MthSc 813 Advanced Linear Programming Spring 1999
Midterm Exam

March 5, 1999

- This take-home exam is due by 3:00 on Tuesday, March 9, 1999. Sign and return this cover sheet with your exam.
- There are a total of 60 points. Point value is listed next to each question.
- You may consult your textbook, class notes, and me. No other sources are allowed.
- Mark your answers clearly. *Show your work.* Unsupported correct answers receive partial credit.
- Be sure to write your name on *each* page.
- Good luck!

Name:____

Student ID #:____________________

I certify that I have not received any unauthorized assistance in completing this examination.

Signature:____

Date:____________________

The first rule of discovery is to have brains and good luck. The second rule of discovery is to sit tight and wait till you get a bright idea. —G. Polya

1. (10 points) A *supporting hyperplane* for a convex polyhedron $P = \{x : Ax \le b\}$ is a hyperplane $H = \{x : a^T x = a_0\}$ such that for *all* $x \in P$, $a^T x \le a_0$ and for *some* $x \in P$, $a^T x = a_0$. A *face* of P is the set $F = P \cap H$ for some supporting hyperplane H.

   (a) Show that H is a supporting hyperplane for P if and only if a_0 is the optimal objective value for the LP $\max\{a^T x : Ax \le b\}$.

   If Let x^* be an optimal solution to the LP, with $a^T x^* = a_0$. Then clearly $x^* \in H$ and for all $x \in P$, $a^T x \le a_0$.

   Only if Suppose $H = \{x : a^T x = a_0\}$ is a supporting hyperplane for P. Then there exists an $x^* \in P$ such that $a^T x^* = a_0$. If x^* is not an optimal solution to the LP, then there exists $\hat{x}$ such that $a^T \hat{x} > a_0$, contradicting the assumption that H is a supporting hyperplane.

   (b) Show that F is a face of P if and only if $F = \{x \in P : A'x = b'\}$, for some subsystem $A'x \le b'$ of $Ax \le b$.

   If Let $F = \{x \in P : A'x = b'\}$ for some subsystem $A'x \le b'$ of $Ax \le b$ (and assume $F \neq \emptyset$). Define $a^T = e^T A'$, $a_0 = e^T b'$, and let $H = \{x : a^T x = a_0\}$. Then every $x \in F$ satisfies $a^T x = a_0$ and every $x \in P$ satisfies $a^T x = e^T A' x \le e^T b' = a_0$.

   Only if Let $F = P \cap H$, where $H = \{x : a^T x = a_0\}$ is a supporting hyperplane, and consider the LP $\max\{a^T x : Ax \le b\}$ and its dual $\min\{b^T y : A^T y = a,\ y \ge 0\}$. Let x^* and y^* be strictly complementary solutions (which exist if H is a supporting hyperplane). Then by complementary slackness, $A_{i.} x^* = b_i$ whenever $y_i^* > 0$, for any optimal x^*, i.e., for any $x^* \in F$. Note that there exists a $y_i^* > 0$ since $a \neq 0$.

2. (10 points) Given a point $\hat{x}$ and a polyhedron $P = \{x : Ax \le b\}$, the *separation problem* is to determine if $\hat{x} \in P$, and if not, to identify a halfspace $H = \{x : a^T x \ge a_0\}$ such that $P \subseteq H$ and $\hat{x} \notin H$. Show that the separation problem is polynomially reducible to the problem of solving linear programs. (**Hint:** Construct a linear program that has an optimal solution if and only if $A\hat{x} \le b$. If this system has no solution, construct a linear program whose solution provides a separating hyperplane for $\{\hat{x}\}$ and P.)

   (a) $\hat{x} \in P$ iff $A\hat{x} \le b$ iff $\min\{0^T w : w = b - A\hat{x},\ w \ge 0\}$ has a feasible solution. (This is a roundabout way to solve a trivial problem, but it's what you have to do if you only have an LP solver.)

(b) If $\hat{x} \notin P$ then the system $x = \hat{x}$, $Ax \le b$ has no solution. Following the construction in Theorem 10.4, the system

$$\begin{bmatrix} I & A^T \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} = 0$$
$$y^2 \ge 0$$
$$\hat{x}^T y^1 + b^T y^2 < 0$$

has a solution. This solution can be found by solving the LP $\min\{\hat{x}^T y^1 + b^T y^2 : y^1 + A^T y^2 = 0,\ y^2 \ge 0\}$. The solver will terminate with a direction of unboundedness that can be taken as the solution $\begin{bmatrix} \hat{y}^1 & \hat{y}^2 \end{bmatrix}$ to the system. (Or you can set bounds on the magnitudes of the components of y^1 and y^2 to guarnateee an optimal solution $\begin{bmatrix} \hat{y}^1 & \hat{y}^2 \end{bmatrix}$.) Then $H = \{x : x^T \hat{y}^1 \ge -b^T \hat{y}^2\}$ is a halfspace containing P but not $\hat{x}$.

We can thus solve the separation problem by solving two LPs whose sizes are obviously polynomial in the size of $\hat{x}$, A and b.

3. (10 points) Let A be an $m \times n$ matrix, and suppose $Ax \le b$ is infeasible. Show that there is an infeasible subsystem $A'x \le b'$ consisting of at most $n+1$ inequalities. (**Hint:** Construct an appropriate theorem of alternative.)

The alternative theorem is just Farkas's Lemma (as given by Vanderbei): $Ax \le b$ is infeasible iff $A^T y = 0$, $y \ge 0$, $b^T y < 0$ is feasible. After eliminating dependent rows, the equality system $A'^T y = 0$ has at most n rows. Thus the LP $\min\{0^T y : A'^T y = 0,\ b^T y = -1,\ y \ge 0\}$ has a feasible solution (found by scaling the components of any solution to the alternative system). By a version of Theorem 3.4 proved in class, this LP has a BFS with at most $n+1$ positive variables. Let $A''x = b''$ be the subsystem of the original inequality system corresponding to the strictly positive components of the BFS found above. By Farkas's lemma again, this system is infeasible.

4. (10 points) For $t \in \mathbb{R}$, let $\zeta^*(t) = \min\{c^T x : Ax = b + tb',\ x \ge 0\}$, and suppose that $\zeta^*(0)$ is finite. Determine if $\zeta^*(t)$ is convex, concave, both, or neither. Justify your answer.

Let t_1 and t_2 be such that the LP is feasible for right-hand sides $b+t_1b'$ and $b+t_2b'$. Then $\zeta^*(t)$ is convex iff $\alpha\zeta^*(t_1) + (1-\alpha)\zeta^*(t_2) \ge \zeta^*(\alpha t_1 + (1-\alpha)t_2)$ for $0 < \alpha < 1$.

Let $\hat{x}^1$ be an optimal solution for the LP with RHS $b + t_1 b'$ and $\hat{x}^2$ be an optimal solution for RHS $b + t_2 b'$. Let $\hat{x} = \alpha\hat{x}^1 + (1-\alpha)\hat{x}^2$. It is easy to show that $\hat{x}$ is feasible for the LP with RHS $b + (\alpha t_1 + (1-\alpha)t_2)b'$, and $\alpha c^T \hat{x}^1 + (1-\alpha)c^T \hat{x}^2 = c^T \hat{x} \ge \zeta^*(\alpha t_1 + (1-\alpha)t_2)$.

5. (10 points) For an LP in bounded-variable form, let the vector of upper bounds be defined by $u + te^{\hat{\jmath}}$ for some fixed $\hat{\jmath}$. Describe a simplex-based algorithm that will compute $\zeta^*(t)$ for all values of t, $l_{\hat{\jmath}} - u_{\hat{\jmath}} \le t \le \infty$.

The equations for a bounded-variable BFS are

$$\begin{aligned}
\zeta &= c_B^T B^{-1} b + (c_L^T - c_B^T B^{-1} L) x_L + (c_U^T - c_B^T B^{-1} U) x_U \\
x_B &= B^{-1} b - B^{-1} L x_L - B^{-1} U x_U \\
x_L &= l_L \\
x_U &= u_U.
\end{aligned}$$

Suppose that the simplex method terminates with a finite optimal solution for $t = l_{\hat{\jmath}} - u_{\hat{\jmath}}$ (i.e., $u_{\hat{\jmath}} = l_{\hat{\jmath}}$) with optimal solution x^*. (If the solution is unbounded for this condition, it is unbounded for all $t > 0$. If it is infeasible, an auxiliary LP can be constructed to identify the smallest value of t for which the LP is feasible. This construction is left as an exercise for the reader.). Otherwise, note that $c_{\hat{\jmath}} - c_B^T B^{-1} A_{\cdot\hat{\jmath}} < 0$ and is unaffected by changes in $u_{\hat{\jmath}}$. Then

(a) Let $t_0 = 0$, $x_0 = x^*$.

(b) If $\hat{\jmath} \in \mathcal{B}$ or $\hat{\jmath} \in \mathcal{L}$, then STOP ($x_0$ is optimal for all $t > t_0$).

(c) Consider increasing $x_{\hat{\jmath}} = u_{\hat{\jmath}}$. We have $x_B = B^{-1} b - B^{-1} A_{\cdot\hat{\jmath}} x_{\hat{\jmath}}$. Perform a bounded-variable ratio test on x_B. If no variable leaves, then STOP (the current basis is optimal for all values of $t > t_0$). Otherwise, suppose x_{i^*} leaves and α^* is the minimum ratio.

(d) Perform a dual ratio test with x_{i^*} leaving at its bound. Let x_{j^*} be the entering variable. If $j^* = \hat{\jmath}$ then STOP (the current solution is optimal for all $t > t_0 + \alpha^*$). Otherwise, set $t_0 := t_0 + \alpha^*$, $x_{\hat{\jmath}}^0 := u_{\hat{\jmath}} := u_{\hat{\jmath}} + \alpha^*$, and GOTO 5b.

6. (10 points) Consider a primal-dual pair of LPs in symmetric form, and let x^* and y^* be strictly complementary primal and dual solutions. Prove: every optimal x has $x_j = 0$ whenever $x_j^* = 0$ and every optimal y has $y_i = 0$ whenever $y_i^* = 0$.

By complementary slackness, *every* optimal x is complementary to *every* optimal y. In particular, every optimal x is complementary to y^*, so has $x_j = 0$ whenever the dual slack $z_j^* > 0$, i.e., whenever $x_j^* = 0$. A similar argument holds for the dual.