Linked Graphs with Restricted Lengths

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Abstract

A graph G is *k-linked* if G has at least $2k$ vertices, and for every sequence $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$ of distinct vertices, G contains k vertex-disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i joins x_i and y_i for $i = 1, 2, \ldots, k$. Moreover, the above defined k-linked graph G is *k-linked modulo $(m_1, m_2, \ldots, m_k)$* if, in addition, for any k-tuple $(d_1, d_2, \ldots, d_k)$ of natural numbers, the paths $P_1, P_2, \ldots, P_k$ can be chosen such that P_i has length d_i modulo m_i for $i = 1, 2, \ldots, k$. Thomassen showed that there exists a function $f(m_1, m_2, \ldots, m_k)$ such that every $f(m_1, m_2, \ldots, m_k)$-connected graph is k-linked modulo $(m_1, m_2, \ldots, m_k)$ provided all m_i are odd. For even moduli, he showed in another article that there exists a natural number $g(2, 2, \cdots, 2)$ such that every $g(2, 2, \cdots, 2)$-connected graph is k-linked modulo $(2, 2, \cdots, 2)$ if deleting any $4k - 3$ vertices leaves a nonbipartite graph.

In this paper, we give linear upper bounds for $f(m_1, m_2, \ldots, m_k)$ and $g(m_1, m_2, \ldots, m_k)$ in terms of $m_1, m_2, \ldots, m_k$, respectively. More specifically, we prove the following two results: (i) For any k-tuple $(m_1, m_2, \ldots, m_k)$ of odd positive integers, every $\max\{14(m_1+\cdots+m_k)-4k, 6(m_1+\cdots+m_k)-4k+36\}$-connected graph is k-linked modulo $(m_1, m_2, \ldots, m_k)$. (ii) Let $1 \le \ell \le k$ and let $(m_1, m_2, \ldots, m_k)$ be a k-tuple of positive integers such that m_i is odd for each i with $\ell + 1 \le i \le k$. If G is $45(m_1 + \cdots + m_k)$-connected, then either G has a vertex set X of order at most $2k + 2\ell - 3 + \delta(m_1, \ldots, m_\ell)$ such that $G - X$ is bipartite or G is k-linked modulo $(2m_1, 2m_2, \ldots, 2m_\ell, m_{\ell+1}, \ldots, m_k)$, where

$$\delta(m_1, \ldots, m_\ell) = \begin{cases} 0 & \text{if } \min\{m_1, \ldots, m_\ell\} = 1, \text{ and} \\ 1 & \text{if } \min\{m_1, \ldots, m_\ell\} \ge 2. \end{cases}$$

Our results generalize several known results on k-parity-linked graphs.

Key words: k-linked, k-linked modulo $(m_1, m_2, \ldots, m_k)$, k-parity-linked

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1 Introduction

We generally follow Diestel [6] for terminology and notation not defined here and consider simple graphs only. Through out this paper, the vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively, while the number of vertices and edges of G are denoted by $|G|$ and $||G||$, respectively. Clearly, $\ell(P) := ||P||$ is the length of P if P is a path. Since we are mainly dealing natural numbers in this paper, we let $[1, k] := \{1, 2, \dots, k\}$ for each natural number k.

The study of cycles and paths with certain lengths modulo a positive integers began with a result of Bollobás [1, 2]: *for every natural number m and every natural number d, every graph G with $||G|| \geq \frac{(m+1)^m-1}{m} \cdot |G|$ contains a cycle of length $2d \bmod m$*, which settled a conjecture of Burr and Erdős [7]. If m is an odd integer, $2d \bmod m$ covers all congruence classes modulo m when d runs over $0,\ 1,\ \cdots,\ m-1$. If m is even and d is odd, all integers of $d \bmod m$ are odd and bipartite graphs do not contain cycles of lengths $d \bmod m$. Thomassen [24] improved the result of Bollobás by showing that, for every natural number m and every natural number d, every graph G with minimum degree $\delta(G) \geq 4d(m+1)$ contains a cycle of length $2d \bmod m$. In the same paper, Thomassen showed that the existence of path systems with prescribed lengths $2d \bmod m$ in graphs of sufficient high connectivity.

A graph is said to be *k-linked* if it has at least $2k$ vertices and for every sequence $x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k$ of distinct vertices there exist k vertex-disjoint paths $P_1, \cdots, P_k$ such that P_i joins x_i and y_i for $i = 1, 2, \dots, k$. Moreover, a graph G is said to be *k-linked modulo* $(m_1, m_2, \dots, m_k)$ if G is k-linked and, in addition, for any k-tuple $(d_1, d_2, \dots, d_k)$ of natural numbers, the paths $P_1, P_2, \dots, P_k$ can be chosen such that P_i has length $d_i \bmod m_i$ for each $i \in [1, k]$.

Theorem 1 (Thomassen [24]) *For any two natural numbers k and p there exists a natural number $\gamma(k, p)$ such that every $\gamma(k, p)$-connected graph G is k-linked modulo $(m_1, m_2, \dots, m_k)$ for any k-tuple $(m_1, m_2, \dots, m_k)$ of old positive integers less than p.*

Thomassen actually gave an explicit bound on $\gamma(k, p)$. In the proof of Theorem 1, a function $\eta(s, t)$ is chosen such that (for some s sufficiently larger than both k and p and $t = p!$) each graph of minimum degree at least $\eta(s, t)$ contains a subdivision H of $K_{s,t}$ such that each edge of $K_{s,t}$ corresponds a path of length $1 \bmod t$ in H. He proves that

$$\gamma(k, p) \leq \underbrace{\xi 2^{\xi} \cdot 2^{\xi 2^{\xi}} \cdot 2^{2^{\xi 2^{\xi}}} \cdots}_{s^2(2t+1)^{2t+1}-s^2 \text{ times}}$$

where $\xi := \xi(s^2(2t+1)^t)$ guarantees that a graph with minimum degree at least ξ contains a subdivision of $K_{s^2(2t+1)^t, s^2(2t+1)^t}$ satisfying the parity condition on the path lengths. This bound contains many levels of exponentiation, so is enormous. We show that $\gamma(k, p)$ can be bounded by a linear function as stated below.

Theorem 2 *Let $(m_1, m_2, \ldots, m_k)$ be a k-tuple of odd positive integers and let*

$$f(m_1, m_2, \ldots, m_k) = \max\{14(m_1+m_2+\cdots+m_k)-4k,\ 6(m_1+m_2+\cdots+m_k)-4k+36\}.$$

Then every $f(m_1, m_2, \ldots, m_k)$-connected graph is k-linked modulo $(m_1, m_2, \ldots, m_k)$.

The condition that each m_i is odd in Theorem 2 is necessary since every complete bipartite graph is not k-linked modulo $(m_1, m_2, \ldots, m_k)$ if some m_i is even. In this case, for any integer d_i and n_i, if $n_i \equiv d_i \bmod m_i$ then n_i and d_i have the same parity. For any connected bipartite graph G and let $u, v \in V(G)$ and P be a path connecting u and v, we note that $\ell(P)$ is always even if both u and v are in the same partite set and $\ell(P)$ is always odd if u and v are in the different partite sets. Let G be a graph. The *bipartite index* $bi(G)$ is the least positive integer b such that there exists an $X \subseteq V(G)$ and $|X| = b$ so that $G - X$ is bipartite.

A graph that is k-linked modulo $(2, 2, \ldots, 2)$ is called *a parity k-linked graph.* Thomassen [25] proved that for any integer k, every $2^{3^{27k}}$-connected graph G is *k-parity-linked* if $bi(G) \geq 4k-3$. Thomassen also showed that lower bound "$4k-3$" of $bi(G)$ is best possible by constructing the following examples G from a large complete bipartite graph by adding the edges of a complete graph on $2k-1$ vertices to one of partite set and the edges of a complete graph on $2k$ vertices minus a perfect matching on the other partite set. Clearly, these graphs G are not k-parity-linked although they may have very high connectivities. But the connectivity $2^{3^{27k}}$ seems to be far from best possible. Thomassen [25] conjectured that the $2^{3^{27k}}$ can be lowered to a linear function of k. Kawarabayashi and Reed [18] verified this conjecture as follows.

Theorem 3 (Kawarabayashi and Reed [18]) *For every natural number k, every $50k$-connected graph G is* k-parity-linked *if $bi(G) \geq 4k-3$.*

For any ℓ-tuple $(m_1, \ldots, m_\ell)$ of natural numbers, let

$$\delta(m_1, \ldots, m_\ell) := \begin{cases} 0 & \text{if } \min\{m_1, \ldots, m_\ell\} = 1, \text{ and} \\ 1 & \text{if } \min\{m_1, \ldots, m_\ell\} \geq 2. \end{cases}$$

Combining the techniques developed in [18] and in the proof of Theorem 2, we prove the following theorem.

Theorem 4 *Let $(m_1, m_2, \ldots, m_k)$ be a k-tuple of natural numbers such that m_i is odd for each $i \in [\ell+1, k]$. If G is $45(m_1 + \cdots + m_k)$-connected and $bi(G) \geq 2k + 2\ell - 3 + \delta(m_1, \ldots, m_\ell)$, then G is k-linked modulo $(m'_1, m'_2, \ldots, m'_k)$ where*

$$m'_i := \begin{cases} 2m_i & \text{if } 1 \leq i \leq \ell, \text{ and} \\ m_i & \text{if } \ell+1 \leq i \leq k. \end{cases}$$

Although we do not believe the connectivity $45(m_1 + \ldots + m_k)$ is best possible, we will demonstrate that $bi(G) \geq 2k + 2\ell - 3 + \delta(m_1, \ldots, m_\ell)$ is sharp by the following two graphs.

(1) Suppose $\min\{m_1, \ldots, m_\ell\} \geq 2$. Let G be a graph obtained from a large complete bipartite graph by embedding a $K_{2\ell-1}$ into a partite set and a K_{2k} to the other partite set. Clearly, $bi(G) \geq 2k + 2\ell - 3$. Let $X := \{x_1, y_1, \ldots, x_k, y_k\}$ be the vertex set of the K_{2k}. Then G does not contain vertex-disjoint paths P_1, P_2, $\ldots$, P_k such that P_i connecting x_i and y_i for each $i \in [1, k]$ and $\ell(P_i) \equiv 3 \bmod 2m_i$ for each $i \in [1, \ell]$. So G is not k-linked modulo $(m'_1, m'_2, \ldots, m'_k)$.

(2) Suppose $\min\{m_1, \ldots, m_\ell\} = 1$. Let H be a graph obtained from a large complete bipartite graph by embedding a $K_{2\ell-1}$ into a partite set and a $K_{2k} - M$ into the other partite set, where M is a perfect matching of K_{2k}. Clearly, $bi(G) = 2k + 2\ell - 4$. Let $M = \{x_i y_i \mid 1 \leq i \leq k\}$. Then it is not difficulty to check that H does not contain vertex-disjoint paths P_1, $\cdots$, P_k such that $\ell(P_i) \equiv 1 \bmod 2m_i$ for each $i \in [1, \ell]$. So H is not k-linked modulo $(2m_1, \ldots, 2m_\ell, m_{\ell+1}, \ldots, m_k)$.

By taking $\ell = k$ and $m_1 = \cdots = m_k = 1$ in Theorem 4, we get the following improvement of Theorem 3.

Theorem 5 *For every natural number k, every $45k$-connected graph G is k-*parity-linked *if $bi(G) \geq 4k - 3$.*

Instead of building one powerful subgraph as did in [24], our basic idea is to establish a path system such that there exist a few segments in each path that can be replaced by paths with difference residue. We prove a number theoretical result on sumsets of integers to ensure the desired path system exists.

The rest of the paper is organized as follows. In the next section, we will state some related results on graph linkages. In Section 3, we will establish the number theoretical result that will serve as a foundation for our needs. We will then prove Theorem 2 in Section 3 and Theorem 4 in Section 4.

2 Related results on connectivities and linkages

Linkage and connectivity are related in a natural way: Every k-linked graph is k-connected. With a slight push, we can show that every k-linked graph is $(2k-1)$-connected. Although the converse is not true, Jung [14] and Larman and Mani [19], independently, proved the existence of a function $f(k)$ such that every $f(k)$-connected graph is k-linked. Bollobás and Thomason [3] showed that every $22k$-connected graph is k-linked, which was the firs linear upper bound on the connectivity implying graphs are k-linked. Recently, Kawarabayashi, Kostochka and Yu [17] proved that every $12k$-connected graph is k-linked. More recently, Thomas and Wollan [23] proved that every $10k$-connected graph is k-linked. Actually, they proved the following stronger statement.

Theorem 6 (Thomas and Wollan [23]) *For every $2k$-connected graph G, if $||G|| \geq 5k|V(G)|$ then G is k-linked.*

The graph obtained from K_{3k-1} by deleting a matching of k edges is not k-linked. Let $g(G)$ be the girth of G. Mader [20] proved the following theorem.

Theorem 7 (Mader [20]) *There is a constant c such that every $2k$-connected graph with $g(G) \geq c$ is k-linked.*

Recently, Kawarabayashi [15] and [16] showed that the above constant c can be as small as 7.

Theorem 8 (Kawarabayashi [16]) *For every $2k$-connected graph G, if $g(G) \geq 7$ and $k \geq 21$, then G is k-linked.*

We should also mentioned some related work on star diameters of graphs. In the definition of k-linked graphs, it is required that the k paths link distinct vertices to distinct vertices. What happens if one allows repetition among the vertices but require the paths be internally disjoint? A special case is to find internally node-disjoint paths from one vertex x to k other vertices $y_1, \cdots, y_k$ (which may have repetition). Any collection of such paths is called a *star container*. The length of a container is the maximum length of its paths. The *star distance* from x to $y_1, \cdots, y_k$, denoted by $d(x; y_1, \cdots, y_w)$, is the minimum length among all the star containers from x to $y_1, \cdots, y_k$. The *star diameter* of G is defined to be the maximum of $d(x; y_1, \cdots, y_k)$ for all vertices $x, y_1, \cdots, y_k$. Containers and star diameters of graphs are studied in several papers, see for example [12, 13] for general graphs and [9, 10] for Cayley graphs over Abelian groups

3 Sumsets of integers

Let $m \geq 2$ and $k \geq 1$ be two integers and let $A_1, A_2, \ldots, A_k$ be k nonempty sets of integers. The sumset $A_1 + A_2 + \cdots + A_k$ is defined as:

$$A_1 + A_2 + \cdots + A_k = \{a_1 + a_2 + \cdots + a_k \mid a_i \in A_i, i \in [1, k]\}.$$

In the following, we require that 0 is in each subset A_i, so the sumset contains all A_i.

For a positive integer m, $\mathbb{Z}/(m)$ denote the congruence classes of integers modulo m. We often use $0, 1, \cdots, m-1$ as the representatives of all the classes and simply write

$\mathbb{Z}/(m) = \{0, 1, \cdots, m-1\}$. For any set A of integers, we denote by $A \bmod m$ for the set $\{a \bmod m | a \in A\}$. We are interested in the case when $\mathbb{Z}/(m) = A_1 + A_2 + \cdots + A_k \bmod m$, that is, every congruence class of $\mathbb{Z}/(m)$ is represented by an integer in $A_1 + A_2 + \cdots + A_k$. Erdős and Heilbronn studied the case when $m = p$ is a prime and each A_i has two elements.

Theorem 9 (Erdős and Heilbronn [8]) *Let p be a prime number and let $A_i = \{0, a_i\}$, $i = 1, 2, \cdots, k$, where a_1, a_2, $\cdots$, a_k are nonzero and distinct modulo p. If $k \geq 3\sqrt{6p}$, then $\mathbb{Z}/(p) = A_1 + A_2 + \cdots + A_k \bmod p$.*

Cauchy and Devenport obtained the following result for the case $m = p$ is a prime.

Theorem 10 (Cauchy and Davenport [4, 5]) *Let p be a prime number and let A and B be two subsets of $\mathbb{Z}/(p)$. Then $|A + B| \geq \min\{p, |A| + |B| - 1\}$.*

Applying the above result, Olson [21] obtained the following result, which was stated more generally in term of the finite abelian group.

Theorem 11 (Olson [21]) *Let p be a prime and let $a_1, \cdots, a_{p-1}$ be $p - 1$ nonzero integers (not necessarily distinct) such that $a_i \not\equiv 0 \bmod p$ for each $i \in [1, p-1]$. Then, $\sum_{i=1}^{\ell}\{0, a_i\} \bmod m = \mathbb{Z}/(m)$.*

We generalize Theorem 11 from prime to general natural numbers. This generalization provides a foundation for our results on linkage with modulo constrains. In order to state our generalization, the following lemma is needed.

Lemma 1 *Let $m \geq 2$ be a positive integer and let $A \subset \mathbb{Z}$ such that for each prime factor p of m there is $a \in A$ such that a is not divisible by p. Then, for any $S \subset \mathbb{Z}$, if $S + A \equiv S \bmod m$ then $S \equiv \{0, 1, ..., m-1\} \mod m$.*

Proof. Since all the computation below is done in $\mathbb{Z}/(m)$, we'll omit the word "mod m". We may assume that the integers in S are reduced modulo m, so S is a subset of $\mathbb{Z}/(m)$. For any prime factor p of m, let $a \in A$ not divisible by p. Then the additive subgroup H_p of $\mathbb{Z}/(m)$ generated by a has order divisible by p^e where p^e is the highest power of p in m. Now the equation $S + A = S$ implies that $a + r$ in S for all $r \in S$. So for any $r \in S$, we have $a + r \in S$, which in turn implies $a + (a + r) = 2a + r \in S$, and $3a + r \in S, \ldots, ma + r \in S$. Hence the coset $r + H_p$ is contained in S. This means that S is a union of cosets of H_p. This implies that $|S|$ is a multiple of $|H_p|$, hence $|S|$ is divisible by p^e. Since this is true for each prime factor p of m, we see that $|S|$ is divisible by m. But S has at most m elements (mod m). We see that S has exactly m elements mod m, so the lemma follows. $\square$

Theorem 12 *Let $m \geq 2$ be a positive integer and $A_1, ..., A_{m-1}$ be subsets of $\mathbb{Z}$ each containing 0. If for each prime factor p of m and for each i there is an element $a_i \in A_i$ not divisible by p, then $\mathbb{Z}/(m) = A_1 + \cdots + A_{m-1} \bmod m$.*

Proof. Let $S_i = A_1 + \cdots + A_i \bmod m$ for each $1 \leq i \leq m-1$. All the sums are computed modulo m, namely in $\mathbb{Z}/(m)$. Since $0 \in A_i$ for each $1 \leq i \leq m-1$, we have

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_{m-1} \subseteq \mathbb{Z}/(m).$$

If $S_i = S_{i+1}$, i.e. $S_i = S_i + A_{i+1}$ for some $i < m-1$, then Lemma 1 tells us that S_i is already equal to $\mathbb{Z}/(m)$, hence $S_{m-1} = \mathbb{Z}/(m)$.

Otherwise S_{i+1} contains at least one more element than S_i for each $i < m-1$. Hence S_{m-1} has at least

$$m - 2 + |S_1| = m - 2 + |A_1| \geq m - 2 + 2 = m$$

elements, which means $S_{m-1} = \mathbb{Z}/(m)$ as claimed. □

4 Proof of Theorem 2

The following lemma plays a key role in our operations on path adjustments to obtain a path with the desired length.

Lemma 2 *Let $m \geq 3$ be an odd integer, G be a graph, and*

$$R := u_1 v_1 w_1 z_1 u_2 v_2 w_2 z_2 \cdots u_{m-1} v_{m-1} w_{m-1} z_{m-1} u_m$$

be a path in G from u_1 to u_m. If there are vertex-disjoint paths $P_1, P_2, \ldots, P_{m-1}$, $Q_1, Q_2, \ldots$, Q_{m-1} such that P_i joins u_i with w_i, Q_i joins v_i with z_i, $V(P_i \cap R) = \{u_i, w_i\}$, and $V(Q_i \cap R) = \{v_i, z_i\}$ for each $i = 1, 2, \ldots, m-1$, then for each integer d there exists a path R_d from u_1 and u_m such that $\ell(R_d) \equiv d \bmod m$.

Proof. Let $p_1, \ldots, p_r$ be all distinct prime factors of m. Since m is odd, $p_j \geq 3$ for each $1 \leq j \leq r$. For each $i = 1, 2, \ldots, m-1$ and $j = 1, \ldots, r$, set

$$R_{ij} = \begin{cases} u_i \overrightarrow{P_i} w_i z_i & \text{if } \ell(P_i) \not\equiv 2 \bmod p_j, \\ u_i v_i \overrightarrow{Q_i} z_i, & \text{if } \ell(Q_i) \not\equiv 2 \bmod p_j \text{, or} \\ u_i \overrightarrow{P_i} w_i v_i \overrightarrow{Q_i} z_i, & \text{if } \ell(P_i) \equiv \ell(Q_i) \equiv 2 \mod p_j. \end{cases}$$

Then, $\ell(R_{ij}) \not\equiv \ell(R[u_i, z_i]) = 3 \mod p_j$. Let a_{ij} be an integer with $1 \leq a_{ij} \leq m-1$ such that

$$a_{ij} \equiv \ell(R_{ij}) - \ell(R[u_i, z_i]) \mod m.$$

Then,

$$a_{ij} \not\equiv 0 \mod p_j. \tag{1}$$

For $i = 1, 2, \ldots, m-1$, set

$$A_i = \{0\} \cup \{a_{ij} \mid \text{for all } 1 \le j \le r\}$$

Note that $a_{ij_1} = a_{ij_2}$ may happen for $j_1 \neq j_2$. By Lemma 12, $A_1 + \cdots + A_{m-1} \bmod m = \mathbb{Z}/(m)$. So, there exists a subset $I \subseteq [1, m-1]$ and there exists $a_{ij_i} \in A_i - \{0\}$ for each $i \in I$ such that

$$d - \ell(R) \equiv \sum_{i \in I} a_{ij_i} \mod m.$$

Let R_d be the path obtained from R by replacing $\cup_{i\in I} R[u_i, z_i]$ with $\cup_{i\in I} R_{ij_i}[u_i, z_i]$. Then,

$$\begin{aligned}
\ell(R_d) &= \ell(R) + \sum_{i\in I}(\ell(R_{ij_i}[u_i, z_i]) - \ell(R[u_i, z_i])) \\
&= \ell(R) + \sum_{i\in I} a_{ij_i} \\
&\equiv d \mod m.
\end{aligned}$$

This completes the proof of Lemma 2. □

Now we are ready to prove Theorem 2. Let G be any $f(m_1, m_2, \ldots, m_k)$-connected graph, X a set of $2k$ specified vertices $\{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k\}$, and $(d_1, d_2, \ldots, d_k)$ be a k-tuple of natural numbers. We wish to find k disjoint paths $T_1, T_2, \ldots, T_k$ such that T_i joins x_i and y_i and $\ell(T_i) \equiv d_i \bmod m$ for each $1 \le i \le k$. We distinguish the following two cases.

Case 1. $G - X$ contains $m_1 + \cdots + m_k - k$ vertex-disjoint cycles $\{C_{ij_i}\}_{1\le i\le k, 1\le j_i\le m_i-1}$ such that $|C_{ij_i}| \le 6$ for each $i \in [1, k]$ and $j_i \in [1, m_i - 1]$.

Let $u_{i1} := x_i$ and $v_{im_i} := y_i$. For each pair (i, j) with $m_i > 1$ and $1 \le j \le m_i - 1$, let v_{ij} be an arbitrary vertex in $V(C_{ij})$ and let

$$u_{i,j+1} = \begin{cases} v_{ij}^+, & \text{if } |C_{ij}| = 3, 4 \\ (v_{ij}^+)^+, & \text{if } |C_{ij}| = 5, 6, \end{cases}$$

where v_{ij}^+ is the successor of v_{ij} in C_{ij}. Set $a_{ij} = \ell(v_{ij}\overleftarrow{C_{ij}}u_{i,j+1}) - \ell(v_{ij}\overrightarrow{C_{ij}}u_{i,j+1})$. Then,

$$a_{ij} = \begin{cases} 1, & \text{if } |C_{ij}| = 3, 5 \\ 2, & \text{if } |C_{ij}| = 4, 6 \end{cases}.$$

Let $X^* := \cup_{i=1}^k \cup_{j=1}^{m_i-1} (V(C_{ij}) - \{v_{ij}, u_{i,j+1}\})$ and $G^* = G - X^*$. Since $|C_{ij}| \le 6$ for each $i \in [1, k]$ and $j \in [1, m_i - 1]$, we have that $|X^*| \le \sum_{i=1}^k 4(m_i - 1)$. Since $(14\sum_{i=1}^k m_i - 4k) - \sum_{i=1}^k 4(m_i - 1) = 10\sum_{i=1}^k m_i$, G^* is $10(m_1 + \cdots + m_k)$-connected. By Theorem 6, G^* is $(m_1 + \cdots + m_k)$-linked and hence there exist $(m_1 + \cdots + m_k)$ vertex-disjoint paths $\{P_{ij_i}\}_{1\le i\le k, 1\le j_i\le m_i}$ in G^* such that P_{ij} joins u_{ij} and v_{ij} (See Figure 1).

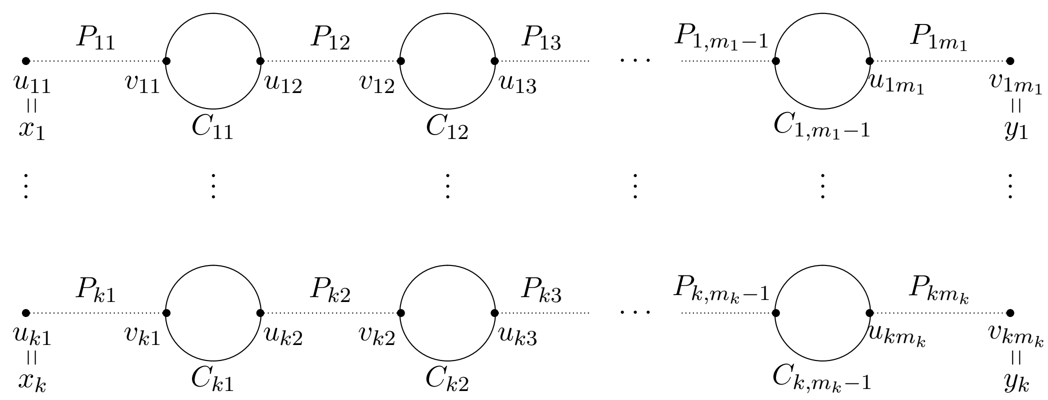

Figure 1

For each $i = 1, 2, \cdots, k$, set

$$R_i := \begin{cases} u_{i1}\overrightarrow{P_{i1}}v_{i1} & \text{if } m_i = 1, \\ u_{i1}\overrightarrow{P_{i1}}v_{i1}\overrightarrow{C_{i1}}u_{i2}\cdots u_{i,m_i-1}\overrightarrow{P_{i,m_i-1}}v_{i,m_i-1}\overrightarrow{C_{i,m_i-1}}u_{im_i}\overrightarrow{P_{im_i}}v_{im_i} & \text{if } m_i > 1. \end{cases}$$

Since $a_{ij} \in \{1, 2\}$, it cannot be divided by any prime factor of m_i. By Theorem 12, there exist $\varepsilon_{i1}, \varepsilon_{i2}, \ldots, \varepsilon_{i,m_i-1} \in \{0, 1\}$ such that

$$\sum_{j=1}^{m_i-1} \varepsilon_{ij}a_{ij} \equiv d_i - \ell(R_i) \mod m_i.$$

Let T_i be the path obtained from R_i by replacing $v_{ij}\overrightarrow{C_{ij}}u_{i,j+1}$ with $v_{ij}\overleftarrow{C_{ij}}u_{i,j+1}$ for each j with $\varepsilon_{ij} = 1$. Then,

$$\ell(T_i) \equiv \ell(R_i) + \sum_{j=1}^{m_i-1} \varepsilon_{ij}a_{ij} \equiv d_i \mod m_i.$$

So, $\{T_1, T_2, \ldots, T_k\}$ is the set of desired paths.

Case 2. $G - X$ contains at most $m_1 + \cdots + m_k - k - 1$ vertex-disjoint cycles of order at most 6.

In this case, there exists an $X^* \subseteq V(G) - X$ such that $|X^*| \leq 6(m_1 + \cdots + m_k - k - 1)$ and the girth of $\hat{G} := G - (X \cup X^*)$ is at least 7. Since G is $\max\{14(m_1 + \cdots + m_k) - 4k,\ 6(m_1 + \cdots + m_k) - 4k + 36\}$-connected, $\hat{G}$ is $\max\{8(m_1 + \cdots + m_k) + 6,\ 42\}$-connected. Since $g(\hat{G}) \geq 7$, by Theorem 8, $\hat{G}$ is $4(m_1 + \cdots + m_k)$-linked. By using $(2m_1 + \cdots + 2m_k - k)$-linkage of $\hat{G}$, we will construct the set $\{T_1, T_2, \ldots, T_k\}$ of required paths in $G - X^*$ as follows.

First, we choose $2k$ distinct vertices $u_{11}, u_{21}, \ldots, u_{k1}, w_{1m_1}, w_{2m_2}, \ldots, w_{km_k}$ in $\hat{G}$ such that for each i, u_{i1} is adjacent to x_i and w_{im_i} is adjacent to y_i. This is doable since every vertex of G has at least $8(m_1 + \cdots + m_k) + 6$ neighbors in $\hat{G}$.

Next, we choose k vertex disjoint path $R_1, R_2, \ldots, R_k$ in $\hat{G} - \{w_{1m_1}, w_{2m_2}, \ldots, w_{km_k}\}$ such that

$$R_i := \begin{cases} u_{i1} & \text{if } m_i = 1, \\ u_{i1}v_{i1}w_{i1}z_{i1} \cdots u_{i,m_i-1}v_{i,m_i-1}w_{i,m_i-1}z_{i,m_i-1}u_{im_i} & \text{if } m_i > 1. \end{cases}$$

This is possible since $\hat{G}$ is $8(m_1 + \cdots + m_k)$-connected.

Since $\hat{G}$ is $4(m_1+\cdots+m_k)$-linked, there exist $(2m_1+2m_2+\cdots+2m_k-k)$ vertex-disjoint paths $\{P_{ij_i}\}_{1\le i\le k, 1\le j_i\le m_i}$ and $\{Q_{ij_i}\}_{1\le i\le k, 1\le j_i\le m_i-1}$ such that for each $i = 1, 2, \ldots, k$

- P_{ij} joins u_{ij} and w_{ij} for $j = 1, 2, \ldots, m_i$ and
- Q_{ij} joins v_{ij} and z_{ij} for $j = 1, 2, \ldots, m_i - 1$.

(See Figure 2).

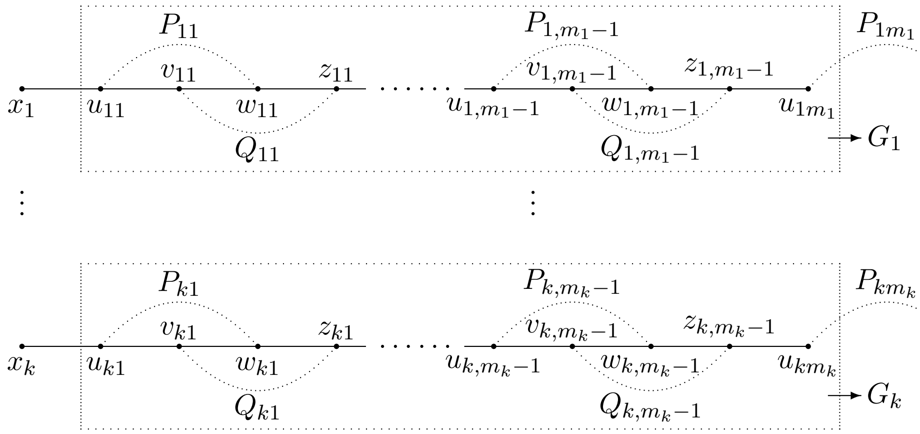

Figure 2

For each i, let G_i be the subgraph of $\hat{G}$ induced by $\cup_{j=1}^{m_i-1}(V(P_{ij} \cup V(Q_{ij}) \cup \{u_{im_i}\})$. Then, $G_1, G_2, \ldots, G_k$ are vertex disjoint subgraphs of $\hat{G} - \{w_{1m_1}, w_{2m_2}, \ldots, w_{km_k}\}$. By applying Lemma 2 with $G := G_i$ and $R := R_i$, we see that G_i contains a path Q_i connecting u_{i1} and u_{im_i} such that

$$\ell(Q_i) \equiv d_i - 2 - \ell(P_{i,m_i}) \mod m_i.$$

Set $T_i := x_i u_{i1} \overrightarrow{Q_i} u_{im_i} \overrightarrow{P_{im_i}} w_{im_i} y_i$. Then, $T_1, T_2, \ldots, T_k$ are k vertex-disjoint paths in G^* such that T_i joins x_i and y_i, and for each $i = 1, 2, \ldots, k$,

$$\ell(T_i) \equiv \ell(Q_i) + 2 + \ell(P_{im_i}) \equiv d_i \mod m_i.$$

This completes the proof of Theorem 2. □

5 Proof of Theorem 4

For a graph G an *odd cycle cover* is a set of vertices $X \subseteq V(G)$ such that $G - X$ is bipartite. Recall that the *bipartite index* $bi(G) = \min\{|X| \mid X \text{ is a odd cycle cover of } G\}$. In order to prove Theorem 4, we need the following three results.

Theorem 13 (Geelen et al, [11]) *For any set S of vertices of a graph G, either*

- *there are k vertex-disjoint odd S-paths, i.e., k disjoint paths each of which has an odd number of edges and both its endpoints in S, or*
- *there is a vertex set X of order at most $2k-2$ such that $G - X$ contains no such paths.*

Theorem 14 *Let G be a triangle-free graph and $c > 0.1$ be a constant such that $|G| \geq 20c$ and $||G|| > 50c|G| - 900c^2$. Then G contains an $\lceil 18c \rceil$-connected subgraph H such that $||H|| \geq \lceil 45c|H| \rceil$ and minimum degree $\delta(H) \geq \lceil 50c \rceil$.*

Proof. Let H be a subgraph of G such that

(a) $|H| \geq 20c$,

(b) $||H|| > 50c|H| - 900c^2$, and

(c) $n := |H|$ is minimal subject to (a) and (b).

We will show that H is as desired.

Claim 1 $|H| > 180c$.

Since H is triangle-free, by the well-known Turan Theorem on extreme graph theory, $||H|| \leq n^2/4$. Solving the inequality $50cn - 900c^2 < n^2/4$, we obtain either $n < 20c$ or $n > 180c$. Then $n > 180c$ by condition (a). □

Claim 2 *The minimum degree of H is more than $50c$.*

Suppose that H has a vertex v with degree at most $50c$, and let H' be the graph obtained from H by deleting v. Then, $||H'|| \geq ||H|| - 50c > 50c|H'| - 900c^2$. Since $c \geq 0.1$, we have $|H'| = n - 1 > 180c - 1 \geq 20c$. So H' satisfies (a) and (b) and $|H'| = |H| - 1$ which contradicts (c). □

Claim 3 $||H|| > 45c|H|$.

Suppose, to the contrary, $||H|| \leq 45c|H|$. Then, $50cn - 900c^2 < 45cn$, which implies $n < 180c$, a contradiction to Claim 1. $\Box$

Claim 4 *H is $\lceil 18c \rceil$-connected.*

Suppose H is not $\lceil 18c \rceil$-connected. Then, H has a separation (A_1, A_2) such that $A_1 \backslash A_2 \neq \emptyset \neq A_2 \backslash A_1$ and $|A_1 \cap A_2| \leq 18c$. By Claim 2, $|A_1|, |A_2| \geq 50c + 1$. For $i \in 1, 2$, let H_i be a subgraph of H with vertex set A_i such that $H = H_1 \cup H_2$ and $E(H_1) \cap E(H_2) = \emptyset$. If $||H_i|| \leq 50c|H_i| - 900c^2$ for both $i = 1, 2$, then

$$\begin{aligned} 50cn - 900c^2 &< ||H_1|| + ||H_2|| \\ &\leq 50c(n + |A_1 \cap A_2|) - 1800c^2 \\ &\leq 50cn - 900c^2, \end{aligned}$$

a contradiction. Hence, we may assume, without loss of generality, that $||H_1|| > 50c|H_1| - 900c^2$. Recall that $|H_1| = |A_1| \geq 50c + 1$. Then H_1 is a subgraph of G that contradicts (c). This completes the proof of Claim 4, so does Theorem 14. $\Box$

The following is an analogue of Lemma 2 for bipartite graphs.

Lemma 3 *Let $m \geq 2$ be an integer, G be a bipartite graph, and $R := u_1 v_1 w_1 z_1 \cdots u_{m-1} v_{m-1} w_{m-1} z_{m-1} u_m$ be paths in G from u_1 to u_m. If there are vertex-disjoint paths $P_1, \ldots, P_{m-1}$, $Q_1, \ldots, Q_{m-1}$ such that P_j joins u_j with w_j, Q_j joins v_j with z_j, $V(P_i \cap R) = \{u_i, w_i\}$, and $V(Q_i \cap R) = \{v_i, z_i\}$ for each $j \in [1, m-1]$, then for each integer d there exists a path R_d in G connecting u_1 and u_m such that $\ell(R_d) \equiv 2d \bmod 2m$.*

Proof. Let $p_1, \ldots, p_r$ be all prime factors of m. For $i = 1, \ldots, m-1$ and $j = 1, \ldots, r$, set

$$R_{ij} = \begin{cases} u_i \overrightarrow{P_i} w_i z_i & \text{if } \ell(P_i) \not\equiv 2 \bmod 2p_j, \\ u_i v_i \overrightarrow{Q_i} z_i & \text{if } \ell(Q_i) \not\equiv 2 \bmod 2p_j, \text{ and} \\ u_i \overrightarrow{P_i} w_i v_i \overrightarrow{Q_i} z_i & \text{if } \ell(P_i) \equiv \ell(Q_i) \equiv 2 \mod 2p_j. \end{cases}$$

Then, $\ell(R_{ij}) \not\equiv \ell(R[u_i, z_i]) = 3 \mod 2p_j$ since $2p_j \geq 4$. Since G is bipartite, $\ell(R_{ij}) - \ell(R[u_i, z_i])$ is even. Let a_{ij} be an integer with $1 \leq a_{ij} \leq m-1$ such that

$$\ell(R_{ij}) - \ell(R[u_i, z_i]) \equiv 2a_{ij} \mod 2m.$$

Then,

$$a_{ij} \not\equiv 0 \mod p_j. \tag{2}$$

By an argument similar to that in the proof of Lemma 2, we can find a path R_d in G connecting u_1 and u_m such that $\ell(R_d) - \ell(R[u_1, u_m]) \equiv 2[d - 2(m-1)] \bmod 2m$. Note that $\ell(R[u_1, u_m]) = 4(m-1)$. So R_d is the desired path. $\square$

We now turn to the proof of Theorem 4. Let X be a set of $2k$ specified vertices $\{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k\}$ of G and let $(d_1, d_2, \ldots, d_k)$ be a k-tuple of nonnegative integers. We wish to find k vertex-disjoint paths $T_1, T_2, \ldots, T_k$ such that T_i joins x_i and y_i and $\ell(T_i) \equiv d_i \bmod m_i'$ for each $1 \le i \le k$. We split the proof into two cases according to $bi(G) \ge 4k + 2\ell - 1$ or $bi(G) \le 4k + 2\ell - 2$.

Case 1. $bi(G) \ge 4k + 2\ell - 1$.

Let H be a spanning bipartite subgraph of G with maximum number of edges. As observed by Erdős, the minimum degree of H is at least $\lceil 45(m_1 + \cdots + m_k)/2 \rceil$, and hence H has at least $\lceil 45(m_1 + \cdots + m_k)|H|/4 \rceil$ edges.

Applying Theorem 14 with $c = 9(m_1 + \cdots + m_k)/40$, we see that H contains a $4(m_1 + \cdots + m_k)$-connected bipartite subgraph K such that $||K|| \ge 10(m_1 + \cdots + m_k)|K|$ and minimum degree $\delta(G) \ge 11(m_1 + \cdots + m_k)$. By Theorem 6, K is $2(m_1 + \cdots + m_k)$-linked.

We say that a path P in G is a *parity breaking path* for K if $E(P) \cap E(K) = \emptyset$ and $K \cup P$ contains an odd cycle. This parity breaking path may be just a single edge. Since K is a connected bipartite graph, for any parity breaking path P for K there exist two distinct vertices x, y in P such that

- $P[x, y]$ is a parity breaking path for K;
- $V(P[x, y]) \cap V(K) = \{x, y\}$;
- For every trail T in K connecting x and y, $\ell(T)$ and $\ell(P)$ have different parities.

Claim 5 *There are at least $2k + \ell$ vertex-disjoint parity breaking paths for K.*

Proof. Let S be one of the partite sets of K. Then $|S| \ge \delta(K) \ge 11(m_1 + \cdots + m_k) \ge 4k + 2\ell + 2$. We shall apply Theorem 13 to G and S. If there are at least $2k + \ell$ vertex-disjoint odd S-paths in G, we can clearly find $2k + \ell$ vertex-disjoint parity breaking paths for K since K is a connected bipartite graph. Otherwise, there is a vertex set R of order at most $4k + 2\ell - 2$ such that $G - R$ has no any odd S-path. Since $|R| \le 4k + 2\ell - 2$ and G is $45(m_1 + \cdots + m_k)$-connected, then graph $G - R$ is 2-connected. If there is an odd cycle C in $G - R$, then we can take two disjoint paths from C to $S - R$, and this would give an odd S-path, a contradiction. This implies that $G - R$ is bipartite, a contradiction to $bi(G) \ge 4k + 2\ell - 1$. $\square$

Let $P_j = P_j[s_j, t_j]$, $j = 1, 2, \ldots, 2k + \ell$, be $2k + \ell$ vertex-disjoint parity breaking paths for K in G such that $V(P_j) \cap V(K) = \{s_j, t_j\}$. We shall construct k vertex-disjoint desired paths by using K and $P_1, P_2, \ldots, P_{2k+\ell}$. Let $E^* := \cup_{j=1}^{2k+\ell} E(P_j)$. Since

G is $45(m_1 + \cdots + m_k)$-connected, there are $2k$ vertex-disjoint paths $\mathcal{W}$=$\{W_1, \ldots, W_{2k}\}$ joining X and K. Choose $\mathcal{W}$ such that $\sum_{i=1}^{2k} |E(W_i) - E^*|$ achieves the minimum value. For $i = 1, 2, \ldots, k$, we assume W_i joins x_i with x_i' and W_{i+k} joining y_i with y_i', where ${x_i}'$ is the only vertex of W_i in K and ${y_i}'$ is the only vertex of W_{i+k} in K. Note that if $x_i \in K$, then $x_i' = x_i$ and $W_i = \{x_i\}$. Similarly, if $y_i \in K$, then $y_i' = y_i$ and $W_{i+k} = \{y_i\}$. Set

$$\begin{array}{rcl} J_0 & = & \{j \mid j \in [1, 2k+\ell] \text{ and } P_j \text{ doesn't intersect any path in } \mathcal{W}\}, \\ J_1 & = & \{j \mid j \in [1, 2k+\ell] \text{ and } P_j \text{ intersects exactly one path in } \mathcal{W}\}, \\ J_2 & = & \{j \mid j \in [1, 2k+\ell] \text{ and } P_j \text{ intersects at least two paths in } \mathcal{W}\}. \end{array}$$

Then,

$$|J_0| + |J_1| + |J_2| = 2k + \ell. \tag{3}$$

For each $j \in J_2$, let W and W' be the paths in $\mathcal{W}$ that intersect P_j as close as possible (on P_j) to s_j and to t_j, respectively. Then, the minimality of $\sum_{i=1}^{2k} |E(W_i) - E^*|$ implies that both W and W' follow the path P_j and end at the end-vertices of P_j. Thus,

$$s_j, t_j \in \{x_1', x_2', \ldots, x_k', y_1', y_2', \ldots, y_k'\}, \ \forall j \in J_2. \tag{4}$$

For each $j \in J_1$, let W be the only path in $\mathcal{W}$ that intersect P_j. Then, the minimality of $\sum_{i=1}^{2k} |E(W_i) - E^*|$ implies that W follow the path P_j and end at one of the end-vertices. Thus,

$$\{s_j, t_j\} \cap \{x_1', x_2', \ldots, x_k', y_1', y_2', \ldots, y_k'\} \neq \emptyset, \ \forall j \in J_1. \tag{5}$$

It follows from (4) and (5) that $|J_1| + 2|J_2| \leq 2k$. This together with (3) implies $|J_0| \geq |J_2| + \ell \geq \ell$.

Renaming $P_1, P_2, \ldots, P_{2k+\ell}$ if necessary, we assume that $J_0 \supseteq [1, \ell]$. By using $\{P_i\}_{1 \leq i \leq \ell}$, $\{W_i\}_{1 \leq i \leq 2k}$, and $2(m_1 + \cdots + m_k)$-linkage of K, we will construct the required paths $T_1, T_2, \ldots, T_k$ in G as follows.

First, we choose k vertex-disjoint paths $R_1, R_2, \ldots, R_k$ in $K^* := K - \cup_{i=1}^{\ell} \{s_i, t_i\} - \cup_{i=1}^{k} \{y_i'\}$ such that

$$R_i := \begin{cases} u_{i1} & \text{if } m_i = 1, \text{ and} \\ u_{i1} v_{i1} w_{i1} z_{i1} \cdots u_{i,m_i-1} v_{i,m_i-1} w_{i,m_i-1} z_{i,m_i-1} u_{im_i} & \text{if } m_i \geq 2, \end{cases}$$

where $u_{i1} := x_i'$. This is possible since minimum degree $\delta(K) \geq 11(m_1 + \cdots + m_k)$.

For each $i \in [1, \ell]$, let

$$\begin{array}{rcl} \alpha_i & = & \begin{cases} 0 & \text{if } x_i' \text{ and } y_i' \text{ are in the same partite set of } K, \text{ and} \\ 1 & \text{otherwise,} \end{cases} \quad \text{and} \\ \beta_i & = & \begin{cases} 0 & \text{if } d_i \equiv \alpha_i + \ell(W_i) + \ell(W_{i+k}) \mod 2, \text{ and} \\ 1 & \text{otherwise.} \end{cases} \end{array}$$

From the definitions above, for each $i \in [1, \ell]$, we have that

$$\alpha_i + \ell(W_i) + \ell(W_{i+k}) \equiv d_i + \beta_i.$$

Set $I_1 := \{i \mid i \in [1,\ell], \beta_i = 1\}$ and $I_0 := [1,k] - I_1$. Since K is $2(m_1 + \cdots + m_k)$-linked, there exist $(2m_1 + 2m_2 + \cdots + 2m_k - |I_0|)$ vertex-disjoint paths $\{P_{ij}, Q_{ij}\}_{1 \le i \le k, 1 \le j \le m_i - 1}$, $\{T_{i0}\}_{i \in I_0}$ and $\{T_{i1}^{(1)}, T_{i1}^{(2)}\}_{i \in I_1}$ in K such that

- P_{ij} joins u_{ij} with w_{ij} and Q_{ij} joins v_{ij} with z_{ij} for each pair (i,j) with $1 \le i \le k$ and $1 \le j \le m_i - 1$,
- T_{i0} joins u_{im_i} with y_i' for each $i \in I_0$, and
- $T_{i1}^{(1)}$ joins u_{im_i} with s_i and $T_{i1}^{(2)}$ joins t_i with y_i' for each $i \in I_1$.

(See Figure 3).

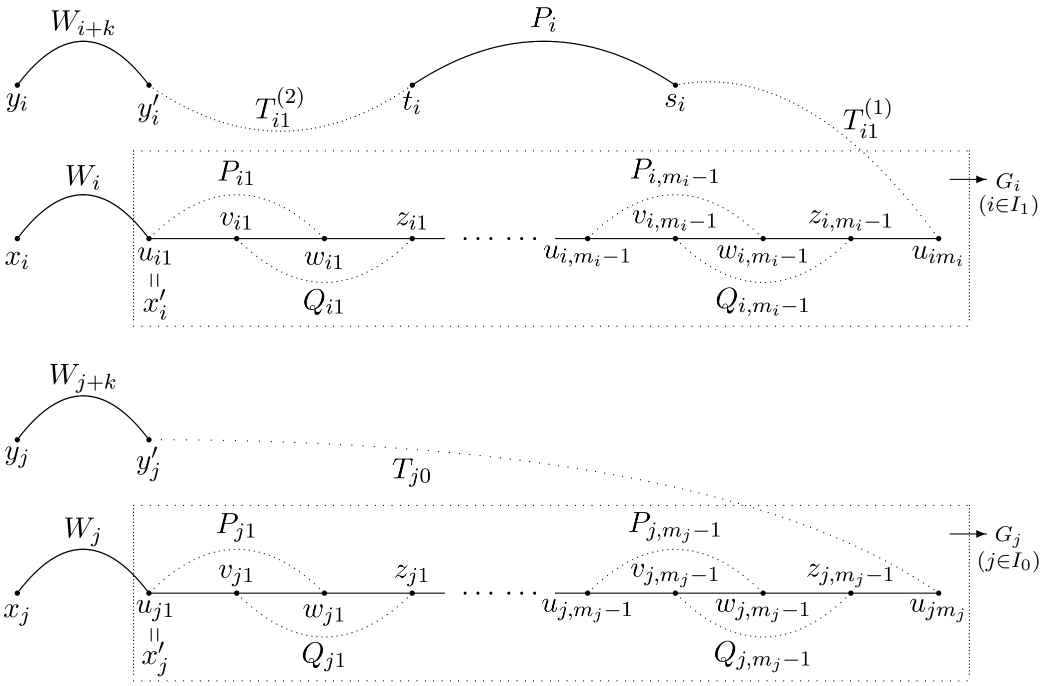

Figure 3

To construct an (x_i, y_i)-path of length d_i modulo m_i', we shall first find an (x_i, y_i)-path with length d_i modulo 2 for each $i \in [1,\ell]$. Set

$$Q_i = \begin{cases} x_i \overrightarrow{W_i} x_i' \overrightarrow{R_i} u_{im_i} \overrightarrow{T_{i0}} y_i' \overleftarrow{W_{i+k}} y_i & \text{if } i \in I_0, \text{ and} \\ x_i \overrightarrow{W_i} x_i' \overrightarrow{R_i} u_{im_i} \overrightarrow{T_{i1}^{(1)}} s_i \overrightarrow{P_i} t_i \overrightarrow{T_{i1}^{(2)}} y_i' \overleftarrow{W_{i+k}} y_i & \text{if } i \in I_1. \end{cases}$$

Claim 6 *For each $i \in [1,\ell]$, Q_i is an (x_i, y_i)-path of length d_i modulo 2.*

Proof. For each $i \in [1, \ell]$, let

$$Q_i' = x_i \overrightarrow{W_i} x_i' \overrightarrow{W} y_i' \overleftarrow{W_{i+k}} y_i,$$

where W is an arbitrary path in K connecting x_i' and y_i'. Since K is a bipartite graph, W has length α_i modulo 2. This implies

$$\ell(Q_i') \equiv \ell(W_i) + \alpha_i + \ell(W_{i+k}) \equiv d_i + \beta_i \mod 2.$$

If $\beta_i = 0$, then $\ell(Q_i) \equiv \ell(Q_i') \equiv d_i + \beta_i \equiv d_i \bmod 2$. Now, assume $\beta_i = 1$. Then,

$$\ell(Q_i) + \ell(Q_i') = 2\ell(W_i) + 2\ell(W_i') + \ell(Q_i[x_i', s_i]) + \ell(P_i) + \ell(Q_i[t_i, y_i']) + \ell(Q_i'[x_i', y_i']).$$

Note that P_i is a parity breaking path for K and $s_i \overleftarrow{Q_i} x_i' \overrightarrow{Q_i'} y_i' \overleftarrow{Q_i} t_i$ is a trail in K, which has the same end vertices as P_i, so that the sum of their lengths is 1 mod 2, i.e.,

$$\ell(Q_i[x_i', s_i]) + \ell(P_i) + \ell(Q_i[t_i, y_i']) + \ell(Q_i'[x_i', y_i']) \equiv 1 \mod 2.$$

Therefore, $\ell(Q_i) + \ell(Q_i') \equiv 1 \bmod 2$, which implies that

$$\ell(Q_i) \equiv \ell(Q_i') + 1 \equiv (d_i + \beta_i) + 1 \equiv d_i \mod 2.$$

This completes the proof of Claim 6. □

It follows from Claim 6 that $Q_1, Q_2, \ldots, Q_\ell$ are ℓ vertex-disjoint paths such that Q_i joins x_i with y_i, and Q_i has length d_i modulo 2 for $i = 1, 2, \ldots, \ell$. So

$$d_i - \ell(Q_i) \equiv 2b_i \mod 2m_i, \quad \text{for each } i = 1, 2, \ldots, \ell, \tag{6}$$

where b_i is an integer with $0 \leq b_i \leq m_i - 1$. Thus, for each $i \in [1, \ell]$ with $m_i = 1$, we have

$$\ell(Q_i) \equiv d_i \mod m_i' \tag{7}$$

Since $m_i' = m_i$ for each $i \geq \ell + 1$, (7) is true for every i with $m_i = 1$.

For each i with $m_i \geq 2$, set

$$G_i := G[\cup_{j=1}^{m_i - 1}(V(P_{ij}) \cup V(Q_{ij}) \cup \{u_{im_i}\})].$$

By using Lemma 3 (for $i \in [1, \ell]$) and Lemma 2 (for $i \in [\ell + 1, k]$) with G_i and R_i, we find a path R_i' in G_i connecting u_{i1} and u_{im_i} such that

$$\ell(R_i') - \ell(R_i) \equiv \begin{cases} 2b_i \mod 2m_i & \text{if } i \in [1, \ell] \\ d_i - \ell(Q_i) \mod m_i & \text{if } i \in [\ell + 1, k] \end{cases} \tag{8}$$

Let T_i be the path obtained from Q_i by replacing R_i with R_i'. By (6) and (8), we have for each i with $m_i > 1$ that

$$\ell(T_i) \equiv \ell(Q_i) + (\ell(R_i') - \ell(R_i)) \equiv d_i \mod m_i'. \tag{9}$$

For each i with $m_i = 1$, set $T_i := Q_i$. By (7) and (9), $T_1, T_2, \ldots, T_k$ are k vertex-disjoint paths in G such that T_i joins x_i with y_i and T_i has length d_i modulo m_i'. So, $T_1, T_2, \ldots, T_k$ are the desired paths. This completes the proof of Case 1 of Theorem 4.

Case 2. $bi(G) \leq 4k + 2\ell - 2$.

In this case, we have $2k + 2\ell - 3 + \delta(m_1, \ldots, m_\ell) \leq bi(G) \leq 4k + 2\ell - 2$. We will use the technic developed by Thommassen in [25] to obtain k desired paths. Let $U = \{u_1, u_2, \ldots, u_t\}$ be a minimum odd cycle cover of G, where $t \in [2k + 2\ell - 3, 4k + 2\ell - 2]$. Then, $H_0 := G - U$ is a $39(m_1 + \cdots + m_k)$-connected bipartite graph. For $i = 1, 2, \ldots, t$, let H_i be the bipartite graph obtained from H_{i-1} by adding u_i to the side of the bipartition of H_{i-1} which has less neighbors of u_i and all edges of G with u_i as one endvertex and the other endvertex on the other partite set of K. Since every vertex of U has at least $39(m_1 + \cdots + m_k)$ neighbors in H, $K := H_t$ is a spanning subgraph of G with connectivity at least $19(m_1 + \cdots + m_k)$. By the definition of K, every edge of $G - E(K)$ joins vertices on the same side of the bipartition of K. Then,

$$\begin{aligned} ||K|| &\geq ||H|| + t \cdot 19(m_1 + \cdots + m_k) \\ &\geq 39/2(m_1 + \cdots + m_k)(|K| - t) + t \cdot 19(m_1 + \cdots + m_k) \\ &\geq 19(m_1 + \cdots + m_k)|K|. \end{aligned}$$

By Theorem 6, K is $3(m_1 + \cdots + m_k)$-linked.

It follows from the choice of K that every edge of $G - E(K)$ is a parity breaking path for K in G. If $G - X - E(K)$ has at least ℓ pairwise independent edges, then by an argument similar to that in the proof of Case 1, we can find k desired paths. So assume that no such ℓ edges exist. Then, $G - X$ has a set A of at most $2\ell - 2$ vertices meeting all edges in $G - X - E(K)$. We may assume that $G - X - E(K)$ has $\ell - 1$ pairwise independent edges whose set of ends in A, since otherwise A has only $2\ell - 4$ vertices and $G - (A \cup X)$ is bipartite, a contradiction to $bi(G) \geq 2k + 2\ell - 3 + \delta(m_1, \ldots, m_\ell)$.

Furthermore, if there are any edge of $G - A - \{x_{\ell+1}, y_{\ell+1}, \ldots, x_k, y_k\} - E(K)$ with only one end in $X_\ell := \{x_1, y_1, x_2, y_2, \ldots, x_\ell, y_\ell\}$ then using such an edge and our set of $\ell - 1$ independent edges in A we can again use the technic developed in Case 1 to find the desired paths. So assume that no such edge exists. So $G - A - (X - \{x_1\})$ is a bipartite graph. Hence

$$2k + 2\ell - 3 + \delta(m_1, \ldots, m_\ell) \leq bi(G) \leq |A| + |X - \{x_1\}| \leq 2k + 2\ell - 3,$$

which in turn shows $\delta(m_1, \ldots, m_\ell) = 0$. Therefore, $\min\{m_1, \ldots, m_\ell\} = 1$. Assume, without loss of generality, $m_1 = 1$.

If any two vertices of X_ℓ are non-adjacent or on opposite side of K, then $G - A - (X - \{x, y\})$ is bipartite contradicting $bi(G) \geq 2k + 2\ell - 3 + \delta(m_1, \ldots, m_\ell)$. We then assume that $G[X_\ell]$ is complete and all vertices of X_ℓ are on the same side of K. Using edge x_1y_1 and our set of $\ell - 1$ independent edges in A we can again find the desired paths by simply setting $T_1 := x_1y_1$, which completes the proof of Theorem 4. $\square$

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