

Computational Bases for RT_k and BDM_k on Triangles

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Abstract

In this article, we derive computational bases for Raviart-Thomas (RT) and Brezzi-Douglas-Marini (BDM) (vector) approximation spaces on a triangulation of a domain in \mathbb{R}^2 . The basis functions, defined on the reference triangle, have a Lagrangian property. The continuity of the normal component of the approximation across the edges in the triangulation is satisfied by the use of the Piola transformation and the Lagrangian property of the basis functions. A numerical example is given demonstrating the approximation property of the bases.

Key words. RT_k basis, BDM_k basis, Piola transformation

AMS Mathematics subject classifications. 41A10, 65N30, 65M60

1 Introduction

In this article we present computational bases for the Raviart-Thomas (RT) and Brezzi-Douglas-Marini (BDM) vector spaces on a triangulation of a domain $\Omega \subset \mathbb{R}^2$. The RT and BDM spaces are commonly used as approximation spaces for $H_{div}(\Omega) := \{\mathbf{u} \in (L^2(\Omega))^2 \mid \nabla \cdot \mathbf{u} \in L^2(\Omega)\}$. The $H_{div}(\Omega)$ space frequently arises in mixed and dual-mixed variational formulations of the solution to partial differential equations. Let \mathcal{T}_h denote a regular triangulation of Ω , $\mathcal{E}_h = \{e_{ij} \mid e_{ij} = \partial T_i \cap \partial T_j, T_i, T_j \in \mathcal{T}_h, i \neq j\}$ (i.e. the interior edges of the triangulation \mathcal{T}_h), and $P_k(T)$ denote the space of bivariate polynomials of degree $\leq k$ on T . Then

$$RT_k(T) := (P_k(T))^2 + \mathbf{x}P_k(T), \quad (1.1)$$

and the RT_k approximation space is defined by

$$RT_k(\Omega) := \{\mathbf{u}_h \mid \mathbf{u}_h \in RT_k(T) \forall T \in \mathcal{T}_h, \mathbf{u}_h \cdot \mathbf{n}_{ij} \text{ is continuous across } e_{ij}, \forall e_{ij} \in \mathcal{E}_h\}, \quad (1.2)$$

where \mathbf{n}_{ij} is a unit normal vector to e_{ij} . The $BDM_k(\Omega)$ approximating space is defined by: $BDM_k(T) := (P_k(T))^2$,

$$BDM_k(\Omega) := \{\mathbf{u}_h \mid \mathbf{u}_h \in BDM_k(T) \forall T \in \mathcal{T}_h, \mathbf{u}_h \cdot \mathbf{n}_{ij} \text{ is continuous across } e_{ij}, \forall e_{ij} \in \mathcal{E}_h\}. \quad (1.3)$$

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In the usual description of the RT_k and BDM_k spaces the continuity of $\mathbf{u}_h \cdot \mathbf{n}_{ij}$ across $e_{ij} \in \mathcal{E}_h$ is depicted graphically by indicating degrees of freedom for $\mathbf{u}_h \cdot \mathbf{n}$ on the edges of $T \in \mathcal{T}_h$ (see [3]).

In typical Finite Element (FE) computations restrictions on the Test Space, such as continuity of the approximation, continuity of the approximation and its derivative, are implicitly imposed by the appropriate choice of the basis functions for the Test Space. Additionally, these basis functions have a Lagrangian property where they have the value one at their nodal point and vanish at the other nodal basis points. Computations are performed using an affine transformation which maps integrals over $T \in \mathcal{T}_h$ to integrals over the reference triangle \hat{T} on which the basis functions are defined.

Finite Element software packages which include RT elements as a approximation choice typically only provide an RT_0 approximation choice. A detailed discussion of the computational implementation of RT_0 elements may be found in [2]. (In the latest release of FreeFEM [4], version 3.13, RT_1 and BDM_1 elements have been added.) Our interest in RT_k , $k \geq 1$, approximations arose from the approximation of coupled Stokes–Darcy flow problems. A common choice for the velocity–pressure approximation in Stokes flow problems is the Taylor–Hood $P_2 - P_1$ pair, i.e. a quadratic approximation for the velocity components and a linear approximation for the pressure. The desire to have a quadratic approximation for the velocity throughout the coupled domain motivated our investigation into approximations using RT_2 .

Regarding the computation using RT and BDM elements, using an affine transformation to map T to \hat{T} is not appropriate. An affine transformation does not preserve orthogonality of vectors and is therefore not suitable for enforcing the continuity of $\mathbf{u}_h \cdot \mathbf{n}_{ij}$ across e_{ij} . Rather, as is well known, the Piola transformation mapping T to \hat{T} should be used in order to implement the continuity of $\mathbf{u}_h \cdot \mathbf{n}_{ij}$ across e_{ij} .

In this article we derive computational bases for RT and BDM approximations. The basis functions are defined relative to the reference triangle \hat{T} and have a Lagrangian property. This approach differs from the work of Arnold, Falk and Winther [1], in which they present an abstract framework for the geometric decomposition of RT and BDM spaces to form local bases for them on any triangle T in the triangulation. In the following section we introduce the notation used and review the Piola transformation. In Section 3 explicit bases are given for RT_0 , RT_1 , RT_2 , and the general RT_k case. Bases for BDM_1 , BDM_2 and BDM_k are derived in Section 4. Section 5 contains, as a numerical illustration, an approximation to a Darcy flow problem. Results where the velocity is approximated using RT_0 , RT_1 , RT_2 , BDM_1 , and BDM_2 are given.

2 Notation

We denote by $P_k(\mathbb{R}^n)$ the space of polynomials in n variable of degree $\leq k$, and for $T \subset \mathbb{R}^n$, $P_k(T)$ the restriction of $P_k(\mathbb{R}^n)$ to T . The reference triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$ is denoted \hat{T} (see Figure 2.1).

From [3], the dimension of the space $RT_k(T)$ is $(k + 1)(k + 3)$, and the dimension of $BDM_k(T)$ is $(k + 1)(k + 2)$.

In describing the bases for RT_k and BDM_k we divide the basis functions into two classes: *Normal Basis Functions* and *Non-Normal Basis Functions*.

2.1 Normal Basis Functions

The Normal Basis Functions are partitioned into three subclasses, each subclass associated with an edge of \widehat{T} . The edge numbering of \widehat{T} is: edge 1 refers to the edge opposite vertex $(0, 0)$, edge 2 refers to the edge opposite vertex $(1, 0)$, and edge 3 refers to the vertex opposite vertex $(0, 1)$ (See Figure 2.1). Normal Basis Functions associated with edge $[i]$ are denoted $\Phi^{[i]}$.

2.2 Non-Normal Basis Functions

For $RT_k(\widehat{T})$ the Non-Normal Basis Functions are partitioned into two subclasses, denoted $\Phi^{[4]}$ and $\Phi^{[5]}$.

For $BDM_k(\widehat{T})$ the Non-Normal Basis Functions are better described as *Tangent Basis Functions* and *Interior Basis Functions*. The Tangent Basis Functions, analogous to the Normal Basis Functions, are partitioned into three subclasses, denoted $\Phi^{[4]}$, $\Phi^{[5]}$ and $\Phi^{[6]}$, associated with edges 1, 2, and 3, respectively. There are two subclasses of Interior Basis Functions for $BDM_k(\widehat{T})$, denoted $\Phi^{[7]}$ and $\Phi^{[8]}$.

2.3 Property of the Basis Functions

Let \mathbf{n}_k , $k = 1, 2, 3$, denote the outer unit normals to the respective edges on \widehat{T} . We have that

$$\begin{aligned} \text{Normal Basis Functions: } \Phi^{[i]} \cdot \mathbf{n}_k &= 0, \text{ along edge } k, \quad i, k = 1, 2, 3, \quad i \neq k, & (2.4) \\ \text{Non-Normal Basis Functions: } \Phi^{[i]} \cdot \mathbf{n}_k &= 0, \text{ along edge } k, \quad k = 1, 2, 3, \quad i = 4, 5, \dots, 8 & (2.5) \end{aligned}$$

2.4 The Piola Transformation

We use the Piola transformation to map functions defined in $RT_k(\widehat{T})$ (or $BDM_k(\widehat{T})$) to functions defined in $RT_k(T)$ (or $BDM_k(T)$).

Consider the affine function F (see Figure 2.1) mapping \widehat{T} onto T .

We have

$$F : \begin{cases} x &= x_1 + (x_2 - x_1)\xi + (x_3 - x_1)\eta \\ y &= y_1 + (y_2 - y_1)\xi + (y_3 - y_1)\eta \end{cases}. \quad (2.6)$$

Let

$$J_T = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} (x_2 - x_1) & (x_3 - x_1) \\ (y_2 - y_1) & (y_3 - y_1) \end{bmatrix}, \quad \text{and } |J_T| = |\det(J_T)|.$$

The Piola transformation $\mathcal{P} : L^2(\widehat{T}) \rightarrow T$ is defined by

$$\hat{\mathbf{q}} \mapsto \mathbf{q}(x) = \mathcal{P}(\hat{\mathbf{q}})(x) := \frac{1}{|J_T|} J_T \hat{\mathbf{q}}(\hat{x}). \quad (2.7)$$

The Piola transformation has the following properties [3, 5]:

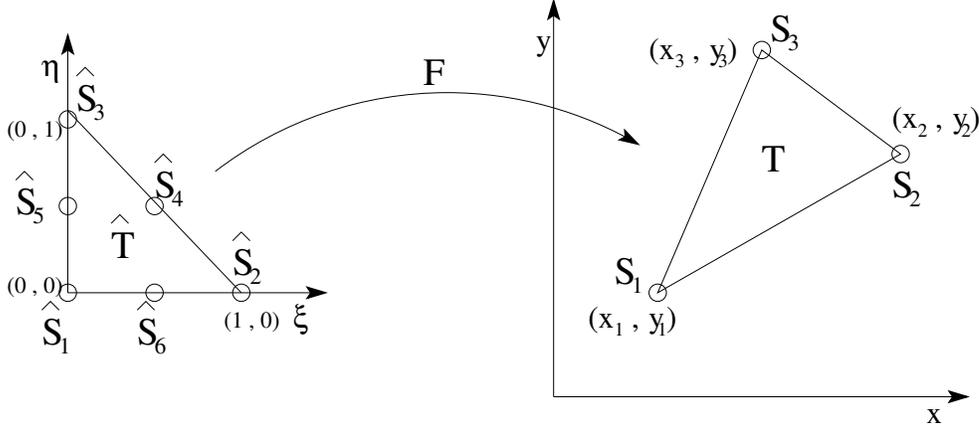


Figure 2.1: Mapping of the reference triangle \hat{T} to the triangle T .

Lemma 1 For $\hat{\mathbf{q}} \in C^1(\hat{T})$, we have that $\mathbf{q} = \mathcal{P}(\hat{\mathbf{q}}) \in C^1(T)$, and

$$\nabla_{x,y} \mathbf{q} = \frac{1}{|J_T|} J_T \nabla_{\xi,\eta} \hat{\mathbf{q}} J_T^{-1}, \quad \text{and} \quad \nabla_{x,y} \cdot \mathbf{q} = \frac{1}{|J_T|} \nabla_{\xi,\eta} \cdot \hat{\mathbf{q}}. \quad (2.8)$$

Lemma 2 For $\hat{v} \in H^1(\hat{T})$, $\hat{\mathbf{q}}, \hat{\mathbf{p}} \in H_{div}(\hat{T})$, $v := \hat{v} \circ F^{-1}$, $\mathbf{p} = \mathcal{P}(\hat{\mathbf{p}})$, and $\mathbf{q} = \mathcal{P}(\hat{\mathbf{q}})$ then

$$\int_T \mathbf{q} \cdot \nabla_{x,y} v \, d\mathbf{x} = \int_{\hat{T}} \hat{\mathbf{q}} \cdot \nabla_{\xi,\eta} \hat{v} \, d\xi, \quad (2.9)$$

$$\int_T \nabla_{x,y} \cdot \mathbf{q} \, v \, d\mathbf{x} = \int_{\hat{T}} \nabla_{\xi,\eta} \cdot \hat{\mathbf{q}} \, \hat{v} \, d\xi, \quad (2.10)$$

$$\int_{\partial T} \mathbf{q} \cdot \mathbf{n} \, v \, ds = \int_{\partial \hat{T}} \hat{\mathbf{q}} \cdot \mathbf{n} \, \hat{v} \, d\hat{s}, \quad (2.11)$$

$$\int_T \mathbf{p} \cdot \mathbf{q} \, d\mathbf{x} = \int_{\hat{T}} J_T \hat{\mathbf{p}} \cdot J_T \hat{\mathbf{q}} \frac{1}{|J_T|} \, d\xi. \quad (2.12)$$

Important Remark: There are a number of affine mappings of a triangle T with vertices V_1, V_2, V_3 , to \hat{T} . From Figure 2.1 the association $S_1 = V_1, S_2 = V_2, S_3 = V_3$, defines one such mapping. Another is given by the association $S_1 = V_2, S_2 = V_3, S_3 = V_1$. In order that $\mathbf{u}_h \cdot \mathbf{n}$ is continuous across the edges of the interior edges of the triangulation we have the following restriction on the choice of the mapping of T to \hat{T} . Suppose that e_{ij} is a shared edge between triangles T_i and T_j , and that under the mapping of T_i to \hat{T} e_{ij} maps to edge 1 of \hat{T} , i.e. the edge of \hat{T} of length $\sqrt{2}$. Then the mapping of T_j to \hat{T} must also map e_{ij} to edge 1. This does not restrict the triangulation, only the choice of the association of S_1, S_2, S_3 with the vertices of T .

3 Basis Functions for $RT_k(T)$

Introduce the following functions:

$$\hat{\mathbf{e}}_1(\xi, \eta) = \sqrt{2} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \quad \hat{\mathbf{e}}_2(\xi, \eta) = \begin{bmatrix} \xi - 1 \\ \eta \end{bmatrix}, \quad \hat{\mathbf{e}}_3(\xi, \eta) = \begin{bmatrix} \xi \\ \eta - 1 \end{bmatrix}, \quad (3.1)$$

$$\hat{\mathbf{e}}_4(\xi, \eta) = \eta \begin{bmatrix} \xi \\ \eta - 1 \end{bmatrix}, \quad \hat{\mathbf{e}}_5(\xi, \eta) = \xi \begin{bmatrix} \xi - 1 \\ \eta \end{bmatrix}. \quad (3.2)$$

3.1 Basis for $RT_0(\hat{T})$

The dimension of $RT_0(\hat{T})$ is 3. There are no Non-Normal Basis Functions. The Normal Basis Functions are:

$$\Phi_1^{[1]}(\xi, \eta) = \hat{\mathbf{e}}_1(\xi, \eta), \quad \Phi_1^{[2]}(\xi, \eta) = \hat{\mathbf{e}}_2(\xi, \eta), \quad \Phi_1^{[3]}(\xi, \eta) = \hat{\mathbf{e}}_3(\xi, \eta). \quad (3.3)$$

Observe that for $\hat{\mathbf{g}}_j$ denoting the midpoint along edge $[j]$,

$$\Phi_1^{[i]}(\hat{\mathbf{g}}_j) \cdot \mathbf{n}_i = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{if } j \neq i. \end{cases} \quad (3.4)$$

Note that $\Phi_1^{[i]}, i = 1, 2, 3$, are in $RT_0(\hat{T})$, and by property (3.4) the functions are linearly independent. As $\dim(RT_0(\hat{T}))$ is 3, then it follows that the $\Phi_1^{[i]}, i = 1, 2, 3$ form a basis for $RT_0(\hat{T})$.

3.2 Basis for $RT_1(\hat{T})$

The dimension of $RT_1(\hat{T})$ is 8. There are six Normal Basis Function, two associated with each side, and two Non-Normal Basis Functions.

Let $g_1 = 1/2 - \sqrt{3}/6$, $g_2 = 1/2 + \sqrt{3}/6$ denote the two Gaussian quadrature points on the interval $[0, 1]$. Introduce the two linear Lagrangian polynomials associated with the Gaussian quadrature points

$$l_1(t) := \frac{(t - g_2)}{(g_1 - g_2)}, \quad \text{and } l_2(t) := \frac{(t - g_1)}{(g_2 - g_1)}. \quad (3.5)$$

The Normal Basis Functions are (defined in a counter clockwise orientation):

$$\Phi_1^{[1]}(\xi, \eta) = l_1(\eta) \hat{\mathbf{e}}_1(\xi, \eta), \quad \Phi_2^{[1]}(\xi, \eta) = l_2(\eta) \hat{\mathbf{e}}_1(\xi, \eta), \quad (3.6)$$

$$\Phi_1^{[2]}(\xi, \eta) = l_2(\eta) \hat{\mathbf{e}}_2(\xi, \eta), \quad \Phi_2^{[2]}(\xi, \eta) = l_1(\eta) \hat{\mathbf{e}}_2(\xi, \eta), \quad (3.7)$$

$$\Phi_1^{[3]}(\xi, \eta) = l_1(\xi) \hat{\mathbf{e}}_3(\xi, \eta), \quad \Phi_2^{[3]}(\xi, \eta) = l_2(\xi) \hat{\mathbf{e}}_3(\xi, \eta). \quad (3.8)$$

The Non-Normal Basis Functions are:

$$\Phi_1^{[4]}(\xi, \eta) = \hat{\mathbf{e}}_4(\xi, \eta), \quad \Phi_1^{[5]}(\xi, \eta) = \hat{\mathbf{e}}_5(\xi, \eta). \quad (3.9)$$

Note that the Normal Basis Function have a Lagrangian property that along the boundary $\Phi_j^{[i]} \cdot \mathbf{n}$ vanishes at all but one of the Gaussian quadrature points. Also, as both Non-Normal Basis Functions satisfy $\Phi_1^{[i]} \cdot \mathbf{n} = 0$ along the boundary then it follows that the Normal Basis Functions are linearly independent. As the two Non-Normal Basis Functions are linearly independent then we have that the set of Normal and Non-Normal Basis Functions is linearly independent. Noting that the $\Phi_j^{[i]}$'s

are all elements of $RT_1(\widehat{T})$ then the given Normal and Non-Normal Basis Functions must form a basis for $RT_1(\widehat{T})$.

Remark: The choice of g_1, g_2 as Gaussian quadrature points is simply for convenience of evaluating integrals along edges of the triangulation. For any choice of $g_1, g_2 \in [0, 1]$, $g_1 \neq g_2$, the above construction generates a basis for $RT_1(\widehat{T})$.

3.3 Basis for $RT_2(\widehat{T})$

The dimension of $RT_1(\widehat{T})$ is 15. There are nine Normal Basis Function, three associated with each side, and six Non-Normal Basis Functions.

Let $g_1 = 1/2 - \sqrt{15}/10$, $g_2 = 1/2$, $g_3 = 1/2 + \sqrt{15}/10$ denote the three Gaussian quadrature points on the interval $[0, 1]$. Introduce the three quadratic Lagrangian polynomials associated with the Gaussian quadrature points

$$q_1(t) := \frac{(t - g_2)(t - g_3)}{(g_1 - g_2)(g_1 - g_3)}, \quad q_2(t) := \frac{(t - g_1)(t - g_3)}{(g_2 - g_1)(g_2 - g_3)}, \quad \text{and} \quad q_3(t) := \frac{(t - g_1)(t - g_2)}{(g_3 - g_1)(g_3 - g_2)}. \quad (3.10)$$

The Normal Basis Functions are (defined in a counter clockwise orientation):

$$\Phi_1^{[1]}(\xi, \eta) = q_1(\eta) \hat{e}_1(\xi, \eta), \quad \Phi_2^{[1]}(\xi, \eta) = q_2(\eta) \hat{e}_1(\xi, \eta), \quad \Phi_3^{[1]}(\xi, \eta) = q_3(\eta) \hat{e}_1(\xi, \eta), \quad (3.11)$$

$$\Phi_1^{[2]}(\xi, \eta) = q_3(\eta) \hat{e}_2(\xi, \eta), \quad \Phi_2^{[2]}(\xi, \eta) = q_2(\eta) \hat{e}_2(\xi, \eta), \quad \Phi_3^{[2]}(\xi, \eta) = q_1(\eta) \hat{e}_2(\xi, \eta), \quad (3.12)$$

$$\Phi_1^{[3]}(\xi, \eta) = q_1(\xi) \hat{e}_3(\xi, \eta), \quad \Phi_2^{[3]}(\xi, \eta) = q_2(\xi) \hat{e}_3(\xi, \eta), \quad \Phi_3^{[3]}(\xi, \eta) = q_3(\xi) \hat{e}_3(\xi, \eta). \quad (3.13)$$

The Non-Normal Basis Functions are:

$$\Phi_1^{[4]}(\xi, \eta) = (1 - \xi - \eta) \hat{e}_4(\xi, \eta), \quad \Phi_2^{[4]}(\xi, \eta) = \xi \hat{e}_4(\xi, \eta), \quad \Phi_3^{[4]}(\xi, \eta) = \eta \hat{e}_4(\xi, \eta), \quad (3.14)$$

$$\Phi_1^{[5]}(\xi, \eta) = (1 - \xi - \eta) \hat{e}_5(\xi, \eta), \quad \Phi_2^{[5]}(\xi, \eta) = \xi \hat{e}_5(\xi, \eta), \quad \Phi_3^{[5]}(\xi, \eta) = \eta \hat{e}_5(\xi, \eta). \quad (3.15)$$

3.4 The General Case: Basis $RT_k(\widehat{T})$

The basis for the general case is constructed in a similar manner.

Let g_n , $n = 1, 2, \dots, k + 1$ denote the $k + 1$ Gaussian quadrature points on the interval $[0, 1]$, and $l_j(t)$ denote the Lagrangian polynomial of degree k such that $l_j(g_n) = \begin{cases} 1, & \text{if } n = j, \\ 0, & \text{if } n \neq j. \end{cases}$

Also, let $\{b_i(\xi, \eta), i = 1, 2, \dots, k(k + 1)/2\}$ denote a basis for $P_{k-1}(\widehat{T})$.

The Normal Basis Functions.

Associated with edge 1 we have the basis functions:

$$\Phi_j^{[1]}(\xi, \eta) = l_j(\eta) \hat{e}_1(\xi, \eta), \quad j = 1, 2, \dots, k + 1. \quad (3.16)$$

Associated with edge 2 we have the basis functions:

$$\Phi_j^{[2]}(\xi, \eta) = l_{k+2-j}(\eta) \hat{\mathbf{e}}_2(\xi, \eta), \quad j = 1, 2, \dots, k+1. \quad (3.17)$$

Associated with edge 3 we have the basis functions:

$$\Phi_j^{[3]}(\xi, \eta) = l_j(\xi) \hat{\mathbf{e}}_3(\xi, \eta), \quad j = 1, 2, \dots, k+1. \quad (3.18)$$

The Non-Normal Basis Functions are:

$$\Phi_j^{[4]}(\xi, \eta) = b_j(\xi, \eta) \hat{\mathbf{e}}_4(\xi, \eta), \quad j = 1, 2, \dots, k(k+1)/2, \quad (3.19)$$

$$\Phi_j^{[5]}(\xi, \eta) = b_j(\xi, \eta) \hat{\mathbf{e}}_5(\xi, \eta), \quad j = 1, 2, \dots, k(k+1)/2. \quad (3.20)$$

Note that the number of Normal Basis Functions plus the number of Non-Normal Basis Functions: $3(k+1) + 2k(k+1)/2 = (k+1)(k+3) = \dim(RT_k(\widehat{T}))$.

The linear independence of the Normal Basis Functions follows from their Lagrangian property and the fact that the Non-Normal Basis Functions satisfy (2.4). Hence, to establish that (3.16)-(3.20) form a basis for $RT_k(\widehat{T})$ what remains is to show that the Non-Normal Basis Functions are linearly independent. A simply calculations shows that the only values for (ξ, η) such that

$$C_1 \hat{\mathbf{e}}_4(\xi, \eta) + C_2 \hat{\mathbf{e}}_5(\xi, \eta) = \mathbf{0}$$

has a nontrivial solution for C_1, C_2 lie along the lines $\xi = 0$, $\eta = 0$, and $\xi + \eta = 1$.

Consider a linear combination of the Non-Normal Basis Functions, $\alpha_j, \beta_j \in \mathbb{R}$, $j = 1, 2, \dots, N := k(k+1)/2$,

$$\begin{aligned} (\alpha_1 b_1(\xi, \eta) + \alpha_2 b_2(\xi, \eta) + \dots + \alpha_N b_N(\xi, \eta)) \hat{\mathbf{e}}_4(\xi, \eta) \\ + (\beta_1 b_1(\xi, \eta) + \beta_2 b_2(\xi, \eta) + \dots + \beta_N b_N(\xi, \eta)) \hat{\mathbf{e}}_5(\xi, \eta) = \mathbf{0} \end{aligned} \quad (3.21)$$

Without loss of generality, suppose that $\{b_i(\xi, \eta), i = 1, 2, \dots, N\}$ is a Lagrangian basis with nodes, $\sigma_j, j = 1, 2, \dots, N$ lying strictly inside \widehat{T} , with $b_i(\sigma_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$

Then, with $(\xi, \eta) = \sigma_j$ (3.21) implies

$$\alpha_j \hat{\mathbf{e}}_4(\sigma_j) + \beta_j \hat{\mathbf{e}}_5(\sigma_j) = \mathbf{0}.$$

As noted above, this then implies that $\alpha_j = \beta_j = 0$. Hence the Non-Normal Basis Functions are linearly independent.

3.5 Basis for $RT_k(T)$

Note that from (2.7),(3.1),(3.2), for $i = 1, 2, 3$,

$$\mathbf{e}_i(x, y) = \mathcal{P}(\hat{\mathbf{e}}_i(F^{-1}(x, y))) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} c_3, \quad \text{where } c_1, c_2, c_3 \in P_0(T) = \mathbb{R}, \quad (3.22)$$

and for $i = 4, 5$,

$$\mathbf{e}_i(x, y) = \mathcal{P}(\hat{\mathbf{e}}_i(F^{-1}(x, y))) = \begin{bmatrix} p_1^i(x, y) \\ p_2^i(x, y) \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} p_3^i(x, y), \quad (3.23)$$

where $p_1^i(x, y), p_2^i(x, y), p_3^i(x, y) \in P_1(T)$.

Also, for $\hat{p}(\xi, \eta) \in P_l(\hat{T})$, then

$$\begin{aligned} \phi_k^{[i]}(x, y) &= \mathcal{P}(\Phi_k^{[i]}(F^{-1}(x, y))) = \mathcal{P}(\hat{p}(F^{-1}(x, y)) \hat{\mathbf{e}}_i(F^{-1}(x, y))) \\ &= p(x, y) \mathcal{P}(\hat{\mathbf{e}}_i(F^{-1}(x, y))), \quad \text{where } p(x, y) \in P_l(T). \end{aligned} \quad (3.24)$$

Theorem 1 *Under the Piola transformation (2.7), the basis given for $RT_k(\hat{T})$ transforms to a basis for $RT_k(T)$.*

Proof: The stated result follows from the definition of $RT_k(T)$, (1.1), and the properties (3.22)-(3.24). ■

4 Basis Functions for $BDM_k(T)$

Introduce the following *edge functions*. We suppress the dependence of the functions on ξ and η .

$$\begin{aligned} \hat{\mathbf{e}}_1(s_1, s_2) &= \frac{\sqrt{2}}{(s_2 - s_1)} \begin{bmatrix} s_2 \xi \\ (s_2 - 1) \eta \end{bmatrix}, \quad \hat{\mathbf{e}}_2(s_1, s_2) = \frac{1}{(s_2 - s_1)} \begin{bmatrix} s_2 \xi + \eta - s_2 \\ (s_2 - 1) \eta \end{bmatrix}, \\ \hat{\mathbf{e}}_3(s_1, s_2) &= \frac{1}{(s_2 - s_1)} \begin{bmatrix} (s_2 - 1) \xi \\ \xi + s_2 \eta - s_2 \end{bmatrix}, \quad \hat{\mathbf{e}}_4(s_1, s_2) = (1 - \xi - \eta) \hat{\mathbf{e}}_1(s_1, s_2), \\ \hat{\mathbf{e}}_5(s_1, s_2) &= \xi \hat{\mathbf{e}}_2(s_1, s_2), \quad \hat{\mathbf{e}}_6(s_1, s_2) = \eta \hat{\mathbf{e}}_3(s_1, s_2). \end{aligned} \quad (4.1)$$

Let $\hat{\mathbf{b}}_1(\xi, \eta), \hat{\mathbf{b}}_2(\xi, \eta)$ denote the interior bubble functions

$$\hat{\mathbf{b}}_1(\xi, \eta) = (1 - \xi - \eta) \xi \eta \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{b}}_2(\xi, \eta) = (1 - \xi - \eta) \xi \eta \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (4.2)$$

Note that

$$\begin{array}{ccc} \text{along edge 1} & \text{along edge 2} & \text{along edge 3} \\ \text{(i.e. } \xi + \eta = 1) & \text{(i.e. } \xi = 0) & \text{(i.e. } \eta = 0) \\ \text{for } \eta = s_1, \hat{\mathbf{e}}_1 \cdot \mathbf{n}_1 = 1, & \text{for } \eta = s_1, \hat{\mathbf{e}}_2 \cdot \mathbf{n}_2 = 1, & \text{for } \xi = s_1, \hat{\mathbf{e}}_3 \cdot \mathbf{n}_3 = 1, \\ \text{for } \eta = s_2, \hat{\mathbf{e}}_1 \cdot \mathbf{n}_1 = 0, & \text{for } \eta = s_2, \hat{\mathbf{e}}_2 \cdot \mathbf{n}_2 = 0, & \text{for } \xi = s_2, \hat{\mathbf{e}}_3 \cdot \mathbf{n}_3 = 0. \end{array} \quad (4.3)$$

Also, $\hat{\mathbf{e}}_i \cdot \mathbf{n}_j = 0$, for $i = 1, 2, \dots, 6, j = 1, 2, 3, i \neq j$, and $\hat{\mathbf{b}}_i(\xi, \eta) \cdot \mathbf{n}_j = 0$, for $i = 1, 2, j = 1, 2, 3$.

4.1 $BDM_1(\widehat{T})$

The dimension of $BDM_1(\widehat{T})$ is 6. The basis consists of six Normal Basis Functions.

Let $g_1 = 1/2 - \sqrt{3}/6$, $g_2 = 1/2 + \sqrt{3}/6$ denote the two Gaussian quadrature points on the interval $[0, 1]$.

The basis functions (defined in a counter clockwise orientation) are:

$$\Phi_1^{[1]}(\xi, \eta) = \hat{\mathbf{e}}_1(g_1, g_2), \quad \Phi_2^{[1]}(\xi, \eta) = \hat{\mathbf{e}}_1(g_2, g_1), \quad (4.4)$$

$$\Phi_1^{[2]}(\xi, \eta) = \hat{\mathbf{e}}_2(g_2, g_1), \quad \Phi_2^{[2]}(\xi, \eta) = \hat{\mathbf{e}}_2(g_1, g_2), \quad (4.5)$$

$$\Phi_1^{[3]}(\xi, \eta) = \hat{\mathbf{e}}_3(g_1, g_2), \quad \Phi_1^{[3]}(\xi, \eta) = \hat{\mathbf{e}}_3(g_2, g_1). \quad (4.6)$$

4.2 $BDM_2(\widehat{T})$

The dimension of $BDM_2(\widehat{T})$ is 12. The basis consists of nine Normal Basis Function and three Tangent Basis Functions.

Let $l(z_1, z_2; t) := \frac{(t-z_2)}{(z_1-z_2)}$, represent the linear function which is equal to 1 when $t = z_1$, and equal to 0 when $t = z_2$.

Let $g_1 = 1/2 - \sqrt{15}/10$, $g_2 = 1/2$, $g_3 = 1/2 + \sqrt{15}/10$ denote the three Gaussian quadrature points on the interval $[0, 1]$.

The Normal Basis Functions are (defined in a counter clockwise orientation):

$$\Phi_1^{[1]}(\xi, \eta) = l(g_1, g_3; \eta) \hat{\mathbf{e}}_1(g_1, g_2), \quad \Phi_2^{[1]}(\xi, \eta) = l(g_2, g_1; \eta) \hat{\mathbf{e}}_1(g_2, g_3), \quad \Phi_3^{[1]}(\xi, \eta) = l(g_3, g_2; \eta) \hat{\mathbf{e}}_1(g_3, g_1), \quad (4.7)$$

$$\Phi_1^{[2]}(\xi, \eta) = l(g_3, g_2; \eta) \hat{\mathbf{e}}_2(g_3, g_1), \quad \Phi_2^{[2]}(\xi, \eta) = l(g_2, g_1; \eta) \hat{\mathbf{e}}_2(g_2, g_3), \quad \Phi_3^{[2]}(\xi, \eta) = l(g_1, g_3; \eta) \hat{\mathbf{e}}_2(g_1, g_2), \quad (4.8)$$

$$\Phi_1^{[3]}(\xi, \eta) = l(g_1, g_3; \xi) \hat{\mathbf{e}}_3(g_1, g_2), \quad \Phi_2^{[3]}(\xi, \eta) = l(g_2, g_1; \xi) \hat{\mathbf{e}}_3(g_2, g_3), \quad \Phi_3^{[3]}(\xi, \eta) = l(g_3, g_2; \xi) \hat{\mathbf{e}}_3(g_3, g_1). \quad (4.9)$$

The Tangent Basis Functions are:

$$\Phi_1^{[4]}(\xi, \eta) = \hat{\mathbf{e}}_4(g_1, g_2), \quad \Phi_1^{[5]}(\xi, \eta) = \hat{\mathbf{e}}_5(g_1, g_2), \quad \Phi_1^{[6]}(\xi, \eta) = \hat{\mathbf{e}}_6(g_1, g_2). \quad (4.10)$$

To establish that the $\Phi_j^{[i]}(\xi, \eta)$, defined in (4.7)-(4.10), form a basis for $BDM_2(\widehat{T})$ it suffices to show that $\Phi_1^{[4]}$, $\Phi_1^{[5]}$, $\Phi_1^{[6]}$, are linearly independent. Consider

$$\alpha_0 \Phi_1^{[4]}(\xi, \eta) + \beta_0 \Phi_1^{[5]}(\xi, \eta) + \gamma_0 \Phi_1^{[6]}(\xi, \eta) = \mathbf{0}. \quad (4.11)$$

Along edge 1, the unit tangent is $\hat{\mathbf{t}} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\xi = 1 - \eta$. Taking the dot product of (4.11) with respect to $\hat{\mathbf{t}}$, substituting $\xi = 1 - \eta$, and simplifying we obtain the equation

$$\beta_0 - \gamma_0 = 0. \quad (4.12)$$

Along edge 2, the unit tangent is $\hat{\mathbf{t}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, and $\xi = 0$. Taking the dot product of (4.11) with respect to $\hat{\mathbf{t}}$, substituting $\xi = 0$, and simplifying we obtain the equation

$$\sqrt{2}(g_2 - 1)\alpha_0 - g_2\gamma_0 = 0. \quad (4.13)$$

Similarly, along edge 3, the unit tangent is $\hat{\mathbf{t}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\eta = 0$. Taking the dot product of (4.11) with respect to $\hat{\mathbf{t}}$, substituting $\eta = 0$, and simplifying we obtain the equation

$$\sqrt{2}\alpha_0 - \beta_0 = 0. \quad (4.14)$$

From (4.12)-(4.14) it follows that $\alpha_0 = \beta_0 = \gamma_0 = 0$.

Hence the $\Phi_j^{[i]}(\xi, \eta)$ for a basis for $BDM_2(\hat{T})$.

4.3 The General Case: $BDM_k(\hat{T})$

The basis for the general case is constructed in a similar manner.

Let g_n , $n = 1, 2, \dots, k+1$ denote the $k+1$ Gaussian quadrature points on the interval $[0, 1]$. Also, for ease of explanation of the basis, let $g_{k+1+i} = g_i$, $i = 1, 2, \dots, k+1$.

Let $l_j(t)$ denote the Lagrangian polynomial of degree $k-1$, constructed using the k points $g_j, g_{j+2}, g_{j+3}, \dots, g_{j+k}$ such that $l_j(g_n) = \begin{cases} 1, & \text{if } n = j, \\ 0, & \text{if } n = j+2, j+3, \dots, j+k. \end{cases}$

Also, let $\{\rho_i(\xi), i = 0, 1, \dots, (k-2)\}$ denote a basis for $P_{k-2}(\mathbb{R})$, and $\{v_i(\xi, \eta), i = 1, 2, \dots, (k-2)(k-1)/2\}$ denote a basis for $P_{k-3}(\hat{T})$.

The Normal Basis Functions.

Associated with Edge 1 we have the basis functions:

$$\Phi_j^{[1]}(\xi, \eta) = l_j(\eta) \hat{\mathbf{e}}_1(g_j, g_{j+1}), \quad j = 1, 2, \dots, k+1. \quad (4.15)$$

Associated with Edge 2 we have the basis functions:

$$\Phi_j^{[2]}(\xi, \eta) = l_j(\eta) \hat{\mathbf{e}}_2(g_j, g_{j+1}), \quad j = k+1, k, \dots, 1. \quad (4.16)$$

Associated with Edge 3 we have the basis functions:

$$\Phi_j^{[3]}(\xi, \eta) = l_j(\xi) \hat{\mathbf{e}}_3(g_j, g_{j+1}), \quad j = 1, 2, \dots, k+1. \quad (4.17)$$

The Tangent Basis Functions are:

$$\Phi_j^{[4]}(\xi, \eta) = \rho_j(\eta) \hat{\mathbf{e}}_4(g_1, g_2), \quad \Phi_j^{[5]}(\xi, \eta) = \rho_j(\xi) \hat{\mathbf{e}}_5(g_1, g_2), \quad \Phi_j^{[6]}(\xi, \eta) = \rho_j(\eta) \hat{\mathbf{e}}_6(g_1, g_2), \quad (4.18)$$

$j = 0, 1, \dots, k-2$.

The Interior Basis Functions are:

$$\Phi_j^{[7]}(\xi, \eta) = v_j(\xi, \eta) \hat{\mathbf{b}}_1(\xi, \eta), \quad \Phi_j^{[8]}(\xi, \eta) = v_j(\xi, \eta) \hat{\mathbf{b}}_2(\xi, \eta), \quad j = 1, 2, \dots, (k-2)(k-1)/2. \quad (4.19)$$

Note that the number of Normal Basis Functions plus Tangent Basis Functions plus Interior Basis Functions = $3(k+1) + 3(k-1) + (k-2)(k-1) = (k+1)(k+2) = \dim(BDM_k)$.

The linear independence of the Normal Basis Functions follows from the Lagrangian property and the fact that the Non-Normal Basis Functions satisfy (2.5). To show that the Non-Normal Basis Functions are linearly independent first note that the Interior Basis Functions all vanish on the boundary of \hat{T} . We proceed by considering a linear combination of the Tangent Basis Functions along the edges of \hat{T} .

Let the linear combination of the Tangent Basis Functions be given by

$$\begin{aligned} (\alpha_0 + \alpha_1\eta + \dots + \alpha_{k-2}\eta^{k-2})\hat{\mathbf{e}}_4(g_1, g_2) &+ (\beta_0 + \beta_1\xi + \dots + \beta_{k-2}\xi^{k-2})\hat{\mathbf{e}}_5(g_1, g_2) \\ &+ (\gamma_0 + \gamma_1\eta + \dots + \gamma_{k-2}\eta^{k-2})\hat{\mathbf{e}}_6(g_1, g_2) = \mathbf{0} \end{aligned} \quad (4.20)$$

As in the discussion above for the basis for BDM_2 , we consider (4.20) along each of the edges of \hat{T} .

Along edge 3 (i.e. $\hat{\mathbf{t}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\eta = 0$) from (4.20) we have that

$$\begin{aligned} (\sqrt{2}\alpha_0 - \beta_0) - \beta_1\xi - \beta_2\xi^2 - \dots - \beta_{k-2}\xi^{k-2} &= 0, \\ \implies \sqrt{2}\alpha_0 - \beta_0 = 0, \quad \beta_j = 0, \quad j = 1, 2, \dots, k-2. \end{aligned} \quad (4.21)$$

With $\beta_j = 0$, $j = 1, 2, \dots, k-2$, along edge 1 (i.e. $\hat{\mathbf{t}} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\xi = 1 - \eta$), from (4.20) it follows that

$$\beta_0 - \gamma_0 = 0, \quad \gamma_j = 0, \quad j = 1, 2, \dots, k-2. \quad (4.22)$$

With $\gamma_j = 0$, $j = 1, 2, \dots, k-2$, along edge 2 (i.e. $\hat{\mathbf{t}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $\xi = 0$) from (4.20) it follows that

$$\sqrt{2}(g_2 - 1)\alpha_0 - g_2\gamma_0 = 0, \quad \alpha_j = 0, \quad j = 1, 2, \dots, k-2. \quad (4.23)$$

In view of (4.12)-(4.14) we also have $\alpha_0 = \beta_0 = \gamma_0 = 0$.

The linear independence of the Interior Basis Functions is obvious.

Hence the $\Phi_j^{[i]}(\xi, \eta)$ for a basis for $BDM_k(\hat{T})$.

Theorem 2 *Under the Piola transformation (2.7), the basis given for $BDM_k(\hat{T})$ transforms to a basis for $BDM_k(T)$.*

Proof: The stated result follows from the definition of $BDM_k(T)$, and that the Piola transformation is an affine invertible mapping. ■

5 Numerical Example

In this section we briefly discuss the numerical implementation and present a numerical example.

5.1 Numerical Implementation

The numerical implementation of an approximation using the bases described above follow in a similar manner to the usual Finite Element computations with two caveats.

Recall that for a continuous (scalar) linear representation in Finite Elements we write

$$w_h(x, y) = \sum_k c_k \phi_k(x, y) = \sum_{T \in \mathcal{T}_h} \sum_{j=1}^3 c_{\kappa(j, T)} \hat{l}_j(F^{-1}(x, y)), \quad (5.1)$$

where $\hat{l}_1 = 1 - \xi - \eta$, $\hat{l}_2 = \xi$, $\hat{l}_3 = \eta$, denote the basis functions on \hat{T} , and $\kappa(j, T)$ denotes an index function. (See [6] for more details.)

Analogously we write

$$\begin{aligned} \mathbf{u}_h(x, y) &= \sum_k c_k \phi_k(x, y) \\ &= \sum_{T \in \mathcal{T}_h} \left[\underbrace{\sum_{i=1}^3 \sum_j c_{\kappa(i, j, T)} n_sgn(i, T) \mathcal{P}(\Phi_j^{[i]}(F^{-1}(x, y)))}_{\text{Normal Basis Functions}} + \underbrace{\sum_{i>3} \sum_j c_{\kappa(i, j, T)} \mathcal{P}(\Phi_j^{[i]}(F^{-1}(x, y)))}_{\text{Non-Normal Basis Functions}} \right] \end{aligned} \quad (5.2)$$

where $\kappa(i, j, T)$ denotes an index function.

Let e_{ij} denote the shared edge between triangles T_i and T_j .

1. Sign associated with Normal Basis Functions

The respective Normal Basis Functions for T_i and T_j point in opposite directions across e_{ij} . In order that $\mathbf{u}_h \cdot \mathbf{n}$ is continuous across e_{ij} , we assign a (unit) normal direction to e_{ij} . If the outer (unit) normal to T_i along e_{ij} is the negative of the assigned normal then the corresponding Normal Basis Functions are multiplied by -1 , otherwise there are multiplied by 1. We use the function $n_sgn(i, T) = \pm 1$ to represent this multiplicative factor associated with edge i of triangle T .

2. Restriction of the mapping of T to \hat{T}

For \mathbf{e}_i an edge of T with outer normal \mathbf{n} , let $\hat{\mathbf{e}}_{iT} = F^{-1}(\mathbf{e}_i)$ denote the corresponding edge on \hat{T} . Then, for $(x, y) \in \mathbf{e}_i$

$$\mathbf{u}_h \cdot \mathbf{n}(x, y) = \left[\sum_{i=1}^3 \sum_j c_{\kappa(i, j, T)} n_sgn(i, T) \mathcal{P}(\Phi_j^{[i]}(F^{-1}(x, y))) \right] \cdot \mathbf{n}. \quad (5.3)$$

In particular for $(x, y) = \mathbf{g} = F(\hat{\mathbf{g}})$, $\hat{\mathbf{g}}$ a Gaussian point on $\hat{\mathbf{e}}_{iT}$, $\mathbf{n}_e = n_sgn(i, T)\mathbf{n}$

$$\mathbf{u}_h \cdot \mathbf{n}_e|_{\mathbf{g}} = \frac{\text{length}(\hat{\mathbf{e}}_{iT})}{\text{length}(\mathbf{e}_i)} c_{\mathbf{g}}, \quad (5.4)$$

where $c_{\mathbf{g}}$ denotes the coefficient c_k in (5.3) associated with the nodal point \mathbf{g} .

Consider now e_{ij} . From (5.4)

$$\lim_{(x,y) \in T_i \rightarrow \mathbf{g}} \mathbf{u}_h \cdot \mathbf{n}_e = \frac{\text{length}(\hat{\mathbf{e}}_{iT_i})}{\text{length}(\mathbf{e}_i)} c_{\mathbf{g}},$$

and

$$\lim_{(x,y) \in T_j \rightarrow \mathbf{g}} \mathbf{u}_h \cdot \mathbf{n}_e = \frac{\text{length}(\hat{\mathbf{e}}_{iT_j})}{\text{length}(\mathbf{e}_i)} c_{\mathbf{g}}.$$

Continuity of $\mathbf{u}_h \cdot \mathbf{n}$ across e_{ij} requires that $\text{length}(\hat{\mathbf{e}}_{iT_i}) = \text{length}(\hat{\mathbf{e}}_{iT_j})$. Thus whenever an edge in the triangulation is mapped to the reference triangle its image must be the same size. This restriction does not impose a constraint on the triangulation, but rather on the numerical implementation. (See the **Important Remark** at the end of Section 2.

5.2 Example

Consider the numerical approximation of

$$\mathbf{u} + \eta \nabla p = \mathbf{f}, \text{ in } \Omega, \quad (5.5)$$

$$\nabla \cdot \mathbf{u} = 0, \text{ in } \Omega, \quad (5.6)$$

$$\mathbf{u} \cdot \mathbf{n} = g, \text{ on } \partial\Omega. \quad (5.7)$$

Remark: From (5.6), g must satisfy the compatibility condition $\int_{\partial\Omega} g \, ds = 0$.

We consider (5.5)–(5.7) for $\Omega = (-1, 1) \times (0, 1)$, $\eta = 1$, and known solution $\mathbf{u} = \begin{bmatrix} xy - y^2 \\ x + x^2 - 0.5y^2 \end{bmatrix}$, $p = 2x + 3y - 3/2$. For approximating elements we use $(\mathbf{u}_h, p_h) \in (RT_0, discP_0)$, $(\mathbf{u}_h, p_h) \in (RT_1, discP_1)$, $(\mathbf{u}_h, p_h) \in (RT_2, discP_2)$, $(\mathbf{u}_h, p_h) \in (BDM_1, discP_0)$ and $(\mathbf{u}_h, p_h) \in (BDM_2, discP_1)$. The numerical results on a series of meshes are presented in Tables 5.1–5.5. The mesh corresponding to $h = 1/4$ is given in Figure 5.1.

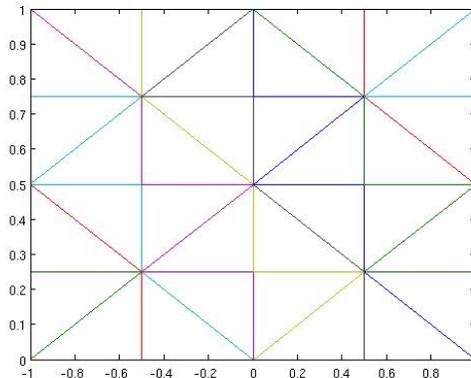


Figure 5.1: Computational mesh corresponding to $h = 1/4$.

$1/h$	$\ \mathbf{u} - \mathbf{u}_h\ _{L^2(\Omega)}$	Cvg. rate	$\ div(\mathbf{u}_h)\ _{L^2(\Omega)}$	$\ p - p_h\ _{L^2(\Omega)}$	Cvg. rate
4	3.706E-1	0.99	8.96E-16	4.168E-1	1.00
6	2.476E-1	1.00	3.92E-15	2.778E-1	1.00
8	1.856E-1	1.00	1.82E-15	2.083E-1	1.00
12	1.236E-1		6.45E-15	1.389E-1	
Pred. rate		1.00			1.00

Table 5.1: Experimental convergence rates for the $(RT_0, discP_0)$ approximation pair.

$1/h$	$\ \mathbf{u} - \mathbf{u}_h\ _{L^2(\Omega)}$	Cvg. rate	$\ div(\mathbf{u}_h)\ _{L^2(\Omega)}$	$\ p - p_h\ _{L^2(\Omega)}$	Cvg. rate
4	2.833E-2	2.00	5.16E-14	5.398E-4	2.97
6	1.258E-2	2.00	2.74E-13	1.617E-4	2.98
8	7.073E-3	2.00	7.04E-13	6.858E-5	2.99
12	3.142E-3		5.03E-12	2.042E-5	
Pred. rate		2.00			2.00

Table 5.2: Experimental convergence rates for the $(RT_1, discP_1)$ approximation pair.

$1/h$	$\ \mathbf{u} - \mathbf{u}_h\ _{L^2(\Omega)}$	Cvg. rate	$\ div(\mathbf{u}_h)\ _{L^2(\Omega)}$	$\ p - p_h\ _{L^2(\Omega)}$	Cvg. rate
4	1.047E-13		8.60E-14	6.612E-14	
6	1.677E-13		2.16E-13	6.965E-14	
8	1.943E-13		6.95E-13	7.056E-14	
12	1.008E-12		2.14E-12	8.421E-14	
Pred. rate		–			–

Table 5.3: Experimental convergence rates for the $(RT_2, discP_2)$ approximation pair.

$1/h$	$\ \mathbf{u} - \mathbf{u}_h\ _{L^2(\Omega)}$	Cvg. rate	$\ div(\mathbf{u}_h)\ _{L^2(\Omega)}$	$\ p - p_h\ _{L^2(\Omega)}$	Cvg. rate
4	2.833E-2	2.00	7.89E-16	4.167E-1	1.00
6	1.258E-2	2.00	2.13E-15	2.778E-1	1.00
8	7.073E-3	2.00	1.75E-15	2.083E-1	1.00
12	3.142E-3		5.29E-15	1.389E-1	
Pred. rate		2			1

Table 5.4: Experimental convergence rates for the $(BDM_1, discP_0)$ approximation pair.

$1/h$	$\ \mathbf{u} - \mathbf{u}_h\ _{L^2(\Omega)}$	Cvg. rate	$\ div(\mathbf{u}_h)\ _{L^2(\Omega)}$	$\ p - p_h\ _{L^2(\Omega)}$	Cvg. rate
4	1.038E-13		3.08E-15	4.235E-14	
6	1.100E-13		5.91E-15	4.349E-14	
8	1.132E-13		8.68E-15	4.384E-14	
12	1.165E-13		1.64E-14	4.423E-14	
Pred. rate		–			–

Table 5.5: Experimental convergence rates for the $(BDM_2, discP_1)$ approximation pair.

6 Concluding Remark

In this article we have introduced a computational basis for the RT_k and BDM_k approximating elements on a triangulation of the domain $\Omega \subset \mathbb{R}^2$. Our computational basis imposes a minor

restriction on the mapping of the triangles in the triangulation to the reference triangle \widehat{T} . (See **Important Remark** at the end of Section 2.) This restriction can be removed if instead of \widehat{T} given by $(0, 0)$, $(1, 0)$, $(0, 1)$ one uses \widehat{T}_{eq} given by $(-1, 0)$, $(0, 1)$, $(0, \sqrt{3})$. The corresponding bases on \widehat{T}_{eq} can be obtained by applying the Piola transformation to the bases defined above. Alternatively, as pointed out by N. Walkington [7], this restriction can be removed and \widehat{T} still used as the reference triangle if the Piola transformation (2.7) is modified for Normal Basis Functions to compensate for the factor $length(\widehat{\mathbf{e}}_i)/length(\mathbf{e}_i)$.

References

- [1] D.N. Arnold, R.S. Falk, and R. Winther. Geometric decompositions and local bases for spaces of finite element differential forms. *Comput. Methods Appl. Mech. Engrg.*, 198(21-26):1660–1672, 2009.
- [2] C. Bahriawati and C. Carstensen. Three MATLAB implementations of the lowest-order Raviart-Thomas MFEM with a posteriori error control. *Comput. Methods Appl. Math.*, 5(4):333–361 (electronic), 2005.
- [3] F. Brezzi and M. Fortin. *Mixed and hybrid finite element methods*. Springer-Verlag, New York, 1991.
- [4] F. Hecht and O. Pironneau. Freefem++. webpage: <http://www.freefem.org>.
- [5] R.H.W. Hoppe. Finite Element Methods – Course Notes (Chapter 7). Technical report, Department of Mathematics, University of Houston, 2005. http://www.math.uh.edu/~rohop/spring_05/downloads/Chapter7.pdf.
- [6] F.-J. Sayas. A gentle introduction to the Finite Element Method – Course Notes. Technical report, Department of Mathematical Sciences, University of Delaware, 2008. <http://www.math.udel.edu/~fjsayas/anIntro2FEM.pdf>.
- [7] N. Walkington. personal communication, 2011.