A Little Set Theory (Never Hurt Anybody)

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1 Introduction

The fundamental ideas of set theory and the algebra of sets are probably the most important concepts across all areas of mathematics other than the algebra of real numbers. Many texts give these ideas too little direct attention, and then utilize concepts that they have not adequately reviewed. In this brief tutorial, we review enough set theory to get the reader through most lower-level college math courses.

2 What is a Set?

A set is one of those fundamental mathematical ideas whose nature we understand without direct reference to other mathematical ideas. Quite simply,

Definition 1 *A* set *is a collection of distinct objects.*

The objects can be real, physical things, or abstract, mathematical things. The number of such objects can be finite or infinite. The only important thing is that the objects be *distinct*, i.e., uniquely identifiable.

This idea of *distinctness* has some subtleties. For example, a collection of identical blue balls in an urn (you meet a lot of these in probability class) is a set because the balls are *in principle* uniquely identifiable: we could paint a different number on each of them. On the other hand, the list of digits 1, 5, 3, 2, 1, 4 is not a set, because there is no way to distinguish the two occurrences of 1 in the list. (Order doesn't matter when listing the elements of a set.)

The most basic way to define a set is to list its members between curly braces: $\{2, 4, 6, 8, 10\}$. We can also define a set by giving a *rule* that determines reliably whether an object is a member or not: $\{x : 0 < x < 1\}$. The ":" is read "such that". (Some authors use "|" instead.) So this rule is read as:

The set of values x such that x is strictly between 0 and 1.

We must be able to determine membership or lack of it for *every* object. Otherwise the rule is not a valid definition of a set.

The other ways to define sets are by building them up from more basic sets. We discuss rules for doing this below. As is common in mathematics, we can refer to a set by naming it with a letter. For example, $S = \{x : 0 < x < 1\}$.

Definition 2 *An object x is an* element *or* member *of a set S, written $x \in S$, if x satisfies the rule defining membership in S.*

We can write $x \notin S$ if x is not an element of S.

Definition 3 *The* empty set *or* null set*, denoted $\emptyset$ or $\{\}$, is the set containing no elements.*

An important property of a set is the number of elements it contains.

Definition 4 *The* cardinality *of a set S, written $|S|$, is the number of elements in the set.*

Note that this number could be infinite (as in $\{0, 2, 4, 6, \ldots\}$ or $\{x : 0 < x < 1\}$). There are issues with doing arithmetic with infinities that we ignore for now—that is, when discussing arithmetic on set cardinalities, we assume the sets are finite.

The cardinality of the empty set is zero ($|\emptyset| = 0$).

3   Subsets and Equality

This section describes *relations* between sets, that is, concepts that we use to compare two sets.

Definition 5 *Two sets S and T are* equal*, written $S = T$, if S and T contain exactly the same elements.*

(This may seem too trivial to even bother writing down, but in mathematics, it's important to agree on even these simple concepts. Everything that comes later is built on them, and we can understand the more complicated ideas only if we clearly understand the simple ideas they are built from.)

Definition 6 *A set S is a* subset *of another set T, written $S \subset T$ if every element of S is also an element of T.*

Note that, by this definition, $S \subset T$ does not exclude the possibility that every element of T is also an element of S.

Definition 7 *A set S is a* proper subset *of T if $S \subset T$ and $S \neq T$.*

That is, if every element of S is also in T, but some element of T is not in S.

A note on notation: Some authors write $S \subseteq T$ to denote subsets that may not be proper and $S \subset T$ to denote proper subsets. Others use $S \subset T$ to denote general subsets and $S \subsetneq T$ to denote proper subsets. There are other combinations of notation in use as well. Be sure you know which convention the author of your textbook uses, and choose one convention and stick to it for your own writing.

Properties If R, S, and T are any sets, the following properties hold:

- $S = S$ (reflexive)

- If $S = T$, then $T = S$ (symmetric).

- If $R = S$ and $S = T$ then $R = T$ (transitive).

- $\emptyset \subset S$

- $S \subset S$ (reflexive)

- If $R \subset S$ and $S \subset T$, then $R \subset T$ (transitive).

- $S \subset T$ and $T \subset S$ if and only if $S = T$.

- If $S \subset T$, then $|S| \leq |T|$, and if $S \subsetneq T$, then $|S| < |T|$.

4   Intersections and Unions

This section and the next discuss *operations* on sets, that is, ways in which sets can be combined to form new sets.

Definition 8 *The* intersection *of two sets S and T, written $S \cap T$ is the set of elements common to both S and T, i.e.,*

$$S \cap T = \{x \,:\, x \in S \text{ and } x \in T\}.$$

This definition reads:

> the set of objects x such that x is an element of S and x is an element of T.

Example 1 If $S = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $T = \{e_0, e_2, e_4, e_6, e_8\}$ then $S \cap T = \{e_2, e_4, e_6\}$.

Example 2 If $S = \{x \,:\, 0 \le x \le 1\}$ and $T = \{x \,:\, .5 < x < 2\}$, then $S \cap T = \{x \,:\, .5 < x \le 1\}$.

Definition 9 *The* union *of two sets S and T, written $S \cup T$, is the set of elements that are in either S or T or both, i.e.,*

$$S \cup T = \{x \,:\, x \in S \text{ or } x \in T\}.$$

Note that when two statements are connected with "or", this means that *at least* one of the statements is true.

Example 3 If $S = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $T = \{e_0, e_2, e_4, e_6, e_8\}$ then $S \cup T = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_8\}$.

Example 4 if $S = \{x \,:\, 0 \le x \le 1\}$ and $T = \{x \,:\, .5 < x < 2\}$, then $S \cup T = \{x \,:\, 0 \le x < 2\}$.

Properties   We can combine multiple operations to create new sets from other sets. Sometimes, two different ways of combining sets produce the same result. In this case, we can replace one combination with the other in an equation. If S is a set constructed from other sets, we can replace S in an expression with the expression that describes the construction of S. (When making such substitutions, it's helpful to enclose the replacement expression

in parentheses.) Repeated application of this principle can lead to profound insights about the relationships between certain sets.

The rules for replacement regarding unions and intersections are listed below. A little reflection will convince you of the validity of most of them.

- $S \cup T = T \cup S$ (commutative)[1]

- $S \cap T = T \cap S$ (commutative)

- $S \subset S \cup T$

- $S \cap T \subset S$

- $S \cup \emptyset = S$

- $S \cap \emptyset = \emptyset$

- $S \cap S = S \cup S = S$

- $R \cup (S \cup T) = (R \cup S) \cup T$ (associative)[2]

- $R \cap (S \cap T) = (R \cap S) \cap T$ (associative)

- $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$ (distributive—$\cup$ over $\cap$)

- $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$ (distributive—$\cap$ over $\cup$)

- $S \cup (S \cap T) = S \cap (S \cup T) = S$

- $S \subset T$ if and only if $S \cup T = T$.

- If $R \subset T$ and $S \subset T$, then $R \cup S \subset T$.

- If $R \subset S$ and $R \subset T$, then $R \subset S \cap T$.

- If $S \subset T$ then $R \cup S \subset R \cup T$ and $R \cap S \subset R \cap T$.

- $S \cup T \neq \emptyset$ if and only if $S \neq \emptyset$ or $T \neq \emptyset$.

- If $S \cap T \neq \emptyset$, then $S \neq \emptyset$ and $T \neq \emptyset$.

- $S = T$ if and only if $S \cup T \subset S \cap T$.

The key property of cardinalities of unions and intersections is this:

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[1]Perform all operations before any comparisons.
[2]Perform all operations in parentheses before any others.

- $|S \cup T| = |S| + |T| - |S \cap T|$.

That is, the number of elements in the union of two sets is the sum of the numbers of elements in each set less the number of elements that appear twice in the two lists.

Example 5 If $S = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $T = \{e_0, e_2, e_4, e_6, e_8\}$ then $S \cup T = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_8\}$ and $S \cap T = \{e_2, e_4, e_6\}$. So $|S| = 6$ and $|T| = 5$. But $|S \cap T| = 3$, because there are three elements on both lists. When we form the union, we combine the two lists of elements, but each duplicate element is listed only once. Thus, $|S \cup T| = 6 + 5 - 3 = 8$.

If two sets have no elements in common, this formula becomes somewhat simpler.

Definition 10 *Two sets S and T are* mutually exclusive *if $S \cap T = \emptyset$.*

If S and T are mutually exclusive, then $|S \cap T| = 0$, so $|S \cup T| = |S| + |T|$.

Example 6 If $S = \{e_1, e_3, e_5\}$ and $T = \{e_0, e_2, e_4, e_6, e_8\}$ then $S \cup T = \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_8\}$ and $S \cap T = \emptyset$. So $|S| = 3$ and $|T| = 5$. But $|S \cap T| = 0$, because no element appears on both lists. When we form the union, we combine the two lists of elements, and there are no duplicates. Thus, $|S \cup T| = 3 + 5 = 8$.

5   Set Differences and Complements

Another way to construct new sets from old is to remove some elements from a set and consider the elements that are left over.

Definition 11 *The* difference *between S and T, written $S \setminus T$, is the set of elements in S but not also in T:*

$$S \setminus T = \{x \ : \ x \in S \ and \ x \notin T\}.$$

Example 7 If $S = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $T = \{e_0, e_2, e_4, e_6, e_8\}$ then $S \setminus T = \{e_1, e_3, e_5\}$, but $T \setminus S = \{e_0, e_8\}$.

Example 8 If $S = \{e_1, e_3, e_5\}$ and $T = \{e_0, e_2, e_4, e_6, e_8\}$ then $S \setminus T = S$ and $T \setminus S = T$.

Example 9 If $S = \{x : 0 \leq x \leq 1\}$ and $T = \{x : .5 < x < 2\}$, then $S \setminus T = \{x : 0 \leq x \leq .5\}$ and $T \setminus S = \{x : 1 < x < 2\}$.

We can see that, in general, $S \setminus T \neq T \setminus S$ (set difference is not commutative).

Properties Two important properties of set differences are:

- $S \setminus T = S \setminus (S \cap T)$

- $|S \setminus T| = |S| - |S \cap T|$

The first property states that the elements of T that are not in S play no role in the construction of $S \setminus T$. The second property states that the number of elements removed from S to produce $S \setminus T$ is the same as the number of elements that S and T have in common.

Definition 12 *The* symmetric difference *between S and T is $(S \cup T) \setminus (S \cap T)$, i.e., the set of elements in S or T, but not both.*

The cardinality of a symmetric difference satisfies:

- $|(S \cup T) \setminus (S \cap T)| = |S| + |T| - 2|S \cap T|$

We pool the member lists of S and T, but we delete *both* copies of duplicates.

Example 10 If $S = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $T = \{e_0, e_2, e_4, e_6, e_8\}$ then $(S \cup T) \setminus (S \cap T) = \{e_0, e_1, e_3, e_5, e_8\}$, and $|(S \cup T) \setminus (S \cap T)| = 6 + 5 - 2 \times 3 = 5$.

Example 11 If $S = \{x : 0 \leq x \leq 1\}$ and $T = \{x : .5 < x < 2\}$, then $(S \cup T) \setminus (S \cap T) = \{x : 0 \leq x \leq .5 \text{ or } 1 < x \leq 2\}$.

In many set problems, all sets are defined to be subsets of some reference set. This reference set is called the *universe*. In the remainder of this section, the discussion refers to a universe U, and we assume that all other sets R, S, and T are subsets of U.

Definition 13 *Relative to a universe U, the* complement *of S, written S', is the set of all elements of the universe not contained in S, i.e.,*

$$S' = \{x : x \in U \text{ and } x \notin S\}$$

Another common notation for the complement of S is $\bar{S}$. Complements of set expressions can be written similarly, for example:[3] $(S \cap T)'$ or $\overline{S \cap T}$.

The important cardinality property of complements is

- $|S'| = |U| - |S|$

Example 12 If $U = \{e_0, e_1, \ldots e_{10}\}$ and $S = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, then $S' = \{e_0, e_7, e_8, e_9, e_{10}\}$. Also, $|U| = 11$ and $|S| = 6$, so $|S'| = 11 - 6 = 5$.

Example 13 If $U = \{x : x \text{ is a real number}\}$ and $S = \{x : 0 \leq x \leq 1\}$, then $S' = \{x : x < 0 \text{ or } x > 1\}$.

The following are important properties of complements and differences.

- $U' = \emptyset$ and $\emptyset' = U$

- $S \cup S' = U$ and $S \cap S' = \emptyset$

- $S \cup U = U$ and $S \cap U = S$

- $S'' = S$

- $S = T'$ if and only if $T = S'$.

- $S \subset S'$ if and only if $S = \emptyset$.

- $S \subset T$ if and only if $T' \subset S'$.

- $S \setminus T = S \cap T'$

- $(S \cup T) \setminus (S \cap T) = (S \setminus T) \cup (T \setminus S)$

- $(S \cup T)' = S' \cap T'$

- $(S \cap T)' = S' \cup T'$

The last two properties are known as *DeMorgan's laws*. They provide a useful way to convert arbitrary set expressions into ones that involve only complements and unions or only complements and intersections.

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[3]Perform complements before performing any of the other set operations. Parentheses or overbars are required around set expressions to be complemented.

6 Conclusion

It is important to know the basic definitions and a few of the key properties described above. But don't spend time memorizing all the properties; they are too abstract and easy to confuse. Instead, keep this guide handy while working problems involving sets, and refer to it liberally. But each time you make use of a property, think carefully about its meaning. You'll find that, with practice, you'll be able to easily re-derive the properties you need, when you need them.