# Stenberg's sufficiency criteria for the LBB condition for Axisymmetric Stokes Flow

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**Abstract**

In this article we investigate the LBB condition for axisymmetric flow problems. Specifically, the sufficiency condition for approximating pairs to satisfy the LBB condition established by Stenberg in the Cartesian coordinate setting is presented for the cylindrical coordinate setting. For the cylindrical coordinate setting, the Taylor-Hood ($k = 2$) and conforming Crouzeix-Raviart elements are shown to be LBB stable. A priori error bounds for approximations to the axisymmetric Stokes flow problem using Taylor-Hood and Crouzeix-Raviart elements are given. The computed numerical convergence rates for the error for an axisymmetric Stokes flow problem support the theoretical results.

**Key words**. axisymmetric flow; LBB condition; Stenberg sufficiency condition

**AMS Mathematics subject classifications**. 65N30

## 1 Introduction

Accurate numerical simulations of 3-D fluid flow problems is a computationally challenging problem, involving the approximate solution of large (sparse) systems of linear equations. However, in the case the domain of the problem is a volume of revolution about a central axis, and the fluid flow is also invariant with respect to rotation about the central axis, a change of variable from a Cartesian to a cylindrical coordinate system significantly reduces the computational complexity. Specifically, the 3-D fluid flow problem decouples into a 2-D fluid flow problem and a scalar flow equation. However, this transformation from the 3-D problem to a 2-D problem results in differential operators with singularities on the the central axis, requiring the analysis to be done in suitably weighted Sobolev spaces.

In the approximation of a fluid flow problem based on a weak formulation of the modeling equations, specifically those modeling Navier-Stokes and Stokes, an important component in the approximation

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algorithm is ensuring that the velocity and pressure approximation spaces, $X_h \subset X$ and $Q_h \subset Q$, respectively, satisfy the LBB condition, i.e.

$$\inf_{q \in Q_h} \sup_{\mathbf{v} \in X_h} \frac{b(q\,,\,\mathbf{v})}{\|q\|_Q\,\|\mathbf{v}\|_X} \;\geq\; \beta\,, \tag{1.1}$$

for some $\beta \in \mathbb{R}^+$, where in Cartesian coordinates

$$b(q\,,\,\mathbf{v}) \;=\; \int_\Omega q\,\nabla\cdot\mathbf{v}\,d\mathbf{x}\,. \tag{1.2}$$

"Compatible pairs" of velocity and pressure approximation spaces for fluid flow problems in Cartesian coordinates are well documented in the literature, see for example [6, 4]. Commonly used elements include the mini-element, $P1isoP2 - P1$, and Taylor-Hood pairs. There are a number of ways of establishing that (1.1) is satisfied for given approximation spaces $X_h$ and $Q_h$ [6]. Of particular interest in this article is the general sufficient condition derived by Stenberg in [9]. Briefly stated, in [9] Stenberg showed that if the partition of the domain can be classified into a finite number of *macroelements* such that for each macroelement, $\mathcal{M}$, the dimension of

$$\mathcal{N}_\mathcal{M} \;=\; \left\{q \in Q_h \,:\, \int_\mathcal{M} q\,\nabla\cdot\mathbf{v}\,d\mathbf{x} \;=\; 0\,, \forall\mathbf{v} \in \{\mathbf{w} \in X_h \,:\, \mathbf{w}|_{\partial\mathcal{M}} = \mathbf{0}\}\right\} \tag{1.3}$$

is equal to one, then (1.1) is satisfied.

In the case of axisymmetric flow in cylindrical coordinates one requires a 2-D LBB condition (1.1) be satisfied. Here however

$$b(q\,,\,\mathbf{v}) \;=\; b_a(q\,,\,\mathbf{v}) \;:=\; \int_\Omega q\,\nabla_a\cdot\mathbf{v}\,r\,d\mathbf{x} \;+\; \int_\Omega q\,v_r\,d\mathbf{x}\,, \tag{1.4}$$

where $\nabla_a = [\partial/\partial r\,,\partial/\partial z]^T$, $\mathbf{v} = [v_r\,,\,v_z]^T$, $d\mathbf{x} = drdz$, and the function spaces (and norms) for $X$ and $Q$ differ significantly from the Cartesian case. Ruas in [8] showed that (1.1)(1.4) was satisfied for rectangular based $Q_2 - discP_1$ elements, and for $P_2 + \text{bubble} - discP_1$ on a restricted triangulation of $\Omega$. In [2] Belhachmi, Bernardi, Deprais showed that (1.1)(1.4) was satisfied on a regular triangulation of $\Omega$ for $P_1isoP_2 - P_1$ elements (which also implied (1.1)(1.4) for Taylor–Hood $P_2 - P_1$ elements).

In this paper we establish that the sufficient condition of Stenberg also applies to (1.1)(1.4). Using this setting we then show that the LBB condition is satisfied by Taylor–Hood $P_2 - P_1$ elements and the conforming Crouzeix-Raviart $P_2 + bubble - discP_1$ elements on a general triangulation of the domain $\Omega$. For applications where mass conservation is of particular importance using $P_2 + bubble - discP_1$ elements is attractive, as the computed approximations are mass conservative over each triangle in the partition of $\Omega$.

The paper is organized as follows. In the following section we present the axisymmetric Stokes flow problem, introduce the appropriate function space setting, give the corresponding weak formulation, and describe the setting for the finite element approximation. Section 3 contains a discussion of Stenberg's sufficiency condition for the LBB condition and shows how it extends to the axisymmetric setting. In Section 4 we use the Stenberg sufficiency condition to show that the Taylor-Hood ($k = 2$) and the conforming Crouzeix-Raviart elements are LBB stable. Combining the approximation properties derived by Belhachmi, Bernardi, Deprais in [2] with the LBB stability, in Section 5 we give a priori error bounds for the approximation to the axisymmetric Stokes flow problem computed using Taylor-Hood and Crouzeix-Raviart elements. A numerical example is given for which the experimental rates of convergence for the approximation error agree with the theoretically predicted rates.

## 2 Mathematical Preliminaries

In this section we give the mathematical framework for the investigation of the LBB condition (1.1)(1.4). We follow the setting used in [2] for the axisymmetric Stokes problem.

### 2.1 Problem Description

Let $\breve{\Omega} \subset \mathbb{R}^3$ denote a domain symmetric with respect to the z-axis. With respect to cylindrical coordinates, $(r, \theta, z)$, we let $\Omega$ denote the half section of $\breve{\Omega}$, $\Omega := \breve{\Omega} \cap \{(r, 0, z) : r > 0, z \in \mathbb{R}\}$. For the description of the boundary we let $\Gamma := \partial\breve{\Omega} \cap \partial\Omega$, and $\Gamma_0$ the intersection of $\breve{\Omega}$ and the z-axis, $\Gamma_0 := \partial\breve{\Omega} \cap \{(0, 0, z) : z \in \mathbb{R}\}$. Note that $\partial\Omega = \Gamma \cup \Gamma_0$. In addition, we assume that $\Omega$ is a simply connected domain with a polygonal boundary. (See Figure 2.1.)

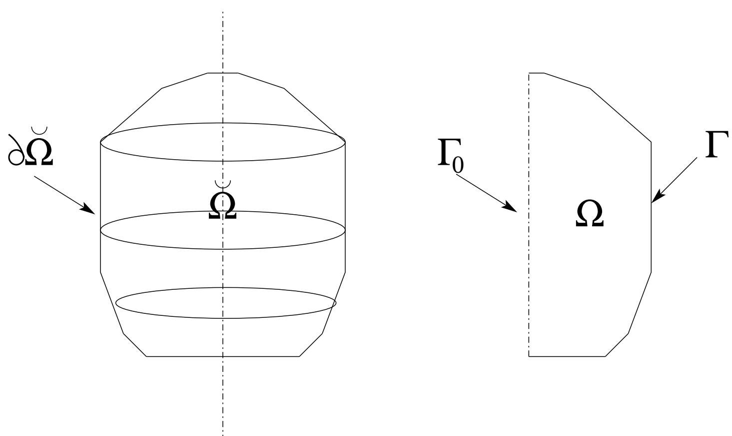


Figure 2.1: Illustration of axisymmetric flow domain.

Consider Stokes equation (in Cartesian coordinates) in $\breve{\Omega}$, subject to homogeneous boundary conditions on $\partial\breve{\Omega}$:

$$-\nabla \cdot \eta \nabla \breve{\mathbf{u}} + \nabla \breve{p} = \breve{\mathbf{f}} \quad \text{in } \breve{\Omega}, \tag{2.1}$$

$$\nabla \cdot \breve{\mathbf{u}} = 0 \quad \text{in } \breve{\Omega}, \tag{2.2}$$

$$\breve{\mathbf{u}} = \mathbf{0} \quad \text{on } \partial\breve{\Omega}, \tag{2.3}$$

where $\breve{\mathbf{u}} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = u_x \mathbf{e}_x + u_y \mathbf{e}_y + u_z \mathbf{e}_z$, for $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ denoting unit vectors in the $x$, $y$ and $z$ directions, respectively.

Multiplying (2.1) through by a suitable smooth function $\breve{\mathbf{v}}$, $\breve{\mathbf{v}}|_{\partial\breve{\Omega}} = \mathbf{0}$, integrating over $\breve{\Omega}$, and multiplying (2.2) through by a suitable smooth function $q$ and integrating over $\breve{\Omega}$ we obtain

$$\int_{\breve{\Omega}} \eta \nabla \breve{\mathbf{u}} : \nabla \breve{\mathbf{v}} \, dV \; - \; \int_{\breve{\Omega}} \breve{p} \nabla \cdot \breve{\mathbf{v}} \, dV \;=\; \int_{\breve{\Omega}} \breve{\mathbf{f}} \cdot \breve{\mathbf{v}} \, dV \qquad (2.4)$$

$$\int_{\breve{\Omega}} \breve{q} \nabla \cdot \breve{\mathbf{u}} \, dV \;=\; 0 \;. \qquad (2.5)$$

Expressing $\breve{\mathbf{u}}$ in cylindrical coordinates, $\breve{\mathbf{u}} = \begin{bmatrix} u_r \\ u_\theta \\ u_z \end{bmatrix} = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z$, and assuming that the flow is axisymmetric, i.e. $\breve{\mathbf{u}}(r, \theta, z) = \mathbf{u}(r, z)$, $\breve{\mathbf{f}}(r, \theta, z) = \mathbf{f}(r, z)$, $\breve{p}(r, \theta, z) = p(r, z)$, $u_r(0, z) = 0$, $u_\theta(0, z) = 0$, equations (2.4)(2.5) transform into

$$\int_\Omega \eta \nabla_a \begin{bmatrix} u_r \\ u_z \end{bmatrix} : \nabla_a \begin{bmatrix} v_r \\ v_z \end{bmatrix} r \, d\mathbf{x} + \int_\Omega \eta \, u_r \, v_r \frac{1}{r} d\mathbf{x} \; - \; \int_\Omega p \, \nabla_a \cdot \begin{bmatrix} v_r \\ v_z \end{bmatrix} r \, d\mathbf{x} \; - \; \int_\Omega p \, v_r \, d\mathbf{x}$$

$$= \int_\Omega \begin{bmatrix} f_r \\ f_z \end{bmatrix} \cdot \begin{bmatrix} v_r \\ v_z \end{bmatrix} r \, d\mathbf{x} \,, \qquad (2.6)$$

$$\int_\Omega \eta \nabla_a \, u_\theta \cdot \nabla_a \, v_\theta \, r \, d\mathbf{x} \; + \; \int_\Omega \eta \, u_\theta \, v_\theta \frac{1}{r} d\mathbf{x} \;=\; \int_\Omega f_\theta \, v_\theta \, r \, d\mathbf{x} \,, \qquad (2.7)$$

$$\int_\Omega q \, \nabla_a \cdot \begin{bmatrix} u_r \\ u_z \end{bmatrix} r \, d\mathbf{x} \; + \; \int_\Omega q \, u_r \, d\mathbf{x} \;=\; 0 \,, \qquad (2.8)$$

$$\text{where} \quad \nabla_a := \begin{bmatrix} \partial/\partial r \\ \partial/\partial z \end{bmatrix} \quad \text{and} \quad d\mathbf{x} := dr \, dz \,.$$

Note that the angular flow equation for $u_\theta$ is decoupled from the flow equations for $u_r$ and $u_z$. For simplicity of our discussion of the LBB condition we will assume $u_\theta = 0$, and let $\mathbf{u} = \begin{bmatrix} u_r \\ u_z \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_r \\ v_z \end{bmatrix}$, etc.

## 2.2 Function Spaces and Weak Formulation

Let $\Theta$ denote a domain in $\mathbb{R}^2$. For any real $\alpha$ and $1 \le p < \infty$, the space ${}_\alpha L^p(\Theta)$ is defined as the set of measurable functions $w$ such that

$$\|w\|_{{}_\alpha L^p(\Theta)} \;=\; \left( \int_\Theta |w|^p \, r^\alpha \, d\mathbf{x} \right)^{1/p} < \infty \,,$$

where $r = r(\mathbf{x})$ is the radial coordinate of $\mathbf{x}$, i.e. the distance of a point $\mathbf{x}$ in $\Theta$ from the symmetry axis. The subspace ${}_1L^2_0(\Theta)$ of ${}_1L^2(\Theta)$ denotes the functions $q$ with weighted integral equal to zero:

$$\int_\Theta q \, r \, d\mathbf{x} \;=\; 0 \,.$$

We define the weighted Sobolev space ${}_1W^{l,p}(\Theta)$ as the space of functions in ${}_1L^p(\Theta)$ such that their partial derivatives of order less that or equal to $l$ belong to ${}_1L^p(\Theta)$. Associated with ${}_1W^{l,p}(\Theta)$ is the semi-norm $|\cdot|_{{}_1W^{l,p}(\Theta)}$ and norm $\|\cdot\|_{{}_1W^{l,p}(\Theta)}$ defined by

$$|w|_{{}_1W^{l,p}(\Theta)} \;=\; \left(\sum_{k=0}^{l} \|\partial_r^k \partial_z^{l-k} w\|^p_{{}_1L^p(\Theta)}\right)^{1/p} , \quad \|w\|_{{}_1W^{l,p}(\Theta)} \;=\; \left(\sum_{k=0}^{l} |w|^p_{{}_1W^{k,p}(\Theta)}\right)^{1/p} .$$

When $p = 2$, we denote ${}_1W^{l,2}(\Theta)$ as ${}_1H^l(\Theta)$. Also used in the analysis is the space ${}_1V^1(\Theta)$, a subset of ${}_1H^l(\Theta)$, given by

$$\begin{aligned} {}_1V^1(\Theta) \;&=\; \left\{w \in {}_1H^1(\Theta) \,:\, w \in {}_{-1}L^2(\Theta)\right\} , \\ \text{with norm} \quad \|w\|_{{}_1V^1(\Theta)} \;&=\; \left(|w|^2_{{}_1H^1(\Theta)} \,+\, \|w\|^2_{{}_{-1}L^2(\Theta)}\right)^{1/2} . \end{aligned}$$

It can be proven that all functions in ${}_1V^1(\Omega)$ have a null trace on $\Gamma_0$, [2, 7].

In order to incorporate the homogeneous boundary condition for the velocity on $\Gamma$, let

$${}_1H^1_\Diamond(\Omega) \;=\; \left\{w \in {}_1H^1(\Omega) \,:\, w \,=\, 0 \;\text{ on } \Gamma\right\} , \quad \text{and} \quad {}_1V^1_\Diamond(\Omega) \;=\; \left\{w \in {}_1V^1(\Omega) \,:\, w \,=\, 0 \;\text{ on } \Gamma\right\} .$$

For convenience of notation, let $X \;:=\; {}_1V^1_\Diamond(\Omega) \times {}_1H^1_\Diamond(\Omega)$ and for $\mathbf{v} \;=\; [v_r,\, v_z]^T$, $\|\mathbf{v}\|_{X(\Theta)} \;=\; \left(\|v_r\|^2_{{}_1V^1(\Theta)} \,+\, |v_z|^2_{{}_1H^1(\Theta)}\right)^{1/2}$, and $Q \;:=\; {}_1L^2_0(\Omega)$ with $\|\cdot\|_Q \;=\; \|\cdot\|_{{}_1L^2(\Omega)}$. When $\Theta = \Omega$, we write $\|\mathbf{v}\|_X \;:=\; \|\mathbf{v}\|_{X(\Theta)}$. With $X$ we associate the innerproduct

$$\langle \mathbf{v},\, \mathbf{w}\rangle_X \;=\; \int_\Omega \left(\nabla_a \mathbf{v} : \nabla_a \mathbf{w} \,+\, \frac{v_r}{r}\frac{w_r}{r}\right) r\, d\mathbf{x} . \tag{2.9}$$

Using as the pivot space $\left({}_1L^2(\Omega)\right)^2$ with innerproduct $\langle \mathbf{f},\, \mathbf{g}\rangle \;:=\; \int_\Omega \mathbf{f}\cdot\mathbf{g}\, r\, d\mathbf{x}$, let $X^*$ denote the dual space of $X$, i.e. $X^*$ is the completion of $\left({}_1L^2(\Omega)\right)^2$ with respect to the norm

$$\|\mathbf{f}\|_{X^*} \;=\; sup_{\mathbf{g}\in X} \frac{\langle \mathbf{f},\, \mathbf{g}\rangle}{\|\mathbf{g}\|_X} .$$

For $\Theta$ a domain in $\mathbb{R}^n$, $n \,=\, 2,\, 3$, we use the standard definitions for $L^2(\Theta)$, $L^2_0(\Theta)$, $H^k(\Theta)$, and $H^k_0(\Theta)$ (see [1]).

The weak axisymmetric formulation for the Stokes equations can be stated as: *Given* $\mathbf{f} \in X^*$, *determine* $(\mathbf{u},\, p) \in (X \times Q)$ *satisfying*

$$a(\mathbf{u},\, \mathbf{v}) \,-\, b_a(p,\, \mathbf{v}) \;=\; \langle \mathbf{f}, \mathbf{v}\rangle_{X^*,X} \quad \forall \mathbf{v} \in X , \tag{2.10}$$

$$b_a(q,\, \mathbf{u}) \;=\; 0 , \quad \forall q \in Q , \tag{2.11}$$

where

$$a(\mathbf{u},\, \mathbf{v}) \;:=\; \int_\Omega \eta \nabla_a \mathbf{u} : \nabla_a \mathbf{v}\, r\, d\mathbf{x} \;+\; \int_\Omega \eta\, u_r\, v_r \frac{1}{r} d\mathbf{x} , \tag{2.12}$$

$$b_a(q,\, \mathbf{v}) \;:=\; \int_\Omega q\, \nabla_a \cdot \mathbf{v}\, r\, d\mathbf{x} \;+\; \int_\Omega q\, v_r\, d\mathbf{x} , \tag{2.13}$$

and $\langle \cdot, \cdot \rangle_{X^*,X}$ denotes the duality pairing between $X$ and $X^*$.

For the discussion of existence and uniqueness of (2.10)(2.11) see [3, 2]. In particular we note that there exists $\beta > 0$ such that

$$\inf_{q \in Q} \sup_{\mathbf{v} \in X} \frac{b_a(q, \mathbf{v})}{\|q\|_Q \, \|\mathbf{v}\|_X} \ \geq \ \beta \,. \tag{2.14}$$

### 2.3 Finite Element Approximation Setting

In this section we describe, as in [2], the setting for the finite element approximation to (2.10)(2.11).

We assume that $\Omega$ is a convex polygonal domain and $(\mathcal{T}_h)_h$ denotes a family of uniformly regular triangulations of $\Omega$ satisfying:

(i) The domain $\overline{\Omega}$ is the union of the triangles of $\mathcal{T}_h$.

(ii) $T_k \cap T_j$ is a side, a node, or empty for all triangles $T_k$, $T_j$, $k \neq j$, in $\mathcal{T}_h$.

(iii) There exists a constant $\sigma$, independent of $h$, such that for all $T \in \mathcal{T}_h$ its diameter $h_T$ is smaller that $h$ and $T$ contains a circle of radius $\sigma \, h_T$.

Additionally we assume that each triangle $T$ in $\mathcal{T}_h$ has at least one vertex inside $\Omega$ (i.e. not on $\Gamma \cap \Gamma_0$).

The properties that $\Omega$ is convex and the triangulations uniformly regular are used in the proof of Lemma 2.

Let $P_k(T)$ denote the set of restriction to $T$ of polynomials of degree less than or equal to $k$. For the velocity approximation space we consider

$$X_h \ = \ \{\mathbf{w} \in (C^0(\overline{\Omega}))^2 \ : \ \mathbf{w}|_{\Gamma} = \mathbf{0} \,, w_r|_{\Gamma_0} = 0 \,, \mathbf{w}|_T \in (P_k(T))^2, \, \forall T \in \mathcal{T}_h\} \subset X \,. \tag{2.15}$$

For the pressure space,

$$Q_h \ = \ \{q \in C^0(\overline{\Omega}) \ : \ \int_{\Omega} q \, r \, d\mathbf{x} = 0 \,, q|_T \in P_{k-1}(T), \, \forall T \in \mathcal{T}_h\} \subset Q \,. \tag{2.16}$$

The approximation pair $(X_h, \, Q_h)$ given by (2.15)(2.16) with $k = 2$ represent the Taylor-Hood $P_2$–$P_1$ pair.

For $T \in \mathcal{T}_h$, let $(\lambda_1(x,y), \, \lambda_2(x,y), \, \lambda_3(x,y))$ denote the normalized (i.e. $\lambda_1 + \lambda_2 + \lambda_3 \ = \ 1$) barycentric coordinates of $(x, y) \in T$. Introduce the *bubble function* on $T$,

$$b_T(x,y) \ := \ 27\lambda_1(x,y)\lambda_2(x,y)\lambda_3(x,y), \quad \text{and} \quad B_T \ := \ span\{b_T\}. \tag{2.17}$$

The approximation pair

$$X_h \ = \ \{\mathbf{w} \in (C^0(\overline{\Omega}))^2 \ : \ \mathbf{w}|_{\Gamma} = \mathbf{0} \,, w_r|_{\Gamma_0} = 0 \,, \mathbf{w}|_T \in (P_2(T) \oplus B_T)^2, \, \forall T \in \mathcal{T}_h\} \subset X \,, \tag{2.18}$$

$$Q_h \ = \ \{q \ : \ \int_{\Omega} q \, r \, d\mathbf{x} = 0 \,, q|_T \in P_1(T), \, \forall T \in \mathcal{T}_h\} \subset Q \,, \tag{2.19}$$

correspond to the conforming Crouzeix-Raviart mixed finite element pair.

In Section 4 we show that the pairs (2.15)(2.16),for $k = 2$, and (2.18)(2.19) are both LBB stable.

Below, all constants $C, C_1, C_2, \ldots$ used are independent of $h$. However their values may change from line to line.

# 3 Mathematical Preliminaries

In [9] Stenberg established a sufficient condition on the family of partitions $(\mathcal{T}_h)_h$ and the approximation spaces $X_h$, and $Q_h$, for the LBB condition (1.1)(1.2) to be satisfied. For the axisymmetric flow formulation we have a different operator $b(\cdot, \cdot)$ and different velocity and pressure spaces.

The proof of the Stenberg sufficiency condition in [9] follows easily from two lemmas, generalized as Lemma 1 and Lemma 2 below. The proof of Lemma 1 follows as in [9]. However, because of the singular operators and different norms arising in the axisymmetric formulation, the proof of Lemma 2 is considerably more complicated. As in [9], the proof of the sufficiency condition follows from Lemmas 1 and 2.

We discuss the case for a triangulation of the domain $\Omega$. The results can be extended to a partition of the domain into regular quadrilateral elements.

## 3.1 Stenberg sufficient condition

A *macroelement* $M$ is said to be equivalent to a reference macroelement $\widehat{M}$ if there is a mapping $F_M : \widehat{M} \to M$ satisfying the conditions:

(i) $F_M$ is continuous and one-to-one.

(ii) $F_M(\widehat{M}) = M$.

(iii) If $\widehat{M} = \cup_{j=1}^m \widehat{T}_j$, where $\widehat{T}_j$, $j = 1, 2, \ldots, m$, are the triangles in $\widehat{M}$, then $T_j = F_M(\widehat{T}_j)$, $j = 1, 2, \ldots, m$, are the triangles in $M$.

(iv) $F_{M|_{\widehat{T}_j}} = F_{T_j} \circ F_{\widehat{T}_j}^{-1}$, $j = 1, 2, \ldots, m$, where $F_{\widehat{T}_j}$ and $F_{T_j}$ are the affine mappings from the reference triangle with vertices (0, 0), (1, 0) and (0, 1) onto $\widehat{T}_j$ and $T_j$, respectively.

The family of macroelements equivalent with $\widehat{M}$ is denoted $\mathcal{E}_{\widehat{M}}$.

For a macroelement $M$ define the spaces $X_{h,M}$, $Q_{h,M}$ and $N_{h,M}$ as

$$X_{h,M} = \{\mathbf{w} \in (C^0(\overline{\Omega}))^2 \,:\, \mathbf{w}|_\Gamma = \mathbf{0}\,, w_r|_{\Gamma_0} = 0\,, \mathbf{w}|_{\overline{\Omega}\setminus M} = \mathbf{0},\ \mathbf{w}|_T \in (P_k(T))^2,\ \forall T \in M\} \subset X, \tag{3.1}$$

$$Q_{h,M} = \{q \,:\, \int_\Omega q\, r\, d\mathbf{x} = 0\,, q|_T \in P_l(T),\ \forall T \in M\} \subset {}_1L_0^2(M)\,, \tag{3.2}$$

$$N_{h,M} = \{q \in Q_{h,M} \,:\, b_a(q\,, \mathbf{w}) = 0\,,\ \ \forall \mathbf{w} \in X_{h,M}\}\,. \tag{3.3}$$

**Theorem 1** *[9] [Stenberg Sufficiency Condition] If* *(i) there exists a finite set of classes $\mathcal{E}_{\widehat{M_i}}$, $i = 1, \ldots, n$, $n \geq 1$, such that for each $M \in \mathcal{E}_{\widehat{M_i}}$, $i = 1, \ldots, n$, the space $N_M$ is one dimensional consisting of functions which are constant on $M$,*

*(ii) for each $\mathcal{T}_h \in (\mathcal{T}_h)_h$, the triangles can be grouped together to form macroelements $M_j$, $j = 1, \ldots, m$, such that the so obtained macroelement partitioning of $\overline{\Omega}$, $\mathcal{M}_h$ satisfies that $M_j$ belongs to some $\mathcal{E}_{\widehat{M_i}}$, for all $M_j \in \mathcal{M}_h$,*

*then (1.1)(1.2) is satisfied.*

*In the case linear elements are used for the velocity approximation there is one additional constraint on $\mathcal{T}_h$.*

*(iii) If $\gamma$ is the common part of two macroelements in (ii) then $\gamma$ is connected and contains at least two edges of triangles in $\mathcal{T}_h$.*

**Remark**: The stated theorem trivially extends to the case where the velocity approximating space is enriched with bubble functions, i.e. $\mathbf{w}|_T \in (P_k(T) \oplus B_k(T))^2$, where $B_k(T) = \{\mathbf{v} \in (P_{k+1}(T))^2 : \mathbf{v} = \lambda_1\lambda_2\lambda_3\mathbf{w}\,,\ \mathbf{w} \in (P_{k-2}(T))^2\}$.

The following two lemmas are analogues of the key lemmas used by Stenberg in [9].

Let $\Pi_h$ denote the projection, with respect to the innerproduct $\langle q,\, p\rangle := \int_\Omega q\,p\,r\,d\mathbf{x}$, from $Q_h$ onto the space

$$Q_h^C := \{q \in Q \,:\, q|_M \text{ is constant } \forall M \in \mathcal{M}_h\}\ . \tag{3.4}$$

**Lemma 1** *[See [9], Lemma 3.2] Under the conditions of Theorem 1, there is a constant $C > 0$ such that for all $q_h \in Q_h$ there is a $\mathbf{v}_h \in X_h$ satisfying*

$$\begin{aligned}
b_a(q_h\,,\,\mathbf{v}_h) \;&=\; \int_\Omega q_h\,\nabla_a\cdot\mathbf{v}_h\,r\,d\mathbf{x} \;+\; \int_\Omega q_h\,v_{h\,r}\,d\mathbf{x} \\
&=\; \int_\Omega (I\,-\,\Pi_h)q_h\,\nabla_a\cdot\mathbf{v}_h\,d\mathbf{x} \;+\; \int_\Omega (I\,-\,\Pi_h)q_h\,v_{h\,r}\,d\mathbf{x} \\
&\geq\; C\|(I\,-\,\Pi_h)q_h\|_Q^2\,, \qquad (3.5)\\
\textit{and}\quad \|\mathbf{v}_h\|_X \;&\leq\; \|(I\,-\,\Pi_h)q_h\|_Q\,. \qquad (3.6)
\end{aligned}$$

**Proof**: The proof of this lemma follows as that of Lemma 3.2 in [9].

■

**Lemma 2** *[[9], Lemma 3.3] Under the conditions of Theorem 1, there is a constant $C > 0$ such that for all $q_h \in Q_h$ there is a $\mathbf{v}_h \in X_h$ satisfying*

$$\begin{aligned}
b_a(q_h\,,\,\mathbf{v}_h) \;&=\; \int_\Omega q_h\,\nabla_a\cdot\mathbf{v}_h\,r\,d\mathbf{x} \;+\; \int_\Omega q_h\,v_{h\,r}\,d\mathbf{x} \;=\; \|\Pi_h q_h\|_Q^2\,, \qquad (3.7)\\
\textit{and}\quad \|\mathbf{v}_h\|_X \;&\leq\; C\,\|\Pi_h q_h\|_Q\,. \qquad (3.8)
\end{aligned}$$

The proof of Lemma 2 involves three steps. First, for $q_h \in Q_h$ given, the identification of $\mathbf{v} \in X$ satisfying $b_a(q_h,\,\mathbf{v}) = \Pi_h q_h$, and $\|\mathbf{v}\|_X \leq C\|\Pi_h q_h\|_Q$. Step 2 is the construction of an approximation $\mathbf{v}_h \in X_h$ of $\mathbf{v}$ such that $b_a(q_h,\,\mathbf{v}_h) = b_a(q_h,\,\mathbf{v})$. The third step involves establishing that

$\|\mathbf{v}_h\|_X \le C\|\mathbf{v}\|_X$. Because of the norms involved, it is this step that differs significantly from [9]. To do step 3 we follow the approach from Ruas in [8].

**Steps 1 and 2 in proof of Lemma 2**
Let $q_h^0 \in Q_h$ be given. As $\Pi_h q_h^0 \in Q_h$ from (2.14) we have that there exists a $\mathbf{v}^0 \in X$ satisfying

$$\nabla_a \cdot \mathbf{v}^0 + \frac{1}{r} v_r^0 \;=\; \Pi_h q_h^0 \,, \quad \text{and } \|\mathbf{v}^0\|_X \;\le\; \frac{1}{\beta}\|\Pi_h q_h^0\|_Q \,. \tag{3.9}$$

Let $P_h : X \to X_h$ denote the orthogonal projection defined by $\langle\cdot,\cdot\rangle_X$. Let $a_i$, $i = 1,\ldots,n_e$ denote a labeling of the triangle edges in the triangulation $\mathcal{T}_h$, with $\mathbf{n}_i$ and $\boldsymbol{\tau}_i$ a unit normal and tangent vector to $a_i$, respectively.

Assume that we have a Lagrangian basis for $X_h$, and that along each edge, $a_i$, the nodes are located at the endpoints and the Gaussian quadrature points. (For $k = 3$ modified Gaussian quadrature points are used. See (3.25).) Let $M_i$ denote an interior nodal point on $a_i$, with $\phi_{M_i}$ the associated local basis function, such that

$$\int_{a_i} \phi_{M_i}\, r\, ds \;\ne\; 0 \,. \tag{3.10}$$

Denote the other nodal points as $S_1,\ldots,S_{n_b}$.

Introduce $R_h : X \to X_h$ an approximation operator defined by

$$R_h\mathbf{v}(S_j) \;=\; P_h\mathbf{v}(S_j), \quad j = 1,\ldots,n_b, \tag{3.11}$$

$$R_h\mathbf{v}(M_i)\cdot\boldsymbol{\tau}_i \;=\; P_h\mathbf{v}(M_i)\cdot\boldsymbol{\tau}_i, \quad i = i,\ldots,n_e, \tag{3.12}$$

$$\int_{a_i} R_h\mathbf{v}\cdot\mathbf{n}_i\, r\, ds \;=\; \int_{a_i} \mathbf{v}\cdot\mathbf{n}_i\, r\, ds, \quad i = i,\ldots,n_e. \tag{3.13}$$

For $\mathbf{v}^0$ defined in (3.9), let

$$\mathbf{v}_h^0 \;=\; R_h\mathbf{v}^0 \,, \quad \mathbf{e}_h^0 \;=\; \mathbf{v}_h^0 - P_h\mathbf{v}^0 \,, \quad \text{and } \mathbf{e}^0 \;=\; \mathbf{v}^0 - P_h\mathbf{v}^0 \,,$$

By construction

$$\|\Pi_h q_h^0\|_Q^2 \;=\; b_a(q_h^0\,,\,\mathbf{v}^0) \;=\; b_a(q_h^0\,,\,\mathbf{v}_h^0) \,.$$

Also,

$$\|\mathbf{v}_h^0\|_{X(T)} \;\le\; \|\mathbf{e}_h^0\|_{X(T)} + \|P_h\mathbf{v}^0\|_{X(T)} \;\le\; \|\mathbf{e}_h^0\|_{X(T)} + \|\mathbf{v}^0\|_{X(T)} \,,$$

■

To complete the proof it suffices to show that $\|\mathbf{e}_h^0\|_X \le C\,\|\mathbf{e}^0\|_X$. To establish this inequality requires us to look closely at the triangulation of $\Omega$ and the interpolation. We introduce the additional notation. For $T \in \mathcal{T}_h$, (see Figure 3.1) let

$$F_T(\boldsymbol{\xi}) \;=\; J_T\boldsymbol{\xi} + \begin{bmatrix} r_1 \\ z_1 \end{bmatrix}, \quad J_T \;=\; \begin{bmatrix} (r_2 - r_1) & (r_3 - r_1) \\ (z_2 - z_1) & (z_3 - z_1) \end{bmatrix}, \tag{3.14}$$

$a_i^T$, $i = 1,2,3$, denote the edges of $T$, with $M_i^T$ denoting the associated edge point used in (3.12), $\mathbf{n}_i^T$ the unit normal used in (3.13), $l_i^T$ its length, and $I_e^T \subset 1,2,3$ an index set such that $i \in I_e^T$ implies that $a_i^T \not\subset \Gamma_0$, i.e. $a_i^T$ does not lie on the $z$-axis.

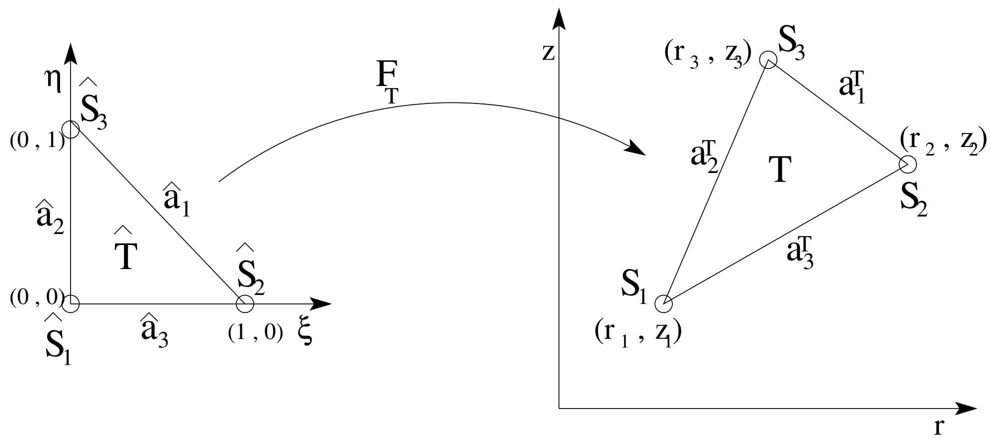


Figure 3.1: Mapping of the reference triangle $\widehat{T}$ to triangle $T$.

By assumption of a regular triangulation, there exists constants $c_J$, $C_J > 0$ such that

$$c_J h_T^2 \le |det(J_T)| = |J_T| \le C_J h_T^2 .$$

For $\Theta \subset \Omega$, let $r_{max}(\Theta) := max\{r : (r, z) \in \bar{\Theta}\}$, and $r_{min}(\Theta) := min\{r : (r, z) \in \bar{\Theta}\}$ .

As $\mathcal{T}_h$ is a regular triangulation of $\Omega$ we have that there exists $c_1 > 0$ such that

$$r_{max}(T) \ge c_1 h_T , \quad \text{for all } T \in \mathcal{T}_h . \tag{3.15}$$

It is useful to categorization the triangles $T$ of $\mathcal{T}_h$ into three types. For constants $c_2$, $c_3$, $c_4 > 0$ the following inequalities hold.

**Type 1:** $T \cap \Gamma_0$ **is empty.** For these triangles we have that

$$r_{min}(T) \ge c_2 h_T . \tag{3.16}$$

**Type 2:** $T \cap \Gamma_0$ **is a side.** For these triangles, without loss of generality (WLOG), we assume that the local counter-clockwise labeling of $T$ is such that the vertices $S_1$ and $S_3$ (equivalently $a_2^T$) lie on $\Gamma_0$. Then, under the transformation $F_T$,

$$r = r_2 \xi = r_{max}(T) \xi , \quad \text{and } r_{max}(T) \le c_3 h_T . \tag{3.17}$$

**Type 3:** $T \cap \Gamma_0$ **is a point.** For these triangles, WLOG, we assume that the local counter-clockwise labeling of $T$ is such that the vertex $S_1$ lies on $\Gamma_0$. Under the transformation $F_T$,

$$r = r_2 \xi + r_3 \eta \ge \min\{r_2, r_3\}(\xi + \eta) \ge c_4 h_T(\xi + \eta) . \tag{3.18}$$

**Step 3 in proof of Lemma 2**

For each $T \in \mathcal{T}_h$ we now estimate $\|\mathbf{e}_h^0\|_{X(T)}$. As $T \in \mathcal{T}_h$ is considered fixed we omit the superscript $T$ in the notation of $a_i^T$, $M_i^T$, etc.

We have

$$\mathbf{e}_h^0|_T \;=\; \sum_{i=1}^{3} \mathbf{e}_h^0(M_i)\cdot\mathbf{n}_i\,\phi_i\,\mathbf{n}_i\,,$$

where $\phi_i$ is the canonical local basis function of $X_h$ associated with $M_i$. Thus,

$$\begin{aligned}
\|\mathbf{e}_h^0\|_{X(T)}^2 &= \|e^0_{h\,r}\|^2_{{}_1V^1(T)} + |e^0_{h\,z}|^2_{{}_1H^1(T)} \\
&= \int_T \left|\sum_{i=1}^{3} \mathbf{e}_h^0(M_i)\cdot\mathbf{n}_i\,\nabla_a\phi_i\,n_{i\,r}\right|^2 r\,d\mathbf{x} + \int_T \left|\sum_{i=1}^{3} \mathbf{e}_h^0(M_i)\cdot\mathbf{n}_i\,\phi_i\,n_{i\,r}\right|^2 \frac{1}{r^2}\,r\,d\mathbf{x} \\
&\quad + \int_T \left|\sum_{i=1}^{3} \mathbf{e}_h^0(M_i)\cdot\mathbf{n}_i\,\nabla_a\phi_i\,n_{i\,z}\right|^2 r\,d\mathbf{x} \\
&\le \sum_{i=1}^{3}\left(\mathbf{e}_h^0(M_i)\cdot\mathbf{n}_i\right)^2 \sum_{i=1}^{3}\left(|\phi_i|^2_{{}_1H^1(T)} + \|\phi_i\|^2_{{}_{-1}L^2(T)}\right). \qquad (3.19)
\end{aligned}$$

Mapping from $T$ to the reference triangle $\widehat{T}$ we have

$$\begin{aligned}
|\phi_i|^2_{{}_1H^1(T)} &= \int_T |\nabla_a\phi_i|^2 r\,d\mathbf{x} \;\le\; C\int_{\widehat{T}}\left(|\frac{\partial\hat{\phi}_i}{\partial\xi}|^2 + |\frac{\partial\hat{\phi}_i}{\partial\eta}|^2\right) h_T^{-2}\,\hat{r}\,h_T^2\,d\boldsymbol{\xi} \\
&\le C\,r_{max}(T)\,. \qquad (3.20)
\end{aligned}$$

For $T$ a **Type 1** triangle,

$$\begin{aligned}
\|\phi_i\|^2_{{}_{-1}L^2(T)} &= \int_T |\phi_i|^2\frac{1}{r^2}\,r\,d\mathbf{x} \;=\; \int_{\widehat{T}} |\hat{\phi}_i|^2\frac{1}{\hat{r}}\,h_T^2\,d\boldsymbol{\xi} \\
&\le C_1\,\frac{1}{r_{min}(T)}h_T^2 \;\le\; C\,h_T\,. \qquad (3.21)
\end{aligned}$$

For $T$ a **Type 2** triangle, for $i \in I_e$, $\hat{\phi}_i$ vanishes along $\xi = 0$ thus $\hat{\phi}_i \;=\; \xi\hat{\psi}_i$, with $\hat{\psi}_i \in P_{k-1}(T)$.

$$\begin{aligned}
\|\phi_i\|^2_{{}_{-1}L^2(T)} &= \int_T |\phi_i|^2\frac{1}{r^2}\,r\,d\mathbf{x} \;=\; \int_{\widehat{T}} |\hat{\phi}_i|^2\frac{1}{r_{max}(T)\,\xi}\,h_T^2\,d\boldsymbol{\xi} \\
&\le C_1\int_{\widehat{T}} \xi\,|\hat{\psi}_i|^2\,h_T\,d\boldsymbol{\xi} \;\le\; C\,h_T\,. \qquad (3.22)
\end{aligned}$$

For $T$ a **Type 3** triangle,

$$\begin{aligned}
\|\phi_i\|^2_{{}_{-1}L^2(T)} &= \int_T |\phi_i|^2\frac{1}{r^2}\,r\,d\mathbf{x} \;=\; \int_{\widehat{T}} |\hat{\phi}_i|^2\frac{1}{r_2\xi + r_3\eta}\,h_T^2\,d\boldsymbol{\xi} \\
&\le C_1\int_{\widehat{T}} |\hat{\phi}_i|^2\frac{1}{\xi + \eta}\,h_T\,d\boldsymbol{\xi} \;\le\; C\,h_T\,. \qquad (3.23)
\end{aligned}$$

Thus, combining (3.20)–(3.23), we have for $i \in I_e$

$$|\phi_i|^2_{{}_1H^1(T)} + \|\phi_i\|^2_{{}_{-1}L^2(T)} \;\le\; C\,r_{max}(T)\,. \qquad (3.24)$$

Next we need to construct an estimate for $|\mathbf{e}_h^0(M_i)\cdot\mathbf{n}_i|$.

By construction of $\mathbf{v}_h^0$,

$$\begin{aligned}\int_{a_i}\mathbf{e}_h^0(M_i)\cdot\mathbf{n}_i\,\phi_i\,r\,ds &= \int_{a_i}\mathbf{e}^0\cdot\,\mathbf{n}_i\,r\,ds\,,\\ \text{i.e.}\quad |\mathbf{e}_h^0(M_i)\cdot\mathbf{n}_i|\,|\int_{a_i}\phi_i\,r\,ds| &\le \int_{a_i}|\mathbf{e}^0\cdot\,\mathbf{n}_i|\,r\,ds\,.\end{aligned}$$

Note that $\phi_i\,r$ is a polynomial of degree $\le k+1$ in $s$ along $a_i$ which vanishes at the endpoints. For $k=2$ and $k\ge 4$ the $k-1$ Gaussian quadrature formula exactly evaluates $\int_{a_i}\phi_i\,r\,ds$. For $k=3$ the modified Gaussian quadrature formula,

$$\int_{-1}^{1}f(t)\,dt\;\sim\;\frac{5}{6}f(-1/\sqrt{5})\,+\,\frac{5}{6}f(1/\sqrt{5})\,,\tag{3.25}$$

exactly evaluates the integral.

With $r_{M_i}$ the $r$ coordinate of $M_i$, applying the quadrature formula we have that there exists $c>0$ such that

$$\left|c\,r_{M_i}\,l_i\,\mathbf{e}_h^0(M_i)\cdot\mathbf{n}_i\right|\;\le\;\int_{a_i}|\mathbf{e}^0\cdot\,\mathbf{n}_i|\,r\,ds\,.$$

For $a_i\subset\Gamma_0$, $\mathbf{e}_h^0(M_i)\cdot\mathbf{n}_i\;=\;0$. Otherwise, for $a_i\not\subset\Gamma_0$, there exists a constant $C>0$ such that $r_{M_i}\;\ge\;C\,r_{max}(T)$, and $l_i\;\ge\;2\sigma h_T$. Thus

$$\left|\mathbf{e}_h^0(M_i)\cdot\mathbf{n}_i\right|\;\le\;C\,(r_{max}(T)\,h_T)^{-1}\int_{a_i}|\mathbf{e}^0\cdot\,\mathbf{n}_i|\,r\,ds\,.\tag{3.26}$$

Next we use the obvious bound

$$\int_{a_i}|\mathbf{e}^0\cdot\,\mathbf{n}_i|\,r\,ds\;\le\;\int_{a_i}|e_r^0|\,r\,ds\,+\,\int_{a_i}|e_z^0|\,r\,ds\,.\tag{3.27}$$

For $\int_{a_i}|e_r^0|\,r\,ds$, again using the mapping of the triangle $T$ to the reference triangle $\widehat{T}$

$$\int_{a_i}|e_r^0|\,r\,ds\;\le\;h_T\int_{\widehat{a_i}}|\widehat{re_r^0}|\,d\hat{s}\;\le\;h_T\,\|\widehat{re_r^0}\|_{L^2(\partial\widehat{T})}\,.$$

Applying the Trace Theorem to $\widehat{T}$, and using $|J_T|\;\ge\;c_J\,h_T^2$, we then have

$$\begin{aligned}\int_{a_i}|e_r^0|\,r\,ds &\le C\,h_T\,\|\widehat{re_r^0}\|_{H^1(\widehat{T})}\\ &\le C\,h_T\left(\int_T|re_r^0|^2\,|J_T|^{-1}\,d\mathbf{x}\;+\;\int_T|\nabla_a\,re_r^0|^2\,h_T^2\,|J_T|^{-1}\,d\mathbf{x}\right)^{1/2}\\ &\le C\left(\int_T|e_r^0|^2\,r\,r\,d\mathbf{x}\;+\;h_T^2\int_T|\nabla_a\,e_r^0|^2\,r\,r\,d\mathbf{x}\;+\;h_T^2\int_T\left|\frac{e_r^0}{r}\right|^2\,r\,r\,d\mathbf{x}\right)^{1/2} &(3.28)\\ &\le C\,(r_{max}(T))^{1/2}\left(\|e_r^0\|^2_{{}_1L^2(T)}\;+\;h_T^2\,\|e_r^0\|^2_{{}_1V^1(T)}\right)^{1/2}\,. &(3.29)\end{aligned}$$

In order to bound $\int_{a_i} |e_z^0| \, r \, ds$ we consider the three types for $T \in \mathcal{T}_h$. In each case we establish

$$\int_{a_i} |e_z^0| \, r \, ds \;\leq\; C \, (r_{max}(T))^{1/2} \left( \|e_z^0\|^2_{{}_1L^2(T)} \;+\; h_T^2 \, |e_z^0|^2_{{}_1H^1(T)} \right)^{1/2} . \tag{3.30}$$

**Type 1**. $T \cap \Gamma_0$ is empty.
Estimate (3.30) is established by mapping $T$ to $\widehat{T}$ and applying the Trace Theorem. (See [5] for details.)

**Type 2**. $T \cap \Gamma_0$ is a side.
Estimate (3.30) is established by mapping $T$ to $\widehat{T}$, revolving $\widehat{T}$ around the $\eta$-axis to generate a reference cone $\widehat{E}$, and then applying the Trace Theorem to $\widehat{E}$. (See [5] for details.)

**Type 3**. $T \cap \Gamma_0$ is a point.
In this case, after mapping $T$ to $\widehat{T}$, the integral is split into two pieces. One piece is handled as in case **Type 2**, by forming a reference cone by rotating $\widehat{T}$ around the $\eta$-axis. The other piece is handled similarly, by forming a reference cone by rotating $\widehat{T}$ around the $\xi$-axis. (See [5] for details.)

Combining (3.27),(3.29) and (3.30) we obtain

$$\int_{a_i} |\mathbf{e}^0 \cdot \mathbf{n}_i| \, r \, ds \;\leq\; C \, (r_{max}(T))^{1/2} \left( \|e_r^0\|^2_{{}_1L^2(T)} + h_T^2 \|e_r^0\|^2_{{}_1V^1(T)} + \|e_z^0\|_{{}_1L^2(T)} + h_T^2 |e_z^0|^2_{{}_1H^1(T)} \right)^{1/2} . \tag{3.31}$$

From (3.26) and (3.31)

$$\left| \mathbf{e}_h^0(M_i) \cdot \mathbf{n}_i \right|^2 \;\leq\; C \, (r_{max}(T))^{-1} \; h_T^{-2} \left( \|e_r^0\|^2_{{}_1L^2(T)} + h_T^2 \|e_r^0\|^2_{{}_1V^1(T)} + \|e_z^0\|_{{}_1L^2(T)} + h_T^2 |e_z^0|^2_{{}_1H^1(T)} \right) . \tag{3.32}$$

Combining (3.19),(3.24), and (3.32) yields

$$\|\mathbf{e}_h^0\|^2_{X(T)} \;\leq\; C \, h_T^{-2} \left( \|e_r^0\|^2_{{}_1L^2(T)} + h_T^2 \|e_r^0\|^2_{{}_1V^1(T)} + \|e_z^0\|_{{}_1L^2(T)} + h_T^2 |e_z^0|^2_{{}_1H^1(T)} \right) . \tag{3.33}$$

Summing over the triangles we obtain

$$\|\mathbf{e}_h^0\|^2_{X(T)} \;\leq\; C \left( \sum_{T \in \mathcal{T}_h} h_T^{-2} \|\mathbf{e}^0\|^2_{{}_1L^2(T)} \;+\; \|\mathbf{e}^0\|^2_X \right) . \tag{3.34}$$

Thus, what remains to show is that

$$\sum_{T \in \mathcal{T}_h} h_T^{-2} \|\mathbf{e}^0\|^2_{{}_1L^2(T)} \;\leq\; C \, \|\mathbf{e}^0\|^2_X \, . \tag{3.35}$$

Using the fact that the mesh is a uniformly regular triangulation implies that there exists a constant $c > 0$ such that $c\,h \;\leq\; h_T$. Hence (3.35) may be replaced by showing that

$$\|\mathbf{e}^0\|_{{}_1L^2(\Omega)} \;\leq\; C \, h \, \|\mathbf{e}^0\|_X \, . \tag{3.36}$$

To establish (3.36) we use the following Propositions relating function spaces defined on $\Omega$ to axisymmetric functions, with zero angular component, defined on $\breve{\Omega}$.

**Proposition 1** *[3] The axisymmetric reduction of functions in* $\left(L^2(\breve{\Omega})\right)^3$*, with zero angular component, to functions in* $\left({}_1L^2(\Omega)\right)^2$ *is an isometry (up to a factor of* $\sqrt{2\pi}$*).*

**Proposition 2** *The axisymmetric reduction of functions in* $\left(H^2(\breve{\Omega})\right)^3$*, with zero angular component, to functions in* $\left({}_1H^2(\Omega)\right)^2$ *is a bounded mapping satisfying* $\|\breve{\mathbf{w}}\|_{\left(H^2(\breve{\Omega})\right)^3} \geq \|\mathbf{w}\|_{({}_1H^2(\Omega))^2}$*.*

**Proof**: The proof follows by direct calculation.

■

Let $\breve{\mathbf{e}}^0$ denote the axisymmetric extension of $\mathbf{e}^0$ to $\breve{\Omega}$. From Proposition 1 we have that $\breve{\mathbf{e}}^0 \in \left(L^2(\breve{\Omega})\right)^3$ and $\|\breve{\mathbf{e}}^0\|_{L^2(\breve{\Omega})} = \sqrt{2\pi}\|\mathbf{e}^0\|_{{}_1L^2(\Omega)}$. Let $\breve{\mathbf{w}} \in \left(H_0^1(\breve{\Omega})\right)^3$ be given by

$$\nabla \cdot \nabla \breve{\mathbf{w}} \; = \; \breve{\mathbf{e}}^0 \,, \quad \text{in } \breve{\Omega} \,. \tag{3.37}$$

As $\breve{\mathbf{e}}^0$ is axisymmetric, then $\breve{\mathbf{w}}$ is also. Additionally, as $\breve{\Omega}$ is convex, $\breve{\mathbf{w}} \in \left(H^2(\breve{\Omega})\right)^3$, and $\|\breve{\mathbf{w}}\|_{H^2(\breve{\Omega})} \leq C_1 \|\breve{\mathbf{e}}^0\|_{L^2(\breve{\Omega})} \leq C \|\mathbf{e}^0\|_{{}_1L^2(\Omega)}$.

Let $\mathbf{w}$ be the reduction of $\breve{\mathbf{w}}$ to $\Omega$. From Proposition 2 we have that $\mathbf{w} \in \left({}_1H^2(\Omega)\right)^2$, and $\|\mathbf{w}\|_{({}_1H^2(\Omega))^2} \leq \|\breve{\mathbf{w}}\|_{\left(H^2(\breve{\Omega})\right)^3} \leq C \|\mathbf{e}^0\|_{{}_1L^2(\Omega)}$.

Then

$$\begin{aligned}
2\pi \|\mathbf{e}^0\|^2_{{}_1L^2(\Omega)} &= \|\breve{\mathbf{e}}^0\|^2_{L^2(\breve{\Omega})} \; = \; (\breve{\mathbf{e}}^0, \, \breve{\mathbf{e}}^0)_{L_2(\breve{\Omega})} \\
&= (\nabla\breve{\mathbf{w}}, \, \nabla\breve{\mathbf{e}}^0)_{L^2(\breve{\Omega})} \; = \; 2\pi \, \langle \mathbf{w}, \, \mathbf{e}^0 \rangle_X \\
&= 2\pi \, \langle \mathbf{w} - \chi \, , \, \mathbf{e}^0 \rangle_X \,, \quad \text{for} \quad \chi \in X_h \,, \\
&\leq 2\pi \, \|\mathbf{e}^0\|_X \inf_{\chi \in X_h} \|\mathbf{w} - \chi\|_X \\
&\leq 2\pi \, \|\mathbf{e}^0\|_X \, C \, h \, \|\mathbf{w}\|_{({}_1H^2(\Omega))^2} \quad \text{(from [2], Theorem 5)} \\
&\leq C \, h \, \|\mathbf{e}^0\|_X \, \|\mathbf{e}^0\|_{{}_1L^2(\Omega)} \,.
\end{aligned} \tag{3.38}$$

Thus we have that

$$\|\mathbf{e}^0\|_{{}_1L^2(\Omega)} \; \leq \; C \, h \, \|\mathbf{e}^0\|_X \,. \tag{3.39}$$

■

**Proof of Theorem 1**: In view of Lemmas 1 and 2, the proof of Theorem 1 now follows as in [9].

■

## 4 The LBB condition for Taylor-Hood and Crouzeix-Raviart elements

In this section we show that the Stenberg sufficiency criteria for satisfying the LBB condition (1.1)(1.3) is satisfied for Taylor-Hood $P_2 - P_1$ and the conforming Crouzeix-Raviart approximating elements on triangles.

## 4.1 Taylor-Hood $P_2 - P_1$ approximation pair

We begin by identifying an appropriate macroelement, $M$, and then show that the corresponding vector space $N_{h,M}$ has dimension one.

Let $M$ be given by the collection of three triangles in Figure 4.1.

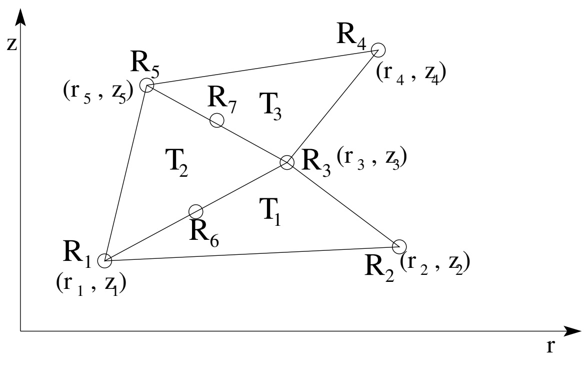


Figure 4.1: Macroelement for Taylor-Hood $P_2 - P_1$.

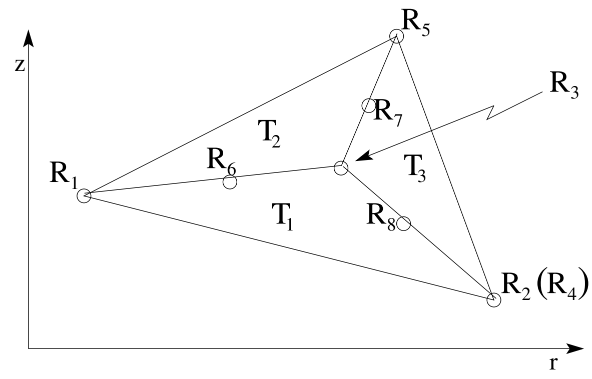


Figure 4.2: Macroelement for Taylor-Hood $P_2 - P_1$.

Let $X_{h,M}$, $Q_{h,M}$, $N_{h,M}$, be given by (3.1)-(3.3) with $k = 2$ and $l = 1$, and

$$X^0_{h,M} = \{\mathbf{w} \in (C^0(\overline{\Omega}))^2 : \mathbf{w}|_{\partial\Omega} = \mathbf{0}, \mathbf{w}|_{\overline{\Omega}\setminus M} = \mathbf{0}, \mathbf{w}|_T \in (P_2(T))^2, \forall T \in M\} \subset X_{h,M}\,, \quad (4.1)$$

$$N^0_{h,M} = \left\{q \in Q_{h,M} : b_a(q\,,\mathbf{w}) = 0\,, \ \ \forall \mathbf{w} \in X^0_{h,M}\right\} \supset N_{h,M}\,. \quad (4.2)$$

As the function $q =$ constant is contained in $N_{h,M}$ and $N^0_{h,M}$, we have $1 \le dim(N_{h,M}) \le dim(N^0_{h,M})$. Hence it suffices to show that $dim(N^0_{h,M}) = 1$.

Note that $X^0_{h,M}$ differs from $X_{h,M}$ in that for $M$ such that $M \cap \Gamma_0 \neq \emptyset$, $\mathbf{w} \in X^0_{h,M}$ satisfies $\mathbf{w}|_{\Gamma_0} = \mathbf{0}$, whereas for $\mathbf{w} \in X_{h,M}$, $w_r|_{\Gamma_0} = 0$. $X^0_{h,M}$ is introduced for convenience so that we do not need to separately consider those macroelements which have a nontrivial intersection with the symmetry boundary, $\Gamma_0$.

For notational convenience we suppress the $h$ subscript and 0 superscript, i.e. $N_M \equiv N^0_{h,M}$ and $X_M \equiv X^0_{h,M}$.

We have that

$$X_M = span\left\{\mathbf{v}_1 = q_6(r,z)\begin{bmatrix}1\\0\end{bmatrix}, \mathbf{v}_2 = q_6(r,z)\begin{bmatrix}0\\1\end{bmatrix}, \mathbf{v}_3 = q_7(r,z)\begin{bmatrix}1\\0\end{bmatrix}, \mathbf{v}_4 = q_7(r,z)\begin{bmatrix}0\\1\end{bmatrix}\right\},$$

where $q_6$, $q_7$ represent the (continuous) Lagrangian quadratic basis function which has value 1 at node $R_6$, $R_7$, respectively, and vanish at all other nodes. $Q_M = span\{l_1(r,z),\, l_2(r,z),\, l_3(r,z),\, l_4(r,z),\, l_5(r,z)\}$, where $l_i$, $i = 1,\ldots,5$, represents the (continuous) Lagrangian linear basis function which has value 1 at node $R_i$ and vanishes at nodes $R_j$, $j = 1,2,\ldots,5$, $j \neq i$.

Note that the defining equation for $N_M$ generates four equations for the five unknown constants $p_1$, $p_2$, $p_3$, $p_4$, $p_5$, where $p(r,z) \;=\; p_1\,l_1(r,z) \,+\, p_2\,l_2(r,z) \,+\, p_3\,l_3(r,z) \,+\, p_4\,l_4(r,z) \,+\, p_5\,l_5(r,z)\,.$

Using Green's theorem,

$$\int_M p\,\nabla_a \cdot \mathbf{v}\, r\, d\mathbf{x} \;+\; \int_M p\, v_r\, d\mathbf{x} \;=\; -\int_M \mathbf{v} \cdot \nabla_a p\, r\, d\mathbf{x} \;=\; -\sum_{j=1}^{3} \int_{T_j} \mathbf{v} \cdot \nabla_a p\, r\, d\mathbf{x}\,. \tag{4.3}$$

Also,

$$\int_{T_j} \mathbf{v} \cdot \nabla_a p\, r\, d\mathbf{x} \;=\; \int_{\hat{T}} \hat{\mathbf{v}} \cdot J_{T_j}^{-t}\, \nabla_{\xi,\eta}\hat{p}\, \hat{r}\, |J_{T_j}|\, d\xi\, d\eta\,, \tag{4.4}$$

where $|J_{T_j}|$ denotes the absolute value of the determinant of $J_{T_j}$, $J_{T_j}^{-t}$ the transpose of the inverse of $J_{T_j}$, and $\hat{g}(\xi,\eta) := g(F_{T_j}^{-1}(r,z))$, (see (3.14)).

For $\mathbf{v} \in X_M$, $p \in Q_M$, $\hat{\mathbf{v}}$ is a (vector) quadratic function, $\nabla_{\xi,\eta}\hat{p}$ is a constant vector, and $J_{T_j}$ is a constant matrix. Hence the integrand in (4.4) is a polynomial of degree $\leq 3$.

Introduce the following Lagrangian quadratic and linear basis functions on $\widehat{T}$.

$$\begin{aligned}
\hat{q}_1(\xi,\,\eta) \;&=\; (1-\xi-\eta)(1-2\xi-2\eta)\,, \qquad & \hat{q}_2(\xi,\,\eta) \;&=\; \xi\,(2\xi-1)\,,\\
\hat{q}_3(\xi,\,\eta) \;&=\; \eta\,(2\eta-1)\,, & \hat{q}_4(\xi,\,\eta) \;&=\; 4\,\xi\,\eta\,,\\
\hat{q}_5(\xi,\,\eta) \;&=\; 4\,\eta\,(1-\xi-\eta)\,, & \hat{q}_6(\xi,\,\eta) \;&=\; 4\,\xi\,(1-\xi-\eta)\,,
\end{aligned}$$

and

$$\hat{l}_1(\xi,\,\eta) \;=\; (1-\xi-\eta)\,, \quad \hat{l}_2(\xi,\,\eta) \;=\; \xi\,, \quad \hat{l}_3(\xi,\,\eta) \;=\; \eta\,. \tag{4.5}$$

Also note that the quadrature formula

$$\begin{aligned}
\int_{\widehat{T}} \hat{f}(\xi,\eta)\, d\xi\, d\eta \;\sim\;& \frac{8}{120}\Big(\hat{f}(1/2,0) + \hat{f}(1/2,1/2) + \hat{f}(0,1/2)\Big) \;+\; \frac{3}{120}\Big(\hat{f}(0,0) + \hat{f}(1,0) + \hat{f}(0,1)\Big)\\
&+ \frac{27}{120}\hat{f}(1/3,1/3)\,,
\end{aligned} \tag{4.6}$$

is exact for polynomials of degree $\leq 3$.

### 4.1.1 Computation of $\int_{T_2} \mathbf{v} \cdot \nabla_a p\, r\, d\mathbf{x}$

In terms of the mapping of $T_2$ to the reference triangle, relative to (3.14), associate $S_1 \equiv R_1$, $S_2 \equiv R_3$, and $S_3 \equiv R_5$.

We have that

$$\begin{aligned}
\mathbf{v}_1(r,z)|_{T_2} \;&=\; \hat{q}_6(F_{T_2}^{-1}(r,z)) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad & \mathbf{v}_2(r,z)|_{T_2} \;&=\; \hat{q}_6(F_{T_2}^{-1}(r,z)) \begin{bmatrix} 0 \\ 1 \end{bmatrix},\\
\mathbf{v}_3(r,z)|_{T_2} \;&=\; \hat{q}_4(F_{T_2}^{-1}(r,z)) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & \mathbf{v}_4(r,z)|_{T_2} \;&=\; \hat{q}_4(F_{T_2}^{-1}(r,z)) \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\end{aligned}$$

$$p(r,z) \;=\; p_1\,\hat{l}_1(F_{T_2}^{-1}(r,z)) + p_3\,\hat{l}_3(F_{T_2}^{-1}(r,z)) + p_5\,\hat{l}_5(F_{T_2}^{-1}(r,z))\,,$$

and $J_{T_2} \;=\; \begin{bmatrix} (r_3 - r_1) & (r_5 - r_1) \\ (z_3 - z_1) & (z_5 - z_1) \end{bmatrix}$.

Using (4.4) and (4.6) we have that

$$\int_{T_2} \mathbf{v}_1 \cdot \nabla_a p\, r\, d\mathbf{x} \;=\; \frac{1}{30}(2r_1 + 2r_3 + r_5)\left((z_3 - z_5)p_1 \,+\, (z_5 - z_1)p_3 \,+\, (z_1 - z_3)p_5\right)\,, \qquad (4.7)$$

$$\int_{T_2} \mathbf{v}_2 \cdot \nabla_a p\, r\, d\mathbf{x} \;=\; -\frac{1}{30}(2r_1 + 2r_3 + r_5)\left((r_3 - r_5)p_1 \,+\, (r_5 - r_1)p_3 \,+\, (r_1 - r_3)p_5\right)\,, \qquad (4.8)$$

$$\int_{T_2} \mathbf{v}_3 \cdot \nabla_a p\, r\, d\mathbf{x} \;=\; \frac{1}{30}(r_1 + 2r_3 + 2r_5)\left((z_3 - z_5)p_1 \,+\, (z_5 - z_1)p_3 \,+\, (z_1 - z_3)p_5\right)\,, \qquad (4.9)$$

and

$$\int_{T_2} \mathbf{v}_4 \cdot \nabla_a p\, r\, d\mathbf{x} \;=\; -\frac{1}{30}(r_1 + 2r_3 + 2r_5)\left((r_3 - r_5)p_1 \,+\, (r_5 - r_1)p_3 \,+\, (r_1 - r_3)p_5\right)\,. \qquad (4.10)$$

Similar equations to (4.7)-(4.10) are obtained from considering $\int_{T_1} \mathbf{v} \cdot \nabla_a p\, r\, d\mathbf{x}$, and $\int_{T_3} \mathbf{v} \cdot \nabla_a p\, r\, d\mathbf{x}$. (See [5] for details.)

### 4.1.2 Dimension of $N_M$

Let

$$\alpha_1 \;:=\; 2r_1 + r_2 + 2r_3\,, \qquad \alpha_2 \;:=\; 2r_1 + 2r_3 + r_5\,,$$
$$\alpha_3 \;:=\; r_1 + 2r_3 + 2r_5\,, \qquad \alpha_4 \;:=\; 2r_3 + r_4 + 2r_5\,.$$

Note, as $r_i \geq 0$, $i = 1, 2, \ldots, 5$, and the geometry of the triangles, that $\alpha_1$, $\alpha_2$, $\alpha_3$ and $\alpha_4 \,>\, 0$. Corresponding to $\int_M p\, \nabla_a \cdot \mathbf{v}\, r\, d\mathbf{x} \;+\; \int_M p\, v_r\, d\mathbf{x} \;=\; -\int_M \mathbf{v} \cdot \nabla_a p\, r\, d\mathbf{x} \;=\; 0\,,\ \ \forall \mathbf{v} \in X_M$, we obtain (after minor simplification) the following linear system of equations, $A\mathbf{p} = \mathbf{0}$, where $A$ is given by

$$\begin{bmatrix} \alpha_1(z_2 - z_3) + \alpha_2(z_3 - z_5) & \alpha_1(-z_1 + z_3) & \alpha_1(z_1 - z_2) + \alpha_2(-z_1 + z_5) & & \alpha_2(z_1 - z_3) \\ \alpha_1(r_2 - r_3) + \alpha_2(r_3 - r_5) & \alpha_1(-r_1 + r_3) & \alpha_1(r_1 - r_2) + \alpha_2(-r_1 + r_5) & & \alpha_2(r_1 - r_3) \\ \alpha_3(z_3 - z_5) & & \alpha_3(-z_1 + z_5) + \alpha_4(z_4 - z_5) & \alpha_4(-z_3 + z_5) & \alpha_3(z_1 - z_3) + \alpha_4(z_3 - z_4) \\ \alpha_3(r_3 - r_5) & & \alpha_3(-r_1 + r_5) + \alpha_4(r_4 - r_5) & \alpha_4(-r_3 + r_5) & \alpha_3(r_1 - r_3) + \alpha_4(r_3 - r_4) \end{bmatrix}. \qquad (4.11)$$

Note that $p_1 \,=\, p_2 \,=\, p_3 \,=\, p_4 \,=\, p_5$ is a solution to (4.11), i.e. $N_M$ contains the constant functions. To show that $dim(N_M) = 1$ it suffices to show that the matrix $A$ has full rank, i.e. the rows of $A$ are linearly independent.

**Lemma 3** *The rows of the matrix $A$ given in (4.11) are linearly independent.*

**Proof**: Let $\tilde{p}_1 \;=\; p_1 + (\alpha_3/\alpha_4)p_4$, $\tilde{p}_5 \;=\; p_5 + (\alpha_2/\alpha_1)p_2$, $\tilde{\mathbf{p}} \;=\; [\tilde{p}_1\ p_2\ p_3\ p_4\ \tilde{p}_5]^T$, and consider in place of $A\mathbf{p} = \mathbf{0}$ the corresponding linear system $\tilde{A}\tilde{\mathbf{p}} = \mathbf{0}$.

To see that the rows of $\tilde{A}$ are linearly independent, consider:

$$C_1\tilde{A}_{1,\cdot} \,+\, C_2\tilde{A}_{2,\cdot} \,+\, C_3\tilde{A}_{3,\cdot} \,+\, C_4\tilde{A}_{4,\cdot} \;=\; \mathbf{0}\,. \qquad (4.12)$$

Corresponding to columns 1 and 2 in (4.12) we have

$$\begin{bmatrix} \alpha_1(z_2 - z_3) + \alpha_2(z_3 - z_5) & \alpha_1(r_2 - r_3) + \alpha_2(r_3 - r_5) \\ \alpha_1(-z_1 + z_3) & \alpha_1(-r_1 + r_3) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Note that if $r_1 = r_3$, as $R_1 \neq R_3$, then the second equation implies $C_1 = 0$. The first equation then gives $C_2 = 0$. Therefore, assume $r_1 \neq r_3$.

A non-trivial solution for $C_1$, $C_2$ requires the determinant of the $2 \times 2$ matrix to be zero. This implies

$$\frac{(z_2 - z_3) + \frac{\alpha_2}{\alpha_1}(z_3 - z_5)}{(r_2 - r_3) + \frac{\alpha_2}{\alpha_1}(r_3 - r_5)} = \frac{z_1 - z_3}{r_1 - r_3}. \tag{4.13}$$

Consider now the quadrilateral formed by $R_1, R_2, R_3, R_P$, where $R_P$ denotes the point on the half-line passing through $R_5$ and terminating at $R_3$ given by

$$R_P := (r_3 - \frac{\alpha_2}{\alpha_1}(r_3 - r_5), \, z_3 - \frac{\alpha_2}{\alpha_1}(z_3 - z_5).$$

Equation (4.13) implies that the vector $\vec{R_P R_2}$ has the same slope as the vector $\vec{R_1 R_3}$, which is impossible as they form opposite diagonals of the quadrilateral. Hence $C_1 = C_2 = 0$.

An analogous argument using columns 4 and 5 in (4.12) leads to $C_3 = C_4 = 0$.

Hence, the rows of $\tilde{A}$ are linearly independent, i.e. $rank(\tilde{A}) = 4 = rank(A)$.

■

By modifying the matrix in (4.11), it is straight forward to show that the three triangles depicted in Figure 4.2 also form a macroelement for Taylor-Hood $P_2 - P_1$ approximation pair. Thus, we could conclude that for any triangulation of the domain of $\Omega$, which can be partitioned into groups of three adjacent triangles, the Taylor-Hood $P_2 - P_1$ approximation pair is LBB stable. However often the number of triangles in a triangulation is not exactly divisible by three. Next we demonstrate that there are many choices of macroelements for the Taylor-Hood $P_2 - P_1$ approximation pair.

**Lemma 4** *Suppose $M$ is a macroelement with $N_M = 1$, consisting of functions which are constant on $M$. Let $\tilde{M}$ be formed from $M$ by adding an adjacent triangle (i.e. sharing an edge with $M$). Then $\tilde{M}$ is also a macroelement with the desired property that $N_{\tilde{M}} = 1$, consisting of functions which are constant on $\tilde{M}$.*

**Proof**: We consider separately the two cases corresponding to $\tilde{M}$ being formed by adding a triangle to $M$ that: (i) shares two edges with $M$, and (ii) shares one edge with $M$.

Case (i): The added triangle shares two edges with $M$. (For example, see Figures 4.1, 4.3.)

Let $A$ and $\tilde{A}$ be the matrices associated with $N_M$ and $N_{\tilde{M}}$, respectively. For $n_{M,Q}$ the dimension of $Q_M$, we have that $rank(A) = n_{M,Q} - 1$, and that $A$ is a $m \times n_{M,Q}$ matrix with $m = \geq n_{M,Q} - 1$. $\tilde{A}$ is therefore a $\tilde{m} \times n_{M,Q}$ matrix with $\tilde{m} \geq n_{M,Q} + 1$, as $\tilde{A}$ must have at least one more interior edge that $M$. Note that every row in $\tilde{A}$ comes from $\int_{\tilde{M}} \mathbf{v}_i \cdot \nabla p \, dA = 0$, for $\mathbf{v}_i \in X_{\tilde{M}}$. As, $\forall \mathbf{v}_i \in X_{\tilde{M}}$, $\int_{\tilde{M}} \mathbf{v}_i \cdot \nabla p \, dA = 0$ is satisfied for $p$ a constant function, then $p_1 = p_2 = \ldots = p_{n_{M,Q}}$ satisfies $\tilde{A}\mathbf{p} = \mathbf{0}$, and $rank(\tilde{A}) \leq n_{M,Q} - 1$. Since the $rank(A) = n_{M,Q} - 1$, this implies that $A$ has $n_{M,Q} - 1$

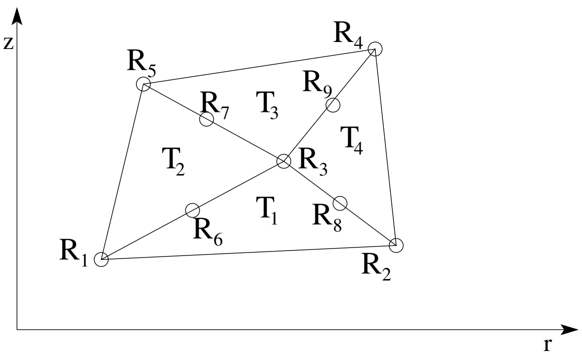


Figure 4.3: New macroelement for Taylor-Hood $P_2 - P_1$.

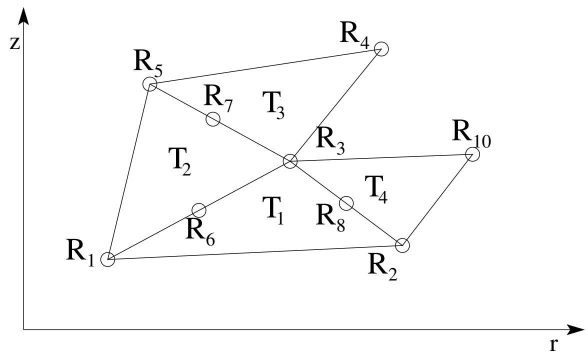


Figure 4.4: New macroelement for Taylor-Hood $P_2 - P_1$.

linearly independent rows. The fact that for $\mathbf{v} \in X_M,\ \mathbf{v}|_{\tilde{M}\backslash M} = \mathbf{0}$; we have $\tilde{A} = \begin{bmatrix} A \\ \cdots \\ B \end{bmatrix}$, which implies that $\tilde{A}$ has at least $n_{M,Q} - 1$ linearly independent rows. Hence $rank(\tilde{A}) = n_{M,Q} - 1$ and the dimension of $N_{\tilde{M}} = 1$.

Case (ii): The added triangle shares one edges with $M$. Let $R_2$ and $R_3$ denote the endpoints of the shared triangle edge. (For example, see Figures 4.1, 4.4.)

In this case, along with two new triangle edges, an additional triangle vertex is added to $M$ in forming $\tilde{M}$. Therefore, the dimension of $Q_{\tilde{M}} = dim(Q_M) + 1$, with the increase in dimension corresponding to the new added vertex. Again, as $\forall \mathbf{v}_i \in X_{\tilde{M}},\ \int_{\tilde{M}} \mathbf{v}_i \cdot \nabla p\, dA = 0$ is satisfied for $p$ a constant function, then $p_1 = p_2 = \ldots = p_{n_{M,Q}+1}$ satisfies $\tilde{A}\mathbf{p} = \mathbf{0}$, which implies that $rank(\tilde{A}) \leq n_{M,Q}$. Also, as for $\mathbf{v} \in X_M,\ \mathbf{v}|_{\tilde{M}\backslash M} = \mathbf{0}$; we have $\tilde{A} = \begin{bmatrix} A & \mathbf{0} \\ \cdots & \cdot \\ B & \mathbf{b} \end{bmatrix}$, where the number of rows in the matrix $B$ is two, corresponds to the velocity basis functions $\tilde{\mathbf{v}}_1$, $\tilde{\mathbf{v}}_2$, associated with the shared triangle edge, added to $Q_M$ to form $Q_{\tilde{M}}$. As the added triangle lies in the support of $\tilde{\mathbf{v}}_1$, and $\tilde{\mathbf{v}}_2$, then from (4.4) (and corresponding minor simplifications) $\mathbf{b} = [\alpha(-z_2 + z_3) \quad \alpha(-r_2 + r_3)]^T$, for $\alpha > 0$. As $R_2 \neq R_3$, the number of independent rows in $\tilde{A}$ must be greater than the number of independent rows in $A = n_{M,Q} - 1$. Hence $rank(\tilde{A}) = n_{M,Q}$ and the dimension of $N_{\tilde{M}} = 1$.

■

**Corollary 1** *The Taylor-Hood $P_2 - P_1$ approximation pair is LBB stable on a regular triangulation of $\Omega$.*

■

### 4.2 Crouzeix-Raviart approximation pair

Again, we begin by identifying a macroelement $M$ for the conforming Crouzeix-Raviart elements. In this case we simply take $M$ to be an arbitrary triangle $T$ in $\mathcal{T}_h$, see Figure 3.1.

With $\hat{l}_i(\xi,\,\eta)$, $i = 1, 2, 3$, defined in (4.5), let

$$\hat{b}(\xi,\,\eta) \;=\; 27\,\hat{l}_1(\xi,\,\eta)\,\hat{l}_2(\xi,\,\eta)\,\hat{l}_3(\xi,\,\eta)\,. \qquad (4.14)$$

$\hat{b}(\xi,\,\eta)$ is the cubic bubble function which vanishes on the boundary of $\widehat{T}$, and is equal to 1 at $(\xi,\,\eta) = (1/3\,,\,1/3)$. With $b_T(x, y)$ as defined in (2.17), let

$$X^0_{h,M} \;=\; span\left\{ b_T \begin{bmatrix} 1 \\ 0 \end{bmatrix} ,\, b_T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \subset X_{h,M}\,, \qquad (4.15)$$

$$N^0_{h,M} \;=\; \left\{ q \in Q_{h,M} \,:\, b_a(q\,,\,\mathbf{w}) \;=\; 0\,,\;\; \forall \mathbf{w} \in X^0_{h,M} \right\} \supset N_{h,M}\,. \qquad (4.16)$$

We note, as commented at the beginning of Section 4.1, that $q =$ constant is contained in $N_{h,M}$ and $N^0_{h,M}$, we have $1 \le dim(N_{h,M}) \le dim(N^0_{h,M})$. Hence it suffices to show that $dim(N^0_{h,M}) = 1$.

Again, for notational convenience we suppress the $h$ subscript and 0 superscript, i.e. $N_M \equiv N^0_{h,M}$ and $X_M \equiv X^0_{h,M}$.

The defining equation for $N_M$ generates two equations for the three unknown constants $p_1$, $p_2$, $p_3$, where $p(r, z) \;=\; p_1\,l_1(r, z) \,+\, p_2\,l_2(r, z) \,+\, p_3\,l_3(r, z)$.

From (4.3)(4.4) we have

$$\int_M p\,\nabla_a \cdot \mathbf{v}\,r\,d\mathbf{x} \;+\; \int_M p\,v_r\,d\mathbf{x} \;=\; \int_{\widehat{T}} \hat{\mathbf{v}} \cdot J^{-t}_{T_j}\,\nabla_{\xi,\eta}\hat{p}\,\hat{r}\,|J_{T_j}|\,d\xi\,d\eta\,.$$

Let $\quad val \;=\; |J_T| \int_{\widehat{T}} \hat{b}(\xi,\,\eta)\,\hat{r}(\xi,\,\eta)\,d\xi\,d\eta \;=\; |J_T| \int_{\widehat{T}} \hat{b}(\xi,\,\eta)\,(r_1 \,+\, (r_2 - r_1)\xi \,+\, (r_3 - r_1)\eta)\,d\xi\,d\eta\,.$

Note that, as $\hat{b}(\xi, \eta)$ and $\hat{r}(\xi, \eta)$ are greater than 0 for $(\xi, \eta) \in T\backslash\partial T$, $val > 0$.

For $\mathbf{v} \;=\; b_T \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we obtain

$$\int_M p\,\nabla_a \cdot \mathbf{v}\,r\,d\mathbf{x} \;+\; \int_M p\,v_r\,d\mathbf{x} \;=\; ((z_3 - z_1)(p_2 - p_1) \;-\; (z_2 - z_1)(p_3 - p_1))\;val\,, \qquad (4.17)$$

and for $\mathbf{v} \;=\; b_T \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$$\int_M p\,\nabla_a \cdot \mathbf{v}\,r\,d\mathbf{x} \;+\; \int_M p\,v_r\,d\mathbf{x} \;=\; -\,((r_3 - r_1)(p_2 - p_1) \;-\; (r_2 - z_1)(p_3 - p_1))\;val\,. \qquad (4.18)$$

From (4.17)(4.18), $N_M$ is given by the solutions to the linear system of equations

$$\begin{bmatrix} z_2 - z_3 & -z_1 + z_3 & z_1 - z_2 \\ r_2 - r_3 & -r_1 + r_3 & r_1 - r_2 \end{bmatrix} \mathbf{p} \;=\; \mathbf{0}\,, \qquad (4.19)$$

where $\mathbf{p} = [p_1\;p_2\;p_3]^T$.

The two rows of the coefficient matrix in (4.19) are linearly independent unless the points $(r_1, z_1)$, $(r_2, z_2)$, $(r_3, z_3)$ all lie along a line. That, however, would contradict the facts that the points form the vertices of a non-degenerate triangle. Hence $dim(N_M) = 1$.

In summary, we have the following.

**Corollary 2** *The conforming Crouzeix-Raviart ($P_2 + bubble - discP_1$) approximation pair is LBB stable on a regular triangulation of* $\Omega$.

■

## 5 Numerical Experiment

From the continuity and positivity of $a(\cdot,\cdot)$, the continuity of $b_a(\cdot,\cdot)$, and the inf-sup condition (1.1)(1.4) we have that approximations $(\mathbf{u}_h, p_h)$ to (2.10)(2.11) satisfy

$$\|\mathbf{u}-\mathbf{u}_h\|_X + \|p-p_h\|_Q \le C\left\{\inf_{\mathbf{v}\in X_h}\|\mathbf{u}-\mathbf{v}\|_X + \inf_{q\in Q_h}\|p-q\|_Q\right\}.$$

From [2], for $X_h$, $Q_h$ given by (2.15),(2.16), respectively, and $k=2$

$$\inf_{\mathbf{v}\in X_h}\|\mathbf{u}-\mathbf{v}\|_X \le C\,h^2\|\mathbf{u}\|_{{}_1H^3(\Omega)} \text{ and } \inf_{q\in Q_h}\|p-q\|_Q \le C\,h^2\|p\|_{{}_1H^2(\Omega)}.$$

Hence, for $\mathbf{u}\in {}_1H^3(\Omega)$, $p\in {}_1H^2(\Omega)$, we have that

$$\|\mathbf{u}-\mathbf{u}_h\|_X + \|p-p_h\|_Q \le C\,h^2. \tag{5.1}$$

We investigate this a priori error estimate in the following example.

Let $\Omega = (0,1/2)\times(-1/2,1/2)$, $\Gamma_0 = \{0\}\times[-1/2,1/2]$, $\Gamma = \partial\Omega\backslash\Gamma_0$. We consider a modified Taylor-Green vortex flow problem

$$\begin{aligned}\mathbf{u}(r,z) &= \begin{bmatrix} -r\cos(\omega\pi r)\sin(\omega\pi z) \\ -\frac{2}{\omega\pi}\cos(\omega\pi r)\cos(\omega\pi z) + r\sin(\omega\pi r)\cos(\omega\pi z)\end{bmatrix},\\ p(r,z) &= \sin(\omega\pi z)(-\cos(\omega\pi r) + 2\omega\pi r\sin(\omega\pi r)).\end{aligned}$$

The computations are performed for $\omega = 1$. A plot of the velocity field $\mathbf{u}$, and the pressure $p$, is given in Figures 5.1 and 5.2, respectively.

For the Taylor-Hood ($k=2$) and Crouzeix-Raviart approximation pairs the errors for the velocity, pressure, and divergence ( $div_{axi}(\mathbf{u}) = \nabla_a\cdot\mathbf{u} + u_r/r$), along with their experimental convergence rates are given in Tables 5.1 and 5.2. (The Crouzeix-Raviart approximation is mass conservative over each triangle $T$ in the triangulation $\mathcal{T}_h$, i.e. $\int_T div_{axi}(\mathbf{u}_h)\,r\,d\mathbf{x} = 0$.) The experimental convergence rates are consistent with those predicted in (5.1).

## References


[1] R.A. Adams and J.J.F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[2] Z. Belhachmi, C. Bernardi, and S. Deparis. Weighted Clément operator and application to the finite element discretization of the axisymmetric Stokes problem. *Numer. Math.*, 105(2):217–247, 2006.

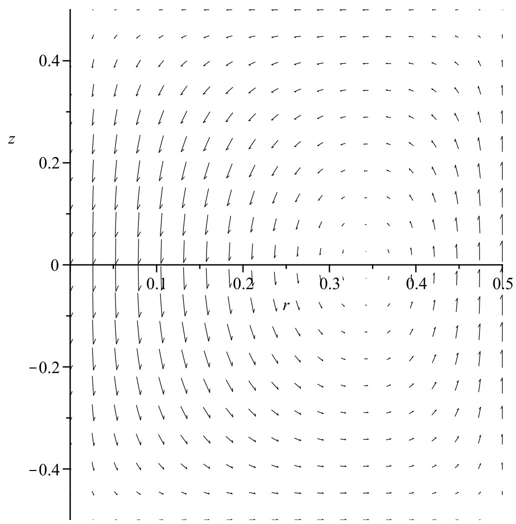


Figure 5.1: Plot of the velocity flow field $\mathbf{u}$.

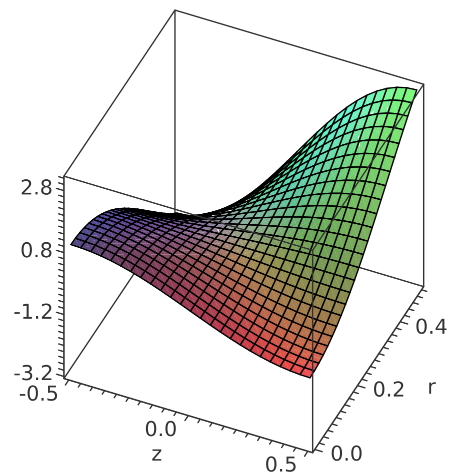


Figure 5.2: Plot of the pressure function $p$.

| $h$ | $\|\mathbf{u}-\mathbf{u}_h\|_X$ | Cvg. rate | $\|p-p_h\|_Q$ | Cvg. rate | $\|div_{axi}(\mathbf{u}_h)\|_{{}_1L^2(\Omega)}$ | Cvg. rate |
|---|---|---|---|---|---|---|
| 1/2 | 2.068E-1 | | 1.585E-1 | | 1.224E-1 | |
| 1/3 | 2.531E-2 | 5.18 | 2.319E-2 | 4.74 | 9.339E-3 | 6.35 |
| 1/4 | 1.410E-2 | 2.03 | 1.215E-2 | 2.25 | 5.236E-3 | 2.01 |
| 1/5 | 8.977E-3 | 2.02 | 7.549E-3 | 2.13 | 3.322E-3 | 2.04 |
| 1/6 | 6.216E-3 | 2.02 | 5.169E-3 | 2.08 | 2.288E-3 | 2.05 |
| 1/7 | 4.558E-3 | 2.01 | 3.769E-3 | 2.05 | 1.668E-3 | 2.05 |
| 1/8 | 3.485E-3 | 2.01 | 2.873E-3 | 2.03 | 1.270E-3 | 2.04 |
| 1/9 | 2.751E-3 | 2.01 | 2.264E-3 | 2.02 | 9.984E-4 | 2.04 |
| 1/10 | 2.227E-3 | 2.01 | 1.830E-3 | 2.02 | 8.055E-4 | 2.04 |

Table 5.1: Experimental convergence rates for the Taylor-Hood ($k = 2$) approximation pair.

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| $h$ | $\|\mathbf{u}-\mathbf{u}_h\|_X$ | Cvg. rate | $\|p-p_h\|_Q$ | Cvg. rate | $\|div_{axi}(\mathbf{u}_h)\|_{{}_1L^2(\Omega)}$ | Cvg. rate |
|---|---|---|---|---|---|---|
| 1/2 | 1.755E-1 | | 2.466E-1 | | 1.082E-1 | |
| 1/3 | 2.043E-2 | 5.30 | 1.890E-2 | 6.33 | 5.273E-3 | 7.45 |
| 1/4 | 1.156E-2 | 1.98 | 1.025E-2 | 2.13 | 2.849E-3 | 2.14 |
| 1/5 | 7.440E-3 | 1.98 | 6.520E-3 | 2.03 | 1.786E-3 | 2.09 |
| 1/6 | 5.188E-3 | 1.98 | 4.539E-3 | 1.99 | 1.224E-3 | 2.07 |
| 1/7 | 3.824E-3 | 1.98 | 3.351E-3 | 1.97 | 8.918E-4 | 2.06 |
| 1/8 | 2.934E-3 | 1.98 | 2.577E-3 | 1.97 | 6.784E-4 | 2.05 |
| 1/9 | 2.323E-3 | 1.99 | 2.044E-3 | 1.97 | 5.334E-4 | 2.04 |
| 1/10 | 1.884E-3 | 1.99 | 1.662E-3 | 1.97 | 4.304E-4 | 2.04 |

Table 5.2: Experimental convergence rates for the Crouzeix-Raviart approximation pair.


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