*Read*: Lax, Chapter 8, pages 101–120.

1. (a) Write the equation $5x_1^2 - 6x_1x_2 + 5x_2^2 = 1$ in the form $x^TAx = 1$.
   (b) Write $A = P^TDP$, where $D$ is a diagonal matrix and $P$ is orthogonal with determinant 1.
   (c) Sketch the graph of the equation $x^TDx = 1$ in the $x_1x_2$-plane.
   (d) Use a geometric argument applied to part (c) to sketch the graph of $x^TAx = 1$.
   (e) Repeat Parts (a)–(d) for the equation $2x_1^2 + 6x_1x_2 + 2x_2^2 = 1$.

2. Let $S$ be the cyclic shift mapping of $\mathbb{C}^n$, that is, $S(z_1, \dots, z_n) = (z_n, z_1, \dots, z_{n-1})$.
   (a) Prove that $S$ is an isometry in the Euclidean norm.
   (b) Determine the eigenvalues and eigenvectors of $S$.
   (c) Verify that the eigenvectors are orthogonal.

   *Hint*: There are very short and elegant solutions for all three parts of this problem! You may find the last problem on HW 9 useful.

3. Let $N\colon X \to X$ be a normal mapping of a Euclidean space. Prove that $||N|| = \max |n_i|$, where the $n_i$s are the eigenvalues of $N$.

4. Let $H, M\colon X \to X$ be self-adjoint mappings, and $M$ positive definite.
   (a) Define a scalar product on $X$ by $\langle x, y\rangle := (x, My)$. Prove that this is an inner product.
   (b) Prove that all the eigenvalues of $M^{-1}H$ are real.
   (c) Prove that if $H$ is positive-definite, then so is $M^{-1}H$. Conclude that all eigenvalues of $M^{-1}H$ are positive.

5. Let $H, M\colon X \to X$ be self-adjoint mappings, and $M$ positive definite. Define

$$R_{H,M}(x) = \frac{(x, Hx)}{(x, Mx)}\,.$$

   (a) Let $\mu = \inf\{R_{H,M}(x) \mid x \in X\}$. Show that $\mu$ exists, and that there is some $v \in X$ for which $R_{H,M}(v) = \mu$, and that $\mu$ and $v$ satisfy $Hv = \mu Mv$.
   (b) Show that the constrained minimum problem

$$\min\{R_{H,M}(x) \mid (x, Mv) = 0\}$$

   has a nonzero solution $w \in X$, and that this solution satisfies $Hw = \kappa Mw$, where $\kappa = R_{H,M}(w)$.

6. Let $H, M\colon X \to X$ be self-adjoint mappings, and $M$ positive definite.

(a) Show that there exists a basis $v_1, \dots, v_n$ of $X$ where each $v_i$ satisfies an equation of the form

$$Hv_i = \mu_i M v_i \quad (\mu_i \text{ real}), \qquad (v_i, Mv_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

(b) Compute $(v_i, Hv_j)$, and show that there is an invertible real matrix $U$ for which $U^* M U = I$ and $U^* H U$ is diagonal.

(c) Characterize the numbers $\mu_1, \dots \mu_n$ by a minimax principle.