\begin{pmatrix}1\\-1\end{pmatrix} = .98\begin{pmatrix}1\\-1\end{pmatrix}$.

Suppose $\vec{x}_0 = \begin{pmatrix}10\\990\end{pmatrix}$.

Write $\vec{x}_0 = \begin{pmatrix}10\\990\end{pmatrix} = c_1\begin{pmatrix}5\\3\end{pmatrix} + c_2\begin{pmatrix}1\\-1\end{pmatrix}$

i.e., solve $\begin{pmatrix}5 & 1\\ 3 & -1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}10\\990\end{pmatrix}$

$$A \quad \vec{c} = \vec{x}_0 \implies \vec{c} = A^{-1}A\vec{c} = A^{-1}\vec{x}_0.$$

and $A^{-1} = \frac{1}{-8}\begin{pmatrix}-1 & -1\\ -3 & 5\end{pmatrix}$, so $\begin{pmatrix}c_1\\c_2\end{pmatrix} = \frac{-1}{8}\begin{pmatrix}-1 & -1\\ 3 & 5\end{pmatrix}\begin{pmatrix}10\\990\end{pmatrix} = \begin{pmatrix}125\\-615\end{pmatrix}$.

Thus, $\begin{pmatrix}10\\990\end{pmatrix} = 125\begin{pmatrix}5\\3\end{pmatrix} - 615\begin{pmatrix}1\\-1\end{pmatrix}$

$$X_t = (1)^t\, 125\begin{pmatrix}5\\3\end{pmatrix} + (.98)^t(-615)\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}625-(615)(.98)^t\\ 375-(615)(.98)^t\end{pmatrix}$$

<u>Question</u>: What is $\lim_{t\to\infty} X_t$?

<u>Ans</u>: $\lim_{t\to\infty} X_t = 125\begin{pmatrix}5\\3\end{pmatrix} = \begin{pmatrix}625\\375\end{pmatrix}$ [since $(.98)^t \to 0$]

<u>Questions</u>: Does this depend on $\vec{x}_0$? What if $\vec{x}_0 = \begin{pmatrix}0\\1000\end{pmatrix}$?