Read: Lax, Chapter 10, pages 143–153, and Appendix 4, pages 313–316.

Let U and V be vector spaces over a field K. The *tensor product* of U and V is the set $U \otimes V$ of finite sums of the form

$$\sum c_i(u_i \otimes v_i)\,,$$

where $c_i \in K$, $u_i \in U$, and $v_i \in V$, subject to the following relations:

$$\begin{aligned} c(u\otimes v) &= (cu)\otimes v = u\otimes(cv)\,,\\ (u+u')\otimes v &= u\otimes v + u'\otimes v\,,\\ u\otimes(v+v') &= u\otimes v + u\otimes v'\,. \end{aligned}$$

Elements of the form $u \otimes v$ are called *pure tensors*.

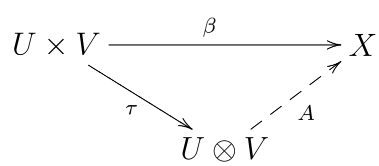
1. Let U, V, and X be vector spaces over a field K. Define a map

$$\tau\colon U\times V \longrightarrow U\otimes V\,,\qquad \tau(u,v)=u\otimes v\,.$$

   (a) Prove that τ is bilinear.

   (b) Prove that for any linear map $A\colon U\otimes V \to X$, the mapping $\alpha := A\circ\tau$ is bilinear from $U\times V$ to X.

   (c) Prove that for any bilinear map $\beta\colon U\times V\to X$, there is a unique linear mapping $A\colon U\otimes V\to X$ such that $\beta = A\circ\tau$. This is the *universal property* of the tensor product, and it says that any bilinear map can be "factored through" it. This is illustrated by the following commutative diagram:



2. If $\{u_1,\dots,u_n\}$ and $\{v_1,\dots,v_m\}$ are bases for U and V, respectively, then it is elementary to show that the pure tensors $\{u_i\otimes v_j \mid 1\le i\le n,\ 1\le j\le m\}$ span $U\otimes V$. Show that these are linearly independent, and conclude that $\dim(U\otimes V) = (\dim U)(\dim V)$. [*Hint*: Use the canonical basis $\{f_{ij}\}$ of the space of bilinear functions $U\times V\to K$, and use the universal property.]

3. Use the universal property of the tensor product to prove the following results:

   (a) $U\otimes V \cong V\otimes U$ (hint: let $X = V\otimes U$);

   (b) $(U\otimes V)\otimes W \cong U\otimes(V\otimes W)$;

   (c) $(U\times V)\otimes W \cong (U\otimes W)\times(V\otimes W)$.