PARTITIONED PENALTY METHODS FOR THE TRANSPORT EQUATION IN THE EVOLUTIONARY STOKES-DARCY-TRANSPORT PROBLEM

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Abstract. There has been a surge of work on models for coupling surface-water with groundwater flows which is at its core the Stokes-Darcy problem, as well as methods for uncoupling the problem into subdomain, subphysics solves. The resulting (Stokes-Darcy) fluid velocity is important because the flow transports contaminants. The numerical analysis and algorithm development for the evolutionary transport problem has, however, focused on a quasi-static Stokes-Darcy model and a single domain (fully coupled) formulation of the transport equation. This report presents a numerical analysis of a partitioned method for contaminant transport for the fully evolutionary system. The algorithm studied is unconditionally stable with one subdomain solve per step. Numerical experiments are given using the proposed algorithm, that investigate the effects of the penalty parameters on the convergence of the approximations.

Key words. Stokes-Darcy, transport, quasi-static, partitioned method

AMS subject classification. 65M12, 65M15, 65M55, 65M60, 35M10, 35Q35, 76D07, 76S05

1. Introduction.

The Stokes-Darcy problem describes the (slow) flow of a fluid across an interface I separating a saturated porous medium Ω_p and a free flowing fluid region Ω_f. Such flow is important because it transports contaminants between surface and groundwater [BC10], [PC06], nutrients and oxygen between capillaries and tissue [AZ11], [QVZ01] and material in industrial filtration systems [EJS09], [HWN06]. It also arises (at higher transport velocities) in modern fuel cells, porous combustors, advanced heat exchangers, the flow of air in the lungs and in the atmospheric boundary layer over vegetation. This report develops partitioned time-stepping methods (non-iterative domain decomposition methods) for the contaminant transport problem. There has been a substantial amount of work on uncoupling the (linear) Stokes-Darcy problem. The Stokes-Darcy-transport problem involves solving one additional (nonlinear) convection-diffusion problem with the Stokes-Darcy velocity passed from a Stokes-Darcy partitioned method. However, this introduces new difficulties into the approximation of the transport problem, (1.1), due to the error in the convecting velocity and its non-zero divergence ($\nabla \cdot u \neq 0$ in Ω_p) when $S_0 \neq 0$.

We therefore consider the equation for the concentration $c(x,t)$ of a transported contaminant with source $s(x,t)$

$$\beta c_t + \nabla \cdot (-D\nabla c + uc) = s \quad \text{in } \Omega := \Omega_f \cup \Omega_p \cup I. \tag{1.1}$$

Here the fluid region's velocity and pressure are u_f and p; the porous media's pressure head and velocity are ϕ and u_p. The transport velocity u in the concentration equation (1.1) is $u = u_f$ in Ω_f and $u = u_p$ in Ω_p. These satisfy, with appropriate boundary, interface (including zero normal jump on I, $[u \cdot \widehat{n}] = 0$) and initial conditions,

$$u_{f,t} - \nu\Delta u_f + \nabla p = f_f \text{ and } \nabla \cdot u_f = 0 \text{ in } \Omega_f, \tag{1.2}$$

$$S_0\phi_t - \nabla \cdot (K\nabla\phi) = f_p \text{ and } u_p = -\beta^{-1}K\nabla\phi \text{ in } \Omega_p. \tag{1.3}$$

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The physical parameters are

$S_0 =$ specific storage	$\nu =$ kinematic viscosity
$K =$ hydraulic conductivity tensor (SPD)	$D =$ dispersion tensor
$\beta =$ volumetric porosity	$f_{f/p}, s =$ body forces and sources

For the concentration $c(x,t)$, on the exterior boundary, $\partial\Omega$, we impose homogeneous Dirichlet boundary conditions (for clarity in the analysis), an initial condition, and across the interface, I, continuity of concentration and fluxes,

$$\begin{aligned} c = 0 \text{ on } \partial\Omega\,, \quad c(x,0) = c^0(x) \text{ in } \Omega\,, \\ [c] = 0 \text{ and } \; [(-D\nabla c + uc)\cdot\widehat{n}] = 0 \text{ on } I\,, \end{aligned} \tag{Jumps}$$

where $\widehat{n}$ denotes a unit normal on I pointing from Ω_f into Ω_p. Since the conservation of mass for the fluid flow implies $[u\cdot\widehat{n}] = 0$, the second jump condition can be simplified to $[D\nabla c\cdot\widehat{n}] = 0$.

Interface conditions on the concentration are not needed in a single domain formulation of (1.1) since such a formulation imposes continuity of concentration and fluxes as natural interface conditions.The structure of (1.1), (1.2), (1.3) is such that a partitioned method for the Stokes-Darcy system (1.2), (1.3) can be applied to find a velocity u_f in Ω_f, u_p in Ω_p which is then passed to the concentration equation (1.1). Thus, we focus on partitioned methods for the concentration equation (1.1) where the transport velocity u is known approximately.

There are four general methods for relaxing of interfacial couplings in a partitioned method: penalties [A99], Lagrange multipliers and mortar elements [GS07], and methods based on discretizing in time the coupling conditions explicitly. To uncouple (1.1) into subdomain solves we impose the coupling across I weakly using penalties. This replaces a conservation condition (a skew symmetric coupling) with a dissipative coupling (deviation from conservation is strongly damped). Sections 3 and 4 give a stability and error analysis of this penalty approximation. The concentration equation can be diffusion dominated, convection dominated or any intermediate state. We therefore develop and analyze in Section 5 a time partitioning algorithm that can be used for differing variational formulations in space, appropriate for the various cases.

The full transport model presents several computational and analytical difficulties. The first is an active nonlinearity in the transport problem. Taking the L^2 inner product of the transport equation with $c(x,t)$ gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \beta c^2\,dx + \int_\Omega D|\nabla c|^2\,dx \;+\; \frac{1}{2}\int_\Omega (\nabla\cdot u)c^2\,dx = \int_\Omega s\;c\;dx. \tag{1.4}$$

The key term involves $\nabla\cdot u$ which is exactly zero (a common assumption in the numerical analysis of convection diffusion equations, [RST96]) when $S_0 = 0$ (quasistatic) and $f_p = 0$. However, for (1.1), (1.2), (1.3)

$$\nabla\cdot u = \begin{cases} 0 \text{ in } \Omega_f, \\ \beta^{-1}\left(-S_0\frac{\partial\phi}{\partial t} + f_p\right) \text{ in } \Omega_p. \end{cases} \tag{1.5}$$

Thus, when $S_0 \neq 0$ this acts like a reaction term causing error growth and couples the growth rate to the error in the discrete convecting velocity u_h.

The regularity needed for u_h to ensure stability is also an important issue. In the continuous time analysis in Section 3 we assume only

$$\begin{aligned} u_h \in L^\infty(0,T;L^2(\Omega)),\, \nabla u_h \in L^2(0,T;L^2(\Omega)), \\ u_h = 0 \text{ on } \partial\Omega, [u_h\cdot\widehat{n}] = 0 \text{ on } I,\; \nabla\cdot u_h = 0 \text{ in } \Omega_f\;, \\ \text{and } \nabla\cdot u_h \in \begin{cases} L^2(0,T;L^2(\Omega_p)) \text{ in } 2d, \\ L^4(0,T;L^2(\Omega_p)) \text{ in } 3d. \end{cases} \end{aligned} \tag{1.6}$$

The assumption $\nabla \cdot u_h = 0$ in Ω_f (rather than small but non-zero) shortens the analysis and can be relaxed. There are also methods of increasing utility that produce exactly incompressible fluid velocities. A recent survey on these, enforcing $\nabla \cdot u_h = 0$ pointwise, is given in [JLM].

For the discrete time approximation in Sections 4 and 5 we assume further that a discrete version of the following holds

$$\nabla \cdot u_h \in L^\infty(0,T; L^2(\Omega_p)). \tag{1.7}$$

A third issue is the multitude of small parameters in the full problem. For example, when $D_{\min} << |u|$ the transport equation reduces to a singularly perturbed, convection diffusion equation with no control on $\nabla \cdot u$, a problem for which methods are comparatively less well developed, [RST96]. Since our focus is the time partitioning, we have studied the discrete time, continuous space approximation for which both the standard Finite Element Method (FEM) and the Streamline Diffusion Finite Element Method (SDFEM) can be used for discretization in space. See [DDD91], [ZYD09], [LY08] for interesting alternate approaches that could be explored for the present application.

1.1. Related work. Porous media transport and transport in a freely flowing fluid describe different physical processes with different variables, time scales, flow rates and uncertainties. There has been an intense effort at developing algorithms that use the subdomain/sub-physics codes to maximum effect to solve the coupled problem, e.g., domain decomposition methods for the equilibrium problem [CGHW11], [CMX09],[D04], [DQ09], [DMQ02], [J09], [LSY] and partitioned methods for the evolutionary problem [CGHW11b], [MZ10], [CGHW08], [LTT13], [SS12], [AZ11]. Partitioned methods for the pure diffusion case (when $u = 0$) have been developed in [CHL09] and partitioned methods based on other principles for convection diffusion equations in [DDD91], [ZYD09].

The reliability of the resulting predictions has spurred analytical study of the coupled model. The quasi-static approximation ($S_0 = 0$), studied in [M12], [EKL15], [CR09] or the fully steady Stokes-Darcy approximation [VY09], treating the transport as a time-dependent, monolithically coupled single domain problem has been studied in [AZ11], [CR09], [SS12], [VY09], [R14]. In these, the quasi-static Stokes-Darcy problem is typically solved by a domain decomposition procedure and a single domain transport problem is solved.

2. Preliminaries. Let the L^2 norms and inner products over Ω_f, Ω_p and I be denoted by $||\cdot||_r, (\cdot,\cdot)_r$, $r \in \{f, p, I\}$, respectively. Recall that $\Omega = \Omega_f \cup \Omega_p \cup I$; the L^2 norm and inner product over Ω will be denoted by $||\cdot||, (\cdot,\cdot)$ (without subscripts). Throughout we use C to denote a generic positive constant, whose actual value may vary from line to line in the analysis. The function space for the concentration is

$$X_{p/f} = \{c \in H^1(\Omega_{p/f}) : c = 0 \text{ on } \partial\Omega_{p/f}\backslash I\} \text{ and } X = \{c : c|_{\Omega_{p/f}} \in X_{p/f}\}.$$

With respect to L^2 duality, we define X^* as the dual space of X.

Due to the exterior boundary conditions, the Poincaré - Friedrichs inequality holds in both sub-domains:

$$||v||_{f/p} \leq C_{PF}(\Omega_{f/p})||\nabla v||_{f/p}, \forall v \in X_{f/p}.$$

We shall also use the following special cases of (combinations of) Sobolev, Poincaré-Friedrichs, interpolation and Gagliardo-Nirenberg inequalities (in $d = 2$ and 3 dimensions) for all $v \in X$

$$||v||_{L^6} \leq C||\nabla v||, \quad ||v||_{L^3} \leq C||v||^{1/2}||\nabla v||^{1/2}, \text{ and } ||v||_{L^4} \leq C||v||^{1-d/4}||\nabla v||^{d/4}. \tag{2.1}$$

We assume $D(x)$ is positive and bounded

$$0 < D_{\min} \leq D(x) \leq D_{\max} < \infty$$

and define a trilinear convective form as follows

$$b(u, c, v) := \frac{1}{2}(u \cdot \nabla c, v) - \frac{1}{2}(u \cdot \nabla v, c) + \frac{1}{2}\int_{\Omega_p} (\nabla \cdot u)cv \, dx.$$

2.1. Regularity. Regularity of the concentration depends on regularity of the Stokes-Darcy variables. The values of $u_{f,t}(0), \phi_t(0)$ at $t = 0$ are defined as

$$u_{f,t}(0) := u_{f,t}(x, 0) = \lim_{t \to 0^+} u_{f,t}(x, t) = \lim_{t \to 0^+} (f_f(x, t) + \nu \Delta u_f(x, t) - \nabla p(x, t))$$
$$\phi_t(0) := \phi_t(x, 0) = \lim_{t \to 0^+} \phi_t(x, t) = S_0^{-1} \lim_{t \to 0^+} (f_p(x, t) + \nabla \cdot (K \nabla \phi(x, t))) .$$

In [M12] Moraiti proved that for $0 < T < \infty$ and data satisfying

$$f_{f/p,t} \in L^2(0, T; H^{-1}(\Omega_{f/p})), \quad u_{f,t}(0) \in L^2(\Omega_f), \quad \phi_t(0) \in L^2(\Omega_p),$$

the following holds and will be assumed herein:

$$u_{f,t} \in L^\infty(0, T; L^2(\Omega_f)),\ \phi_t \in L^\infty(0, T; L^2(\Omega_p)) \text{ and } \nabla \phi_t \in L^2(0, T; L^2(\Omega_p)). \tag{2.2}$$

Additionally we assume

$$c^0 \in H^1(\Omega),\ s \in L^2(0, T; L^2(\Omega)), \nabla \phi_t(0) \in L^2(\Omega_p), \text{ and} \tag{2.3}$$
$$f_f \in L^\infty(0, T; L^2(\Omega_f)), f_p \in L^\infty(0, T; H^1(\Omega_p)), f_{p,t} \in L^2(0, T; L^2(\Omega_p)). \tag{2.4}$$

In [EKL15] the following regularity was proven for the transport problem.

PROPOSITION 2.1 (Regularity of concentration). *Suppose $0 < T < \infty$ and the problem data is such that (2.2)-(2.4) hold. Then*

$$c \in L^\infty(0, T; L^2(\Omega)), \quad \nabla c \in L^\infty(0, T; L^2(\Omega)), \text{ and } c_t \in L^2(0, T; L^2(\Omega)). \tag{2.5}$$

3. The continuous penalty method. Pick a penalty parameter $\delta > 0$ (small), exponent $q \geq 2$ and replace the jump conditions (JUMPS) with a penalty term in the variational formulation. We also replace $(u_h \cdot \nabla c, v)$ with $b(u_h, c, v)$. This results in another jump integral which is also controlled by the penalty term. The resulting solution then depends on δ and is denoted c^δ. The continuous penalty approximation is: *Given an approximate velocity u_h, select $q \geq 2$, and find $c^\delta : [0, T] \to X$ with $c^\delta(0) = c(0)$ and satisfying, for all $v \in X$,*

$$\beta(c_t^\delta, v) + (D \nabla c^\delta, \nabla v) + b(u_h, c^\delta, v) + \delta^{-q} \int_I |[c^\delta]|^{q-2} [c^\delta][v] ds = (s, v). \tag{3.1}$$

Integrating backwards, this penalty approximation is equivalent to replacing the interface jump conditions (JUMPS) with

$$\begin{cases} D \nabla c_p \cdot \vec{n}_p - \frac{1}{2} u_{ph} \cdot \vec{n}_p c_p = \delta^{-q} |[c]|^{q-2} [c], \text{ on } I, \text{ in } \Omega_p, \\ D \nabla c_f \cdot \vec{n}_f - \frac{1}{2} u_{fh} \cdot \vec{n}_f c_f = \delta^{-q} |[c]|^{q-2} [c], \text{ on } I, \text{ in } \Omega_f, \end{cases}$$

where $c_{f/p}$ denotes $c|_{\Omega_{f/p}}$, and $\vec{n}_{f/p}$ denotes the unit outer normal vector with respect to $\Omega_{f/p}$. The above form of the nonlinear penalty term is a natural extension of the linear ($q = 2$) case. The complexity of the analysis for the general $q \geq 2$ case does not increase substantially over the linear $q = 2$ case. As motivation, when $q > 2$, large jumps are over penalized (and small jumps under-weighted) compared to $q = 2$.

We prove in Section 3.1 that $[c^\delta] \to 0$ as $\delta \to 0$ and $c^\delta \to c$, as $\delta \to 0$ and $||u - u_h|| \to 0$.

3.1. Convergence of the continuous in time, penalty approximation. In the convergence analysis, the true concentration is transported by the true velocity $u(x,t)$ while the approximation is transported by the approximate velocity $u_h(x,t)$. The error in the velocity couples to the error in the concentration so that the regularity of the approximate velocity is significant. For (3.1), we assume that u_h satisfies (1.6). We begin by proving an á priori bound that shows the jump on I, $\left[c^\delta\right] \to 0$ as $\delta \to 0$.

PROPOSITION 3.1. *Suppose $0 < T < \infty$ and (1.6) holds. Then, there is a $C = C(T, problem\ data) < \infty$ such that*

$$(i)\ \ c^\delta \in L^\infty(0,T;L^2(\Omega))\,, \quad \nabla c^\delta \in L^2(0,T;L^2(\Omega)), \tag{3.2}$$

$$(ii)\ \ \|[c^\delta]\|^q_{L^q(0,T;L^q(I))} \le C\delta^q \to 0 \ \text{as}\ \delta \to 0, \tag{3.3}$$

$$(iii)\ \ \beta\|c^\delta\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\sqrt{D}\nabla c^\delta\|^2_{L^2(0,T;L^2(\Omega))} + \delta^{-q}\ \|[c^\delta]\|^q_{L^q(0,T;L^q(I))} \le C\left(\|s\|^2_{L^2(0,T;L^2(\Omega))} + \|c^\delta(0)\|^2\right). \tag{3.4}$$

Proof. Setting $v = c^\delta$ in (3.1), and using $b(u_h, c^\delta, c^\delta) = \frac{1}{2}\int_{\Omega_p}(\nabla\cdot u_h)\,(c^\delta)^2\,dx$, we obtain

$$\beta\frac{1}{2}\frac{d}{dt}\|c^\delta\|^2 + \|\sqrt{D}\nabla c^\delta\|^2 + \delta^{-q}\ \|[c^\delta]\|^q_{L^q(I)} = (s, c^\delta) - \frac{1}{2}\int_{\Omega_p}(\nabla\cdot u_h)\left(c^\delta\right)^2\ dx.$$

By the Cauchy-Schwarz inequality,

$$\left|\int_{\Omega_p}(\nabla\cdot u_h)\left(c^\delta\right)^2 dx\right| \le ||\nabla\cdot u_h||\,||c^\delta||^2_{L^4(\Omega)}\,.$$

Using the inequalities (2.1) for $||c||^2_{L^4}$ we have

$$\left|\frac{1}{2}\int_{\Omega_p}(\nabla\cdot u_h)\left(c^\delta\right)^2 dx\right| \le C||\nabla\cdot u_h||\left\{\begin{array}{c} ||c^\delta||||\nabla c^\delta|| \text{ in } 2d, \\ ||c^\delta||^{1/2}||\nabla c^\delta||^{3/2} \text{ in } 3d. \end{array}\right. \tag{3.5}$$

Of the 2 cases in (3.5), we present the $3d$ case. (The $2d$ case follows by analogous steps.) In $3d$ there follows

$$\begin{aligned} &\frac{1}{2}\beta\frac{d}{dt}||c^\delta||^2 + ||\sqrt{D}\nabla c^\delta||^2 \ + \ \delta^{-q}\ \|[c^\delta]\|^q_{L^q(I)} \\ &\qquad \le \frac{1}{2}||s||^2 + \frac{1}{2}||c^\delta||^2 + C||\nabla\cdot u_h||\,||c^\delta||^{1/2}||\nabla c^\delta||^{3/2} \\ &\qquad \le \frac{1}{2}||s||^2 + \frac{1}{2}||c^\delta||^2 + ||\sqrt{D}\nabla c^\delta||^{3/2}\left(CD^{-3/4}_{\min}||\nabla\cdot u_h||\,||c^\delta||^{1/2}\right). \end{aligned} \tag{3.6}$$

For the last term in (3.6) using $ab \le \frac{3}{4}a^{4/3} + \frac{1}{4}b^4$, we have that

$$\begin{aligned} &\frac{1}{2}\beta\frac{d}{dt}||c^\delta||^2 \ + \ \frac{1}{4}||\sqrt{D}\nabla c^\delta||^2 + \delta^{-q}\ \|[c^\delta]\|^q_{L^q(I)} \\ &\qquad\qquad \le \frac{1}{2}||s||^2 \ + C\left(1 + \left(D^{-3/4}_{\min}||\nabla\cdot u_h||\right)^4\right)||c^\delta||^2. \end{aligned}$$

The regularity (1.6) implies that $||\nabla\cdot u_h||^4 \in L^1(0,T)$. Thus, using Grönwall's inequality we establish (3.2)-(3.4). ◻

We now prove convergence of the continuous penalty method to the true concentration and give an error estimate. With u the true velocity, define

$$\lambda := -D\nabla c\cdot\vec{n}_f + \frac{1}{2}u\cdot\vec{n}_f c\Big|_I \;=\; -\left(-D\nabla c\cdot\vec{n}_p + \frac{1}{2}u\cdot\vec{n}_p c\right)\Big|_I.$$

In the analysis (see (3.10) and (3.27) below) we require λ to be bounded in the $L^{q'}(0,T;L^{q'}(I))$ norm, for $1 < q' \le 2$. Hence, we additionally assume that the true solution c satisfies

$$c \in L^2(0,T;H^2(\Omega_{p/f}))\,. \tag{3.7}$$

Since the Stokes-Darcy velocity satisfies $[u\cdot\widehat{n}]_I = 0$, $c(x,t)$ satisfies, for all $v\in X$:

$$\beta(c_t,v) + (D\nabla c,\nabla v) + \frac{1}{2}(u\cdot\nabla c,v) - \frac{1}{2}(u\cdot\nabla v,c) + \frac{1}{2}\int_\Omega(\nabla\cdot u)c\,v\,dx$$
$$+\int_I(-D\nabla c_p\cdot\vec{n}_p + \frac{1}{2}u_p\cdot\vec{n}_p c_p)v_p\,ds + \int_I(-D\nabla c_f\cdot\vec{n}_f + \frac{1}{2}u_f\cdot\vec{n}_f c_f)\,v_f\,ds = (s,v).$$

Thus, the variational formulation for $c(x,t)$ in X (i.e., in terms of the multiplier λ and $[v] = v_f - v_p$) may be written as, for all $v\in X$:

$$\beta(c_t,v) + (D\nabla c,\nabla v) + b(u,c,v) + \int_I \lambda[v]ds = (s,v). \tag{3.8}$$

We begin the error analysis with the linear penalty whose approximation $c^\delta \in X$ satisfies (3.1) with $q=2$

$$\beta(c_t^\delta,v) + (D\nabla c^\delta,\nabla v) + b(u_h,c^\delta,v) + \delta^{-2}\int_I[c^\delta][v]ds = (s,v). \tag{3.9}$$

THEOREM 3.2. *Suppose $0<T<\infty$, $q=2$ and (2.2) and (1.6) hold. Let $e = c - c^\delta$. Then there is a $C = C(T, data)$ such that the error in the linear penalty method (3.9) satisfies*

$$\beta\|e(T)\|^2 + \int_0^T\left(\|D^{1/2}\nabla e\|^2 + \delta^{-2}\|[e]\|^2_{L^2(I)}\right)dt \le C\,\delta^2\int_0^T \|\lambda\|^2_{L^2(I)}dt$$
$$+\,C\,||\nabla\,(u-u_h)\,||^2_{L^2(0,T;L^2(\Omega))} + C\,||\nabla\cdot(u-u_h)\,||^2_{L^4(0,T;L^2(\Omega_p))}. \tag{3.10}$$

Proof. Subtracting (3.9) from (3.8), and setting $v=e$ we obtain

$$\frac{\beta}{2}\frac{d}{dt}\|e\|^2 + \|D^{1/2}\nabla e\|^2 + \delta^{-2}\|[e]\|^2_{L^2(I)} + b(u,c,e) - b(u_h,c^\delta,e) = -\int_I\lambda[e]ds\,, \tag{3.11}$$

in which the core difficulty is the nonlinear term. Adding and subtracting $b(u_h,c,e)$, the nonlinear term can be algebraically rearranged as follows

$$\begin{aligned} b(u,c,e) - b(u_h,c^\delta,e) &= b(u,c,e) - b(u_h,c,e) + b(u_h,c,e) - b(u_h,c^\delta,e)\\ &= b(u-u_h,c,e) \;+\; b(u_h,e,e)\\ &= \frac{1}{2}\int_\Omega(u-u_h)\cdot\nabla c\,e\,dx \;-\; \frac{1}{2}\int_\Omega(u-u_h)\cdot\nabla e\,c\,dx\\ &\quad+\frac{1}{2}\int_{\Omega_p}\nabla\cdot(u-u_h)\,c\,e\,dx \;+\frac{1}{2}\int_{\Omega_p}(\nabla\cdot u_h)\,e^2dx. \end{aligned} \tag{3.12}$$

For the last two terms in (3.12), subtracting and adding $\frac{1}{2}\int_{\Omega_p} \nabla \cdot u\,(c - c^\delta)\,e\,dx$ yields

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega_p} \nabla \cdot (u - u_h)\, c\, e\, dx + \frac{1}{2}\int_{\Omega_p} (\nabla \cdot u_h)\, e^2\, dx \\
&\qquad = \frac{1}{2}\int_{\Omega_p} \nabla \cdot (u - u_h)\, c^\delta\, e\, dx \;+\; \frac{1}{2}\int_{\Omega_p} (\nabla \cdot u)\, e^2 dx.
\end{aligned}
$$

Thus we have

$$
b(u, c, e) - b(u_h, c^\delta, e) = N_1 + N_2 + N_3 + N_4, \tag{3.13}
$$

where

$$
\begin{aligned}
N_1 &= \frac{1}{2}\int_{\Omega} (u - u_h) \cdot \nabla c\, e\, dx, & N_2 &= -\frac{1}{2}\int_{\Omega} (u - u_h) \cdot \nabla e\, c\, dx, \\
N_3 &= \frac{1}{2}\int_{\Omega_p} \nabla \cdot (u - u_h)\, c^\delta\, e\, dx, & N_4 &= \frac{1}{2}\int_{\Omega_p} (\nabla \cdot u)\; e^2\, dx.
\end{aligned}
$$

The term N_1 is estimated using the regularity of $c(x,t)$ and applications of Hölder's, Poincaré-Friedrichs' inequalities, and (2.1) as follows:

$$
\begin{aligned}
|N_1| &= \left|\frac{1}{2}\int_{\Omega} (u - u_h) \cdot \nabla c\, e\, dx\right| \le C\, ||\nabla(u - u_h)||\; ||\nabla c||\, \sqrt{||e||\; ||\nabla e||} \\
&\le \left(C\, ||\nabla(u - u_h)||\, D_{\min}^{-1/4}\, ||e||^{1/2}\right)\left(||\sqrt{D}\nabla e||^{1/2}\right) \\
&\le \frac{1}{8}||\sqrt{D}\nabla e||^2 + C||\nabla(u - u_h)||^{4/3}||e||^{2/3}\,.
\end{aligned} \tag{3.14}
$$

Next using $ab \le \frac{2}{3}a^{3/2} + \frac{1}{3}b^3$,

$$
||\nabla(u - u_h)||^{4/3}\, ||e||^{2/3} \le C\, ||\nabla(u - u_h)||^2 + \frac{1}{3}||e||^2. \tag{3.15}
$$

Thus, combining (3.14) and (3.15),

$$
|N_1| \le \frac{1}{8}||\sqrt{D}\nabla e||^2 + \,C\, ||\nabla(u - u_h)||^2 + \frac{1}{3}||e||^2. \tag{3.16}
$$

The N_2 term is treated similarly using (2.5), resulting in the estimate

$$
\begin{aligned}
|N_2| &= \left|\frac{1}{2}\int_{\Omega} (u - u_h) \cdot \nabla e\, c\, dx\right| \le C\, ||\nabla(u - u_h)||\; ||\nabla e||\sqrt{||c||\; ||\nabla c||} \\
&\le C\, ||\nabla(u - u_h)||^2 + \frac{1}{8}||\sqrt{D}\nabla e||^2\,.
\end{aligned} \tag{3.17}
$$

The terms in (3.16) and (3.17) involving $||\sqrt{D}\nabla e||^2$ are subsumed into the LHS of (3.11) and the terms involving $||e||^2$ are handled using Grönwall's inequality.

For N_4, by Hölder's inequality and the inequalities (2.1) for $||e||^2_{L^4(\Omega_p)}$, and the Poincaré-Friedrichs' inequality

$$
\begin{aligned}
|N_4| &= \left|\frac{1}{2}\int_{\Omega_p} (\nabla \cdot u) e^2 dx\right| \le \frac{1}{2}||\nabla \cdot u||_p ||e||^2_{L^4(\Omega_p)} \\
&\le C||\nabla \cdot u||_p\, ||e||_p^{1/2}\, ||\nabla e||_p^{3/2} \\
&\le \left(||\sqrt{D}\nabla e||_p^{3/2}\right)\left(C\, D_{\min}^{-3/2}||\nabla \cdot u||_p\, ||e||_p^{1/2}\right) \\
&\le \frac{3}{4}||\sqrt{D}\nabla e||_p^2 + \left(C\, ||\nabla \cdot u||_p^4\right) ||e||^2\,,
\end{aligned} \tag{3.18}
$$

where in the last step we have used $ab \le \frac{3}{4}a^{4/3} + \frac{1}{4}b^4$.

For N_3, we have

$$|N_3| \le \frac{1}{2}||\nabla \cdot (u - u_h)\, c^\delta||_{X^*}\, ||\nabla e|| \le \frac{1}{8}||\sqrt{D}\nabla e||^2 + \frac{1}{2}D_{\min}^{-1}||\nabla \cdot (u-u_h)\, c^\delta||_{X^*}^2\,, \tag{3.19}$$
$$\text{where } ||\nabla \cdot (u-u_h)\, c^\delta||_{X^*} := \sup_{v\in X} \frac{(\nabla \cdot (u-u_h)\, c^\delta, v)}{||\nabla v||}.$$

Hölder's inequality, $||v||_{L^6} \le C||\nabla v||$, (1.5) and (1.6) gives

$$\begin{aligned} ||\nabla \cdot (u-u_h)\, c^\delta||_{X^*} &\le \sup \frac{||\nabla \cdot (u-u_h)\, c^\delta||_{L^{6/5}(\Omega)}||v||_{L^6(\Omega)}}{||\nabla v||} \\ &\le C||\nabla \cdot (u-u_h)\, c^\delta||_{L^{6/5}(\Omega_p)}. \end{aligned}$$

Applying Hölder's inequality again,

$$\begin{aligned} ||\nabla \cdot (u-u_h)\, c^\delta||_{L^{6/5}(\Omega_p)}^2 &= \left(\int_{\Omega_p} |\nabla \cdot (u-u_h)\, c^\delta|^{6/5} dx\right)^{5/3} \\ &\le \left(\int_{\Omega_p} |\nabla \cdot (u-u_h)\,|^2 dx\right)\left(\int_{\Omega_p} |c^\delta|^3 dx\right)^{2/3} \\ &\le \|\nabla \cdot (u-u_h)\,\|_p^2\, \|c^\delta\|_{L^3(\Omega_p)}^2 \\ &\le C\,\|\nabla \cdot (u-u_h)\,\|_p^2\, \|c^\delta\|_p\, \|\nabla c^\delta\|_p \quad \text{(using (2.1))} \\ &\le C\,\|\nabla \cdot (u-u_h)\,\|_p^2\, \|\sqrt{D}\nabla c^\delta\| \quad \text{(using (3.4))}. \end{aligned} \tag{3.20}$$

Combining (3.19) and (3.20) we have that

$$|N_3| \le \frac{1}{8}||\sqrt{D}\nabla e||^2 \,+\, C\,\|\nabla \cdot (u-u_h)\,\|_p^2\, \|\sqrt{D}\nabla c^\delta\|\,. \tag{3.21}$$

Next, combining (3.11) with (3.13), (3.16)-(3.18), and (3.21) we have

$$\begin{aligned} &\frac{\beta}{2}\frac{d}{dt}\|e\|^2 + \frac{1}{2}\|D^{1/2}\nabla e\|^2 + \frac{1}{2}\delta^{-2}\|[e]\|_{L^2(I)}^2 \\ &\qquad\le C(\|\nabla \cdot u\|_p^4 + 1)\|e\|^2 \,+\, \frac{1}{2}\delta^2\|\lambda\|_{L^2(I)}^2 \,+\, C\,\|\nabla\,(u-u_h)\,\|^2 \\ &\qquad\quad +\, C\,\|\nabla \cdot (u-u_h)\,\|_p^2\, \|\sqrt{D}\nabla c^\delta\|\,. \end{aligned} \tag{3.22}$$

Equation (1.6) implies that $||\nabla \cdot u||_p^4 \in L^1(0,T)$ so that Grönwall's inequality can be applied to (3.22). This gives, for any $T < \infty$,

$$
\begin{aligned}
&\beta\|e(T)\|^2 + \int_0^T \left(\|D^{1/2}\nabla e\|^2 + \delta^{-2}\|[e]\|^2_{L^2(I)}\right) dt \\
&\le C(T)\int_0^T \delta^2\|\lambda\|^2_{L^2(I)}\,dt + C(T)\int_0^T \|\nabla\left(u-u_h\right)\|^2\,dt \\
&\quad + C(T)\int_0^T \|\nabla\cdot\left(u-u_h\right)\|_p^2\,\|\sqrt{D}\nabla c^\delta\|\,dt \\
&\le C\delta^2\int_0^T \|\lambda\|^2_{L^2(I)}\,dt + C\|\nabla\left(u-u_h\right)\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad + C\left(\int_0^T \|\nabla\cdot\left(u-u_h\right)\|_p^4\,dt\right)^{1/2}\left(\int_0^T \|\sqrt{D}\nabla c^\delta\|^2\,dt\right)^{1/2} \\
&\le C\delta^2\int_0^T \|\lambda\|^2_{L^2(I)}\,dt + C\|\nabla\left(u-u_h\right)\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad + C\|\nabla\cdot\left(u-u_h\right)\|^2_{L^4(0,T;L^2(\Omega_p))}\,,
\end{aligned}
\tag{3.23}
$$

where in the last step we have used the boundedness of $\|\sqrt{D}\nabla c^\delta\|_{L^2(0,T;L^2(\Omega))}$ given by (3.4), establishing (3.10). □

The analysis is similar for the nonlinear penalty method and yields the following.

THEOREM 3.3 (Convergence of the nonlinear penalty method). *Suppose $0 < T < \infty$, $2 \le q < \infty$, $1/q + 1/q' = 1$, and (2.2) and (1.6) hold. Let $e = c - c^\delta$. Then there is a $C = C(T, data)$ such that the error in the nonlinear penalty method (3.9) satisfies*

$$
\begin{aligned}
\beta\|e(T)\|^2 + \int_0^T \left(\|D^{1/2}\nabla e\|^2 + \delta^{-q}\|[e]\|^q_{L^q(I)}\right) dt &\le C\,\delta^{q'}\int_0^T \|\lambda\|^{q'}_{L^{q'}(I)}dt \\
&+ C\,||\nabla\left(u-u_h\right)||^2_{L^2(0,T;L^2(\Omega))} + C\,||\nabla\cdot\left(u-u_h\right)||^2_{L^4(0,T;L^2(\Omega))}.
\end{aligned}
\tag{3.24}
$$

REMARK 3.4 (On L^4 regularity in time). *The assumption of L^4 regularity in time in (1.6) for the approximate Stokes-Darcy velocity in 3d is no issue for the quasi-static Stokes-Darcy approximation or if the velocity u is assumed to be known exactly. If the discrete velocity is calculated so that it satisfies the regularity proven for the continuous velocity then the estimates are also improvable and the "4" can also be improved to "2" in the time regularity in the last term.*

4. The discrete time penalty method. This section considers the time discretized penalty approximation. A partitioned extension of this method is given in Section 5. We let $\triangle t = T/N > 0$ denote the timestep, $t^n := n\triangle t$, $c^n = c^n(x)$ (suppressing the superscript δ) the approximation to $c(t^n, x)$, the solution to (1.1) and $c^n_{p/f}$ denotes $c^n|_{\Omega_{p/f}}$. The approximate Stokes-Darcy velocity at time t^n is denoted $u^n(x)$. In the continuous case $\nabla\cdot u \in L^\infty(0,T;L^2(\Omega))$. We shall assume in this section that the discrete approximation satisfies a discrete version of this condition:

$$
\max_{0\le n\le N} ||\nabla\cdot u^n|| \le C(T, data) < \infty. \tag{4.1}
$$

This is a stronger regularity assumption than the one used for the continuous time case in Section 3.

4.1. Stability of the discrete penalty method. To avoid a timestep restriction of the form $\triangle t\, C(T, data) < 1$ we study a method where $\nabla\cdot u^{n+1}c^{n+1}$ is replaced by $\nabla\cdot u^{n+1}c^{n}$. The resulting time-discrete method is

$$\beta\left(\frac{c^{n+1}-c^n}{\triangle t}, v\right) + (D\nabla c^{n+1}, \nabla v) + \frac{1}{2}(u^{n+1}\cdot\nabla c^{n+1}, v) - \frac{1}{2}(u^{n+1}\cdot\nabla v, c^{n+1}) \\ + \frac{1}{2}\int_{\Omega_p}(\nabla\cdot u^{n+1})c^n v\, dx + \delta^{-q}\int_I ||[c^{n+1}]|^{q-2}[c^{n+1}][v]ds = (s^{n+1}, v),\ \forall v \in X. \tag{4.2}$$

We prove 0-stability without a timestep restriction. The basic tool used will be a version of the discrete Grönwall inequality that does not require a timestep restriction (Lemma 2.4 p.176 of [GR79]).

Theorem 4.1 (0-Stability). *Assume (4.1) holds. The method (4.2) is stable: that is, there is a $C = C(T, data)$ such that*

$$\beta||c^N||^2 + \triangle t||\sqrt{D}\nabla c^N||^2 \\ + 2\triangle t\sum_{n=0}^{N-1}\left(\frac{\beta}{2\triangle t}||c^{n+1}-c^n||^2 + \frac{5}{16}||\sqrt{D}\nabla c^{n+1}||^2 + \delta^{-q}||[c^{n+1}]||^q_{L^q(I)}\right) \\ \le \beta||c^0||^2 + \triangle t||\sqrt{D}\nabla c^0||^2 + C\triangle t\sum_{n=0}^{N-1}||s^{n+1}||^2_{X^*}. \tag{4.3}$$

Proof. Setting $v = c^{n+1}$ in (4.2), using the polarization identity on the first term, the Cauchy-Schwarz and Poincaré-Friedrichs inequalities on the last term yields

$$\frac{\beta}{2\triangle t}\left(||c^{n+1}||^2 - ||c^n||^2\right) + \left(\frac{\beta}{2\triangle t}||c^{n+1}-c^n||^2 + ||\sqrt{D}\nabla c^{n+1}||^2 + \delta^{-q}||[c^{n+1}]||^q_{L^q(I)}\right) \\ + \frac{1}{2}\int_\Omega(\nabla\cdot u^{n+1})c^n\, c^{n+1}\, dx \ \le\ \frac{1}{16}||\sqrt{D}\nabla c^{n+1}||^2 + C||s^{n+1}||^2_{X^*}. \tag{4.4}$$

Next, using Hölder's inequality, (4.1), (2.1) and $ab \le \varepsilon a^{4/3} + C(\varepsilon)b^4$ we have

$$\begin{aligned}\frac{1}{2}\left|\int_\Omega(\nabla\cdot u^{n+1})c^n\, c^{n+1}\, dx\right| &\le \frac{1}{2}||\nabla\cdot u^{n+1}||\, ||c^n||_{L^4(\Omega)}||c^{n+1}||_{L^4(\Omega)} \\ &\le C\,||c^n||^{1/4}||\nabla c^n||^{3/4}||c^{n+1}||^{1/4}||\nabla c^{n+1}||^{3/4} \\ &\le \left(||\sqrt{D}\nabla c^n||^{3/4}||\sqrt{D}\nabla c^{n+1}||^{3/4}\right)\left(C\,||c^n||^{1/4}||c^{n+1}||^{1/4}\right) \\ &\le \frac{1}{8}||\sqrt{D}\nabla c^n||\,||\sqrt{D}\nabla c^{n+1}|| + C\,||c^n||\,||c^{n+1}|| \\ &\le \frac{1}{16}||\sqrt{D}\nabla c^n||^2 + \frac{1}{16}||\sqrt{D}\nabla c^{n+1}||^2 + C\,||c^n||^2 + \varepsilon||c^{n+1}||^2.\end{aligned}$$

By the Poincaré-Friedrichs inequality we can bound the last term as

$$\varepsilon||c^{n+1}||^2 \le \varepsilon\, C\,||\nabla c^{n+1}||^2 \le \varepsilon\, C\,||\sqrt{D}\nabla c^{n+1}||^2.$$

Then, picking ε appropriately, we obtain the bound

$$\frac{1}{2}\left|\int_\Omega(\nabla\cdot u^{n+1})c^n\, c^{n+1}\, dx\right| \le \frac{1}{16}||\sqrt{D}\nabla c^n||^2 + \frac{2}{16}||\sqrt{D}\nabla c^{n+1}||^2 + C\,||c^n||^2. \tag{4.5}$$

Consider next the $||\sqrt{D}\nabla c^{n+1}||^2$ term on the LHS of (4.4). Note that

$$||\sqrt{D}\nabla c^{n+1}||^2 = \frac{1}{2}\left(||\sqrt{D}\nabla c^{n+1}||^2 + ||\sqrt{D}\nabla c^{n}||^2\right) + \frac{1}{2}||\sqrt{D}\nabla c^{n+1}||^2 - \frac{1}{2}||\sqrt{D}\nabla c^{n}||^2. \tag{4.6}$$

Inserting (4.6) on the LHS of (4.4) and using (4.5) yields

$$\begin{aligned}
&\left(\frac{\beta}{2\triangle t}||c^{n+1}||^2 + \frac{1}{2}||\sqrt{D}\nabla c^{n+1}||^2\right) - \left(\frac{\beta}{2\triangle t}||c^{n}||^2 + \frac{1}{2}||\sqrt{D}\nabla c^{n}||^2\right)\\
&\quad + \left(\frac{\beta}{2\triangle t}||c^{n+1} - c^n||^2 + \frac{1}{2}(||\sqrt{D}\nabla c^{n+1}||^2 + ||\sqrt{D}\nabla c^{n}||^2) + \delta^{-q}||[c^{n+1}]||^q_{L^q(I)}\right)\\
&\quad \le C\,||s^{n+1}||^2_{X^*} + \frac{1}{16}||\sqrt{D}\nabla c^{n}||^2 + \frac{3}{16}||\sqrt{D}\nabla c^{n+1}||^2 + C||c^n||^2.
\end{aligned}$$

Thus, collecting terms we have

$$\begin{aligned}
&\left(\frac{\beta}{2\triangle t}||c^{n+1}||^2 + \frac{1}{2}||\sqrt{D}\nabla c^{n+1}||^2\right) - \left(\frac{\beta}{2\triangle t}||c^{n}||^2 + \frac{1}{2}||\sqrt{D}\nabla c^{n}||^2\right)\\
&\quad + \left(\frac{\beta}{2\triangle t}||c^{n+1} - c^n||^2 + \frac{5}{16}(||\sqrt{D}\nabla c^{n+1}||^2 + ||\sqrt{D}\nabla c^{n}||^2) + \delta^{-q}||[c^{n+1}]||^q_{L^q(I)}\right)\\
&\quad \le C||s^{n+1}||^2_{X^*} + C||c^n||^2\,.
\end{aligned} \tag{4.7}$$

Finally, summing (4.7) from $n = 0$ to $N - 1$ and using the discrete Grönwall's inequality we obtain (4.3). ∎

Remark 4.2. *In the stability proof the key term is* $\int(\nabla\cdot u^{n+1})c^n\, c^{n+1}\, dx$*. If the fully implicit method was used this term would instead be* $\int(\nabla\cdot u^{n+1})\,(c^{n+1})^2\, dx$*. In this case a different version of the discrete Grönwall's inequality is needed and from this version a timestep restriction would arise. If reduction to the standard implicit Euler method in the quasistatic case is desired the method can be modified by replacing*

$$\frac{1}{2}\int_{\Omega_p}(\nabla\cdot u^{n+1})c^n v\ dx \ \ by$$

$$\frac{1}{2}\int_{\Omega_p}\beta^{-1} f_p(t^{n+1}, x)c^{n+1} v\ dx - \frac{1}{2}\int_{\Omega_p}\beta^{-1} S_0 \frac{\partial\phi}{\partial t}^{n+1} c^n v\ dx.$$

When $S_0 = 0$ *this becomes the fully implicit method. A timestep restriction linking* $\triangle t$ *to* $f_p(t^{n+1}, x)$ *would reemerge.*

We proceed to an analysis of the error in the semi-discrete method. To derive an equation for $e^n := c(t^n) - c^n$ we first rewrite the continuous problem to identify the method's consistency error.

Definition 4.3. *The consistency error of (4.2) is* $R^{n+1} = R_1^{n+1} + R_2^{n+1}$ *where*

$$\begin{aligned}
R_1^{n+1} &:= \frac{c(t^{n+1}) - c(t^n)}{\triangle t} - c_t(t^{n+1}),\\
R_2^{n+1} &:= \frac{1}{2}(\nabla\cdot u(t^{n+1}))(c(t^n) - c(t^{n+1})).
\end{aligned}$$

Rearranging equation (3.8), the true solution $c(t)$ satisfies

$$
\begin{aligned}
\beta\left(\frac{c(t^{n+1})-c(t^n)}{\triangle t}, v\right) &+ (D\nabla c(t^{n+1}), \nabla v) + \delta^{-q}\int_I |[c(t^{n+1})]|^{q-2}[c(t^{n+1})][v]ds \\
&+ \frac{1}{2}(u(t^{n+1})\cdot\nabla c(t^{n+1}), v) - \frac{1}{2}(u(t^{n+1})\cdot\nabla v, c(t^{n+1})) \\
&+ \frac{1}{2}\int_{\Omega_p}(\nabla\cdot u(t^{n+1}))c(t^n)v\,dx \;+\; \int_I \lambda(t^{n+1})[v]\,ds \\
&= (s(t^{n+1}), v) \;+\; (R_1^{n+1}+R_2^{n+1}, v),\;\; \forall v \in X.
\end{aligned}
\tag{4.8}
$$

We begin by estimating the consistency error terms.

LEMMA 4.4 (Consistency Error: Monolithic Method). *For any* $\varepsilon > 0$ *there is a* $C = C(T, data, \varepsilon) < \infty$ *such that*

$$
\begin{aligned}
(R_1^{n+1}, v) &\le \varepsilon||\sqrt{D}\nabla v||^2 + C\,||R_1^{n+1}||^2_{X^*}, \\
(R_2^{n+1}, v) &\le \varepsilon||\sqrt{D}\nabla v||^2 + C\,||R_2^{n+1}||^2_{X^*}.
\end{aligned}
\tag{4.9}
$$

Further,

$$(R_1^{n+1}, v) \le \varepsilon||\sqrt{D}\nabla v||^2 + C\triangle t^2||c_{tt}||^2_{L^\infty(0,T;L^2(\Omega))}, \tag{4.10}$$

$$(R_2^{n+1}, v) \le \varepsilon||\sqrt{D}\nabla v||^2 + C\triangle t^2||c_t||^2_{L^\infty(0,T;L^4(\Omega))}, \;\; \textit{and} \tag{4.11}$$

$$(R_2^{n+1}, v) \le \varepsilon||\sqrt{D}\nabla v||^2 + C\triangle t^2||c_t||^{1/2}_{L^\infty(0,T;L^2(\Omega))}||\nabla c_t||^{3/2}_{L^\infty(0,T;L^2(\Omega))}. \tag{4.12}$$

Proof. The estimates given by (4.9) and (4.10) are standard. The bounds in (4.11) and (4.12) are established in a similar manner, so we will prove the more involved estimate (4.12). As

$$c(t^{n+1}) - c(t^n) = \int_{t^n}^{t^{n+1}} c_t(t)\,dt,$$

we have, by Hölder's inequality, (1.5), (2.2) and (2.4)

$$
\begin{aligned}
(R_2^{n+1}, v) &= \frac{1}{2}\int_{\Omega_p}\nabla\cdot u(x,t^{n+1})\int_{t^n}^{t^{n+1}} c_t(x,t)\,dt\,v(x)dx \\
&\le ||\nabla\cdot u||_{L^\infty(0,T;L^2(\Omega))}\left\|\int_{t^n}^{t^{n+1}} c_t(t)dt\right\|_{L^4(\Omega)} \|v\|_{L^4(\Omega)} \\
&\le C\left\|\int_{t^n}^{t^{n+1}} c_t(t)dt\right\|_{L^4(\Omega)} \|v\|_{L^4(\Omega)}\;.
\end{aligned}
$$

Using Poincaré-Friedrichs inequality, (2.1) and Hölder's in time we obtain

$$
\begin{aligned}
(R_2^{n+1}, v) &\leq C \left\|\sqrt{D}\nabla v\right\| \left(\int_\Omega \left(\int_{t^n}^{t^{n+1}} 1 \cdot c_t(t)\, dt \right)^4 dx \right)^{1/4} \\
&\leq C \left\|\sqrt{D}\nabla v\right\| \left(\int_\Omega \left(\left(\int_{t^n}^{t^{n+1}} 1^{4/3} dt \right)^{3/4} \left(\int_{t^n}^{t^{n+1}} c_t(t)^4 dt \right)^{1/4} \right)^4 dx \right)^{1/4} \\
&\leq C \left\|\sqrt{D}\nabla v\right\| \left((\triangle t)^3 \int_{t^n}^{t^{n+1}} \|c_t(t)\|^4_{L^4(\Omega)} dt \right)^{1/4} \\
&\leq C \left\|\sqrt{D}\nabla v\right\| \left((\triangle t)^3 \int_{t^n}^{t^{n+1}} \|c_t(t)\| \, \|\nabla c_t(t)\|^3 dt \right)^{1/4} \\
&\leq C \left\|\sqrt{D}\nabla v\right\| \triangle t \, \|c_t\|^{1/4}_{L^\infty(0,T;L^2(\Omega))} \, \|\nabla c_t\|^{3/4}_{L^\infty(0,T;L^2(\Omega))} \\
&\leq \varepsilon ||\sqrt{D}\nabla v||^2 + C\triangle t^2 ||c_t||^{1/2}_{L^\infty(0,T;L^2(\Omega))} ||\nabla c_t||^{3/2}_{L^\infty(0,T;L^2(\Omega))}.
\end{aligned}
$$

□

We now establish the error in the time discrete method when $q = 2$. (When $q > 2$ monotonicity and local Lipschitz continuity are used to replace linearity and boundedness but other details are identical.)

THEOREM 4.5. *Assume (4.1) holds and $q = 2$. Then, there is a $C = C(data, T)$ such that*

$$
\begin{aligned}
&\frac{\beta}{2}||e^N||^2 + \frac{\triangle t}{2}||\sqrt{D}\nabla e^N||^2 \\
&+\frac{\triangle t}{2}\sum_{i=1}^{N-1}\left(\frac{\beta}{\triangle t}||e^{n+1}-e^n||^2 + ||\sqrt{D}\nabla e^{n+1}||^2 + ||\sqrt{D}\nabla e^n||^2 + \delta^{-2}||[e^{n+1}]||^2_{L^2(I)}\right) \\
&\leq C\left(\frac{\beta}{2}||e^0||^2 + \frac{\triangle t}{2}||\sqrt{D}\nabla e^0||^2\right) + C\,\delta^2||\lambda(t^{n+1})||^2_{L^\infty(0,T;L^2(I))} \\
&\quad + C\left(\triangle t^2\,||c_t||^2_{L^\infty(0,T;L^4(\Omega))} + \triangle t^2\,||c_{tt}||^2_{L^\infty(0,T;L^2(\Omega))}\right) \\
&\quad + C\max_{1\leq n\leq N}||\nabla(u(t^{n+1}) - u^{n+1})||^2\,.
\end{aligned}
\tag{4.13}
$$

Proof. Subtracting (4.2) from (4.8), and noting that $[c(t)] = 0$, we have

$$
\begin{aligned}
\beta\left(\frac{e^{n+1}-e^n}{\triangle t}, v\right) &+ (D\nabla e^{n+1}, \nabla v) + \delta^{-2}\int_I [e^{n+1}][v]ds \\
&+ \left(\frac{1}{2}(u(t^{n+1})\cdot\nabla c(t^{n+1}), v) - \frac{1}{2}(u(t^{n+1})\cdot\nabla v, c(t^{n+1}))\right) \\
&- \left(\frac{1}{2}(u^{n+1}\cdot\nabla c^{n+1}, v) - \frac{1}{2}(u^{n+1}\cdot\nabla v, c^{n+1})\right) \\
&+ \frac{1}{2}\int_{\Omega_p}(\nabla\cdot u(t^{n+1}))\, c(t^n)\, v\, dx - \frac{1}{2}\int_{\Omega_p}(\nabla\cdot u^{n+1})\, c^n\, v\, dx \\
&= \left(R_1^{n+1} + R_2^{n+1}, v\right) - \int_I \lambda(t^{n+1})[v]ds.
\end{aligned}
\tag{4.14}
$$

In (4.14) let $v = e^{n+1}$. All the terms in the first line are treated exactly as in the stability proof (Theorem 4.1). The terms in line 5 are treated using the consistency error lemma (Lemma 4.4), and the λ term is treated using the Cauchy-Schwarz inequality with $\varepsilon = \delta^2$. The terms in lines 2, 3 and 4 are treated by adding and subtracting terms and regrouping as in the proof of the error estimate for the continuous time method. To compress the result, define

$$\mathcal{E}^{n+1} := \frac{\beta}{2}||e^{n+1}||^2 + \frac{\triangle t}{2}||\sqrt{D}\nabla e^{n+1}||^2,$$
$$\mathcal{D}^{n+1} := \frac{\beta}{2\triangle t}||e^{n+1} - e^n||^2 + \frac{1}{2}||\sqrt{D}\nabla e^{n+1}||^2 + \frac{1}{2}||\sqrt{D}\nabla e^n||^2 + \frac{\delta}{2}^{-2}||[e^{n+1}]||^2_{L^2(I)}.$$

Then, from (4.14) it follows

$$\begin{aligned}
&\mathcal{E}^{n+1} - \mathcal{E}^n + \triangle t \mathcal{D}^{n+1} + \\
&\frac{\triangle t}{2}\int_\Omega (u(t^{n+1}) - u^{n+1})(\nabla \cdot c^{n+1})\, e^{n+1}\, dx - \frac{\triangle t}{2}\int_\Omega (u(t^{n+1}) - u^{n+1})(\nabla \cdot e^{n+1})\, c^{n+1}\, dx \\
&+\frac{\triangle t}{2}\int_{\Omega_p} \nabla \cdot (u(t^{n+1}) - u^{n+1}) c^n e^{n+1} dx + \frac{\triangle t}{2}\int_{\Omega_p} \nabla \cdot u(t^{n+1}) e^n e^{n+1} dx \\
&\le C\triangle t\left((\triangle t)^2 ||c_{tt}||^2_{L^\infty(0,T;L^2(\Omega))} + (\triangle t)^2 ||c_t||^2_{L^\infty(0,T;L^4(\Omega))}\right) + \varepsilon \triangle t ||\sqrt{D}\nabla e^{n+1}||^2 \\
&+\frac{\triangle t}{2}\delta^2 ||\lambda(t^{n+1})||^2_{L^2(I)}\,.
\end{aligned} \tag{4.15}$$

The terms on line 2 of (4.15) are bounded in an analogous fashion to N_1 and N_2 in the proof of Theorem 3.2. With (4.10) and (4.11) we then obtain

$$\begin{aligned}
&\mathcal{E}^{n+1} - \mathcal{E}^n + \triangle t \mathcal{D}^{n+1} \\
&+\frac{\triangle t}{2}\int_\Omega \nabla \cdot (u(t^{n+1}) - u^{n+1}) c^n e^{n+1} dx + \frac{\triangle t}{2}\int_{\Omega_p} \left(\nabla \cdot u(t^{n+1})\right) e^n e^{n+1} dx \\
&\le C\triangle t\left((\triangle t)^2 ||c_{tt}||^2_{L^\infty(0,T;L^2(\Omega))} + (\triangle t)^2 ||c_t||^2_{L^\infty(0,T;L^4(\Omega))}\right) + 3\varepsilon \triangle t ||\sqrt{D}\nabla e^{n+1}||^2 \\
&+C\triangle t \|e^{n+1}\|^2 + \frac{\triangle t}{2}\delta^2 ||\lambda(t^{n+1})||^2_{L^2(I)} + C\,\triangle t\, ||\nabla(u(t^{n+1}) - u^{n+1})||^2\,.
\end{aligned} \tag{4.16}$$

We consider now the two nonlinear terms in line 2 of (4.16). For the second term, we have (using, respectively, Hölder's inequality, $\nabla \cdot u \in L^\infty(0,T;L^2(\Omega))$, the inequality (2.1) and the arithmetic-geometric mean inequality), for any $\varepsilon > 0$,

$$\begin{aligned}
&\left|\frac{1}{2}\int_{\Omega_p} \left(\nabla \cdot u(t^{n+1})\right) e^n e^{n+1} dx\right| \le \frac{1}{2}||\nabla \cdot u(t^{n+1})||_p\, ||e^n||_{L^4(\Omega_p)} ||e^{n+1}||_{L^4(\Omega_p)} \\
&\le C||e^n||_{L^4(\Omega_p)} ||e^{n+1}||_{L^4(\Omega_p)} \le C||e^n||^{1/4} ||\nabla e^n||^{3/4} ||e^{n+1}||^{1/4} ||\nabla e^{n+1}||^{3/4} \\
&\le \varepsilon \left(||\sqrt{D}\nabla e^n||^2 + ||\sqrt{D}\nabla e^{n+1}||^2\right) + C\left(||e^n||\, ||e^{n+1}||\right).
\end{aligned} \tag{4.17}$$

On the last term in (4.17) we use the Poincaré-Friedrichs inequality, and again the arithmetic-geometric mean inequality to obtain

$$\begin{aligned}
\left|\frac{1}{2}\int_{\Omega_p} \left(\nabla \cdot u(t^{n+1})\right) e^n e^{n+1} dx\right| &\le \varepsilon \left(||\sqrt{D}\nabla e^n||^2 + ||\sqrt{D}\nabla e^{n+1}||^2\right) \\
&+C\, ||e^n||^2.
\end{aligned} \tag{4.18}$$

For the first term in line 2 of (4.16) using (2.1) and the arithmetic-geometric mean inequality several times we have, for any $\varepsilon > 0$,

$$\begin{aligned}
&\left|\frac{1}{2}\int_{\Omega_p} \nabla\cdot(u(t^{n+1})-u^{n+1})c^n e^{n+1}dx\right| \le \frac{1}{2}||\nabla\cdot(u(t^{n+1})-u^{n+1})||_{\,p}||c^n||_{L^3(\Omega_p)}||e^{n+1}||_{L^6(\Omega_p)}\\
&\le C\,||\nabla\cdot(u(t^{n+1})-u^{n+1})||_p\,||c^n||_p^{1/2}||\nabla c^n||_p^{1/2}||\sqrt{D}\nabla e^{n+1}||_p\\
&\le \varepsilon||\sqrt{D}\nabla e^{n+1}||^2 + C\,||\nabla\cdot(u(t^{n+1})-u^{n+1})||_p^2||c^n||\,||\sqrt{D}\nabla c^n||\\
&\le \varepsilon||\sqrt{D}\nabla e^{n+1}||^2 + C\,\left(||c^n||^2 + ||\sqrt{D}\nabla c^n||^2\right)||\nabla\cdot(u(t^{n+1})-u^{n+1})||_p^2\,. \qquad (4.19)
\end{aligned}$$

With estimates (4.18) and (4.19), and picking ε so as to subsume the terms in the LHS of (4.16), for some fixed $\alpha > 0$ we have

$$\begin{aligned}
&\mathcal{E}^{n+1} - \mathcal{E}^n + \triangle t\,\alpha\,\mathcal{D}^{n+1}\\
&\le C\triangle t\,\mathcal{E}^n + C\,\triangle t\left((\triangle t)^2||c_{tt}||^2_{L^\infty(0,T;L^2(\Omega))} + (\triangle t)^2||c_t||^2_{L^\infty(0,T;L^4(\Omega))}\right)\\
&\quad+\frac{\triangle t}{2}\delta^2||\lambda(t^{n+1})||^2_{L^2(I)} + C\,\triangle t\,||\nabla(u(t^{n+1})-u^{n+1})||^2\\
&\quad+C\,\triangle t\,\left(||c^n||^2 + ||\sqrt{D}\nabla c^n||^2\right)||\nabla\cdot(u(t^{n+1})-u^{n+1})||_p^2. \qquad (4.20)
\end{aligned}$$

The result now follows from the discrete Grönwall's inequality and the stability estimate (4.3). ∎

5. The partitioned method. In previous sections we have analyzed the error in imposing the jump conditions linking the concentration in the two domains as a penalty term. Herein, we give a first-order accurate and unconditionally 0-stable partitioned method for the resulting penalized system. This method decouples the approximation of (4.2) into separate solves in the fluid and porous regions. In $\Omega_{f/p}$ respectively, we solve the following systems at each timestep.

$$\begin{aligned}
&\beta\left(\frac{c_f^n - c_f^{n-1}}{\triangle t}, v_f\right)_f + (D\nabla c_f^n, \nabla v_f)_f\\
&+\frac{1}{2}(u_f^n\cdot\nabla c_f^n, v_f)_f - \frac{1}{2}(u_f^n\cdot\nabla v_f, c_f^n)_f\\
&+\ \delta^{-q}\int_I\left(|[c^{n-1}]|^{q-2}c_f^n v_f - \sqrt{|[c^{n-1}]|^{q-2}|[c^{n-2}]|^{q-2}}\,c_p^{n-1}\,v_f\right)ds \qquad (5.1)\\
&=\ (f_f^n, v_f)_f,\ \ \forall v_f \in X_f^h\,,
\end{aligned}$$

$$\begin{aligned}
&\beta\left(\frac{c_p^n - c_p^{n-1}}{\triangle t}, v_p\right)_p + (D\nabla c_p^n, \nabla v_p)_p\\
&+\ \frac{1}{2}(u_p^n\cdot\nabla c_p^n, v_p)_p - \frac{1}{2}(u_p^n\cdot\nabla v_p, c_p^n)_p +\ \frac{1}{2}\int_{\Omega_p}(\nabla\cdot u_p^n)c_p^n v_p\ dx\\
&+\ \delta^{-q}\int_I\left(|[c^{n-1}]|^{q-2}c_p^n v_p - \sqrt{|[c^{n-1}]|^{q-2}|[c^{n-2}]|^{q-2}}\,c_f^{n-1}\,v_p\right)ds \qquad (5.2)\\
&=\ (f_p^n, v_p)_p,\ \ \forall v_p \in X_p^h.
\end{aligned}$$

This method has an $\mathcal{O}(\triangle t)$ consistency error, is linearly implicit, and uncouples into one subdomain solve on each subdomain per timestep. The treatment of the nonlinearity is inspired by [CHL12]. Our tests in Section 6 compare approximations using $q = 2$ with $q = 4$.

5.1. Stability analysis. We establish 0-stability of the partitioned method (5.1), (5.2).

THEOREM 5.1. *Assume $\nabla\cdot u_p^n \in l^\infty(0,T;L^2(\Omega_p))$. The method (5.1) and (5.2) is unconditionally 0-stable, in the sense that for $n \geq 2$ and $C = C(T, data)$*

$$
\begin{aligned}
&\beta(\|c_f^n\|_f^2 + \|c_p^n\|_p^2) + \frac{\triangle t}{\delta^q}\int_I |[c^{n-1}]|^{q-2}((c_f^n)^2 + (c_p^n)^2)ds \\
&+ \beta\sum_{i=2}^{n}(\|c_f^i - c_f^{i-1}\|_f^2 + \|c_p^i - c_p^{i-1}\|_p^2 + \|D^{1/2}\nabla c_f^i\|^2 + \|D^{1/2}\nabla c_p^i\|_f^2) \\
&+ \frac{\triangle t}{\delta^q}\sum_{i=2}^{n}\int_I (|[c^{i-2}]|^{\frac{q-2}{2}}c_f^{i-1} - |[c^{i-1}]|^{\frac{q-2}{2}}c_p^i)^2 + (|[c^{i-2}]|^{\frac{q-2}{2}}c_p^{i-1} - |[c^{i-1}]|^{\frac{q-2}{2}}c_f^i)^2 ds \\
&\leq C\left(\triangle t\sum_{i=2}^{n}\|f^i\|^2 \;+\; \beta(\|c_f^1\|_f^2 + \|c_p^1\|_p^2) + \frac{\triangle t}{\delta^q}\int_I |[c^0]|^{q-2}((c_f^1)^2 + (c_p^1)^2)\,ds\right).
\end{aligned}
$$

Proof. Letting $v_f = 2\triangle t c_f^n$ in (5.1), we deduce

$$
\begin{aligned}
&\beta\|c_f^n\|_f^2 - \beta\|c_f^{n-1}\|_f^2 + \beta\|c_f^n - c_f^{n-1}\|_f^2 + 2\triangle t\|D^{1/2}\nabla c_f^n\|_f^2 \\
&+2\frac{\triangle t}{\delta^q}\int_I |[c^{n-1}]|^{q-2}(c_f^n)^2\,ds - 2\frac{\triangle t}{\delta^q}\int_I |[c^{n-1}]|^{\frac{(q-2)}{2}}c_f^n\,|[c^{n-2}]|^{\frac{(q-2)}{2}}c_p^{n-1}\,ds \\
&= 2\triangle t(f_f^n, c_f^n)_f.
\end{aligned} \tag{5.3}
$$

The nonlinear terms satisfy the algebraic identity

$$
\begin{aligned}
&\int_I |[c^{n-1}]|^{q-2}(c_f^n)^2\,ds - \int_I |[c^{n-1}]|^{\frac{(q-2)}{2}}c_f^n\,|[c^{n-2}]|^{\frac{(q-2)}{2}}c_p^{n-1}\,ds \\
&= \frac{1}{2}\int_I |[c^{n-1}]|^{q-2}(c_f^n)^2\,ds - \frac{1}{2}\int_I |[c^{n-2}]|^{q-2}(c_p^{n-1})^2\,ds \\
&+ \frac{1}{2}\int_I (|[c^{n-2}]|^{\frac{(q-2)}{2}}c_p^{n-1} - |[c^{n-1}]|^{\frac{(q-2)}{2}}c_f^n)^2\,ds.
\end{aligned}
$$

Combining with (5.3), we get

$$
\begin{aligned}
&\beta\|c_f^n\|_f^2 - \beta\|c_f^{n-1}\|_f^2 + \\
&+\beta\|c_f^n - c_f^{n-1}\|_f^2 + 2\triangle t\|D^{1/2}\nabla c_f^n\|_f^2 + \frac{\triangle t}{\delta^q}\int_I |[c^{n-1}]|^{q-2}(c_f^n)^2\,ds \\
&-\frac{\triangle t}{\delta^q}\int_I |[c^{n-2}]|^{q-2}(c_p^{n-1})^2\,ds + \frac{\triangle t}{\delta^q}\int_I (|[c^{n-2}]|^{\frac{q-2}{2}}c_p^{n-1} - |[c^{n-1}]|^{\frac{q-2}{2}}c_f^n)^2\,ds \\
&\leq 2\triangle t(f_f^n, c_f^n)_f.
\end{aligned} \tag{5.4}
$$

Letting $v_p = 2\triangle t c_p^n$ in (5.2), we obtain

$$
\begin{aligned}
&\beta\|c_p^n\|_p^2 - \beta\|c_p^{n-1}\|_p^2 + \beta\|c_p^n - c_p^{n-1}\|_p^2 + 2\triangle t\|D^{1/2}\nabla c_p^n\|_p^2 + \triangle t((\nabla\cdot u_p^n)c_p^n, c_p^n)_p \\
&+2\frac{\triangle t}{\delta^q}\int_I |[c^{n-1}]|^{p-2}(c_p^n)^2\,ds - 2\frac{\triangle t}{\delta^q}\int_I |[c^{n-1}]|^{\frac{q-2}{2}}c_p^n\,|[c^{n-2}]|^{\frac{q-2}{2}}c_f^{n-1}\,ds \\
&= 2\triangle t(f_p^n, c_p^n)_p.
\end{aligned}
$$

Similarly, we have the algebraic identity

$$\begin{aligned}
&\int_I |[c^{n-1}]|^{q-2}(c_p^n)^2\, ds - \int_I |[c^{n-1}]|^{\frac{q-2}{2}} c_p^n |[c^{n-2}]|^{\frac{q-2}{2}} c_f^{n-1}\, ds \\
&= \frac{1}{2}\int_I |[c^{n-1}]|^{q-2}(c_p^n)^2\, ds - \frac{1}{2}\int_I |[c^{n-2}]|^{q-2}(c_f^{n-1})^2\, ds \\
&\quad + \frac{1}{2}\int_I (|[c^{n-2}]|^{\frac{q-2}{2}} c_f^{n-1} - |[c^{n-1}]|^{\frac{q-2}{2}} c_p^n)^2\, ds.
\end{aligned}$$

The term $|((\nabla\cdot u_p^n)c_p^n, c_p^n)_p|$ is bounded as follows using Hölder's inequality, (2.1), (1.7), and $ab \le \frac{3}{4}a^{4/3} + \frac{1}{4}b^4$. This gives

$$\begin{aligned}
|((\nabla\cdot u_p^n)c_p^n, c_p^n)_p| &\le ||\nabla\cdot u_p^n||_p \|c_p^n\|^2_{L^4(\Omega_p)} \\
&\le C\, ||\nabla\cdot u_p^n||_p\, ||c_p^n||_p^{1/2} ||\nabla c_p^n||_p^{3/2} \le\; C\,\|c_p^n\|_p^2 + \|D^{1/2}\nabla c_p^n\|_p^2\,.
\end{aligned}$$

Then, we deduce

$$\begin{aligned}
&\beta\|c_p^n\|_p^2 - \beta\|c_p^{n-1}\|_p^2 + \beta\|c_p^n - c_p^{n-1}\|_p^2 + \triangle t\|D^{1/2}\nabla c_p^n\|_p^2 + \frac{\triangle t}{\delta^q}\int_I |[c^{n-1}]|^{q-2}(c_p^n)^2\, ds \\
&- \frac{\triangle t}{\delta^q}\int_I |[c^{n-2}]|^{q-2}(c_f^{n-1})^2\, ds + \frac{\triangle t}{\delta^q}\int_I (|[c^{n-2}]|^{\frac{q-2}{2}} c_f^{n-1} - |[c^{n-1}]|^{\frac{q-2}{2}} c_p^n)^2\, ds \\
&\qquad \le 2\triangle t(f^n, c_p^n)_p + C\,\triangle t\|c_p^n\|_p^2. \qquad (5.5)
\end{aligned}$$

With $(f^n, c_r^n)_r \le 1/2\|f^n\|_r^2 \;+\; 1/2\|c_r^n\|_r^2$, adding (5.4) and (5.5) gives

$$\begin{aligned}
&\left[\beta(\|c_f^n\|_f^2 + \|c_p^n\|_p^2) + \frac{\triangle t}{\delta^q}\int_I |[c^{n-1}]|^{q-2}((c_f^n)^2 + (c_p^n)^2)\, ds\right] \\
&- \left[\beta(\|c_f^{n-1}\|_f^2 + \|c_p^{n-1}\|_p^2) + \frac{\triangle t}{\delta^q}\int_I |[c^{n-2}]|^{q-2}((c_f^{n-1})^2 + (c_p^{n-1})^2)\, ds\right] \\
&+\beta(\|c_f^n - c_f^{n-1}\|_f^2 + \|c_p^n - c_p^{n-1}\|_p^2) \\
&+\triangle t(\|D^{1/2}\nabla c_f^n\|_f^2 + \|D^{1/2}\nabla c_p^n\|_p^2) \\
&+ \frac{\triangle t}{\delta^q}\int_I (|[c^{n-2}]|^{\frac{(q-2)}{2}} c_f^{n-1} - |[c^{n-1}]|^{\frac{(q-2)}{2}} c_p^n)^2\, ds \\
&+ \frac{\triangle t}{\delta^q}\int_I (|[c^{n-2}]|^{\frac{(q-2)}{2}} c_p^{n-1} - |[c^{n-1}]|^{\frac{(q-2)}{2}} c_f^n)^2\, ds \\
&\le \triangle t\|f^n\|_f^2 + \triangle t\|f_n\|_p^2 + C\,\triangle t(\|c_f^n\|^2 + \|c_p^n\|^2).
\end{aligned}$$

Summing over the above inequality and using a discrete Grönwall lemma, we get the following, completing the proof. For $n \ge 2$,

$$\begin{aligned}
&\beta(\|c_f^n\|_f^2 + \|c_p^n\|_p^2) + \frac{\triangle t}{\delta^q}\int_I |[c^{n-1}]|^{q-2}((c_f^n)^2 + (c_p^n)^2)\, ds \\
&+\beta\sum_{i=2}^n(\|c_f^i - c_f^{i-1}\|_f^2 + \|c_p^i - c_p^{i-1}\|_p^2) + \triangle t\sum_{i=2}^n(\|D^{1/2}\nabla c_f^i\|_f^2 + \|D^{1/2}\nabla c_p^i\|_p^2) \\
&+\frac{\triangle t}{\delta^q}\sum_{i=2}^n\int_I (|[c^{i-2}]|^{\frac{q-2}{2}} c_f^{i-1} - |[c^{i-1}]|^{\frac{q-2}{2}} c_p^i)^2 \;+ (|[c^{i-2}]|^{\frac{q-2}{2}} c_p^{i-1} - |[c^{i-1}]|^{\frac{q-2}{2}} c_f^i)^2 ds \\
&\le\; C\left(\triangle t\sum_{i=2}^n \|f^i\|^2 \;+\; \beta(\|c_f^1\|_f^2 + \|c_p^1\|_p^2) + \frac{\triangle t}{\delta^q}\int_I |[c^0]|^{q-2}((c_f^1)^2 + (c_p^1)^2)\, ds\right).
\end{aligned}$$

□

6. Numerical results. In this section we investigate the numerical algorithms described above. Two series of numerical experiments are presented. In the first series of experiments, Section 6.1, we focus our investigation on the influence of the penalty parameters δ and q. The second series of numerical experiments, Section 6.2, investigates the performance of the partitioned method described in (5.1), (5.2).

Continuous, piecewise linear, and piecewise quadratic approximations are computed. For the numerical experiments we take $\Omega_f = (0,1) \times (0.5,1)$, $\Omega_p = (0,1) \times (0, 0.5)$, and $I = \{(x, 1/2) : 0 < x < 1\}$. We use for the numerical experiments [VY09]

$$c(x, y, t) \;=\; t\left(\cos(\pi x) \,+ \cos(\pi y)\right)/\pi\,, \text{ and } u(x, y, t) \;=\; \left[\begin{array}{c} \sin\left(\frac{x}{G} \,+ \omega\right)\, e^{y/G} \\ -\cos\left(\frac{x}{G} \,+ \omega\right)\, e^{y/G} \end{array}\right],$$

where $G = 2\sqrt{0.1}$, and $\omega = 1.05$. Additionally, for the parameters in the modeling equation (1.1), we use $\beta = 1$ and $D = 1$. The initial time is taken to be $t = 0$, and final time $t = T = 1$. A fixed time step $\Delta t = 0.01$ is used.

Illustrated in Figure 6.1 is the computational mesh corresponding to $h = 1/8$.

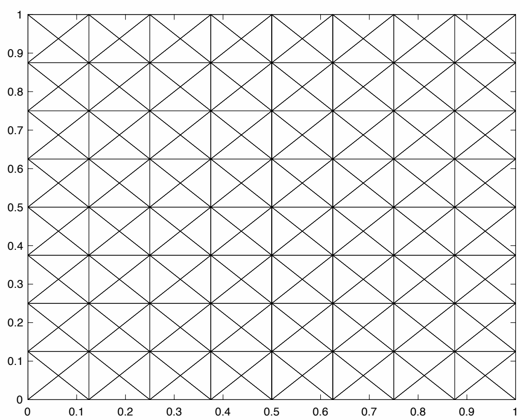

FIG. 6.1. *Computational mesh for Ω corresponsding to $h = 1/8$.*

The tables present the discrete $L^\infty(0,T;\, L^2(\Omega_p))$ norm for $(c - c_p)$ and the discrete $L^2(0,T;\, L^2(\Omega_p))$ norm for $(\nabla c - \nabla c_p)$, together with their experimental convergence rates. (Similar results were also obtained for $(c - c_f)$ in Ω_f.) The discrete $L^2(0,T;\, L^2(\Omega_p))$ norm for $(c - c_p)$ behaved in a similar manner to its discrete $L^\infty(0,T;\, L^2(\Omega_p))$ norm. The discrete $L^2(0,T;\, L^2(I))$ norms for $(c_f - c_p)$ (on the interface) is also given.

For compactness in the table headings, we use

$\|c^n - c_p^n\|_{n1} \;:=\; \max_{n=1,\ldots,N_T} \|c^n - c_p^n\|_{\Omega_p}$, $\|\nabla(c^n - c_p^n)\|_{n2} \;:=\; \left(\Delta t \sum_{n=1}^{N_T} \|\nabla(c^n - c_p^n)\|^2_{\Omega_p}\right)^{1/2}$, and $\|c_f - c_p\|_{m2} \;:=\; \left(\Delta t \sum_{n=1}^{N_T} \|c_f^n - c_p^n\|^2_I\right)^{1/2}$.

6.1. Influence of the penalty parameters δ and q. In this section we investigate the influence of the penalty parameters δ and q using algorithm (4.2). Section 6.1.1 presents computations for $q = 2$ (linear algorithm) using fixed values of δ, for continuous piecewise linear and quadratic approximations. Section 6.1.2 presents similar computations to Section 6.1.1 for the penalty parameter $q = 4$ (nonlinear algorithm).

In Tables 6.1 - 6.9, the given expected convergence rate assumes that the penalty parameter δ is sufficiently small and Δt sufficiently small such that the approximation error is due to the spatial discretization.

6.1.1. Investigation of δ for $q = 2$. In this section, with $q = 2$, we investigate the influence of using a fixed value of δ for continuous piecewise linear and quadratic approximations. Results are presented in Tables 6.1 - 6.5.

Remarks:
For δ fixed the norm of the error on the interface approaches a fixed, nonzero value. For δ sufficiently small, with respect to the smallest mesh size, the approximation converges optimally. For δ fixed, under repeated mesh refinements, the error on the interface will eventually significantly influence the approximation errors in the subdomains, with the effect that the errors in the subdomains also approach a fixed, nonzero value. The condition number of the approximating linear system scales $\approx C\,\delta^{-q}$.

h	$\|c^n - c_p^n\|_{n1}$	Cvg. rate	$\|\nabla(c^n - c_p^n)\|_{n2}$	Cvg. rate	$\|c_f - c_p\|_{m2}$	Cond. Num.
1/4	2.340E-02	-8.63E-02	1.086E-01	3.19E-01	8.282E-02	1.445E+01
1/6	2.423E-02	-6.66E-02	9.540E-02	1.86E-01	8.423E-02	1.748E+01
1/8	2.470E-02	-4.42E-02	9.044E-02	1.17E-01	8.480E-02	2.077E+01
1/10	2.495E-02	-3.07E-02	8.811E-02	7.88E-02	8.508E-02	2.435E+01
1/12	2.509E-02	-2.24E-02	8.685E-02	5.59E-02	8.525E-02	2.823E+01
1/14	2.517E-02	-1.69E-02	8.611E-02	4.13E-02	8.536E-02	3.730E+01
1/16	2.523E-02		8.563E-02		8.543E-02	4.766E+01
Exptd.		2.0		1.0		

Table 6.1
Example 1. Convergence rates for a linear approximation with $\delta = 0.5$ and $q = 2$.

h	$\|c^n - c_p^n\|_{n1}$	Cvg. rate	$\|\nabla(c^n - c_p^n)\|_{n2}$	Cvg. rate	$\|c_f - c_p\|_{m2}$	Cond. Num.
1/4	8.945E-03	2.04E+00	7.563E-02	9.91E-01	1.247E-03	3.004E+02
1/6	3.905E-03	2.08E+00	5.060E-02	9.95E-01	1.264E-03	2.702E+02
1/8	2.145E-03	2.11E+00	3.801E-02	9.96E-01	1.274E-03	2.543E+02
1/10	1.341E-03	2.09E+00	3.044E-02	9.95E-01	1.280E-03	2.455E+02
1/12	9.153E-04	2.00E+00	2.539E-02	9.95E-01	1.284E-03	2.401E+02
1/14	6.722E-04	1.81E+00	2.178E-02	9.94E-01	1.286E-03	2.746E+02
1/16	5.280E-04		1.907E-02		1.289E-03	3.112E+02
Exptd.		2.0		1.0		

Table 6.2
Example 1. Convergence rates for a linear approximation with $\delta = 0.05$ and $q = 2$.

h	$\|c^n - c_p^n\|_{n1}$	Cvg. rate	$\|\nabla(c^n - c_p^n)\|_{n2}$	Cvg. rate	$\|c_f - c_p\|_{m2}$	Cond. Num.
1/4	9.118E-03	1.99E+00	7.567E-02	9.92E-01	1.253E-05	2.930E+04
1/6	4.061E-03	2.00E+00	5.062E-02	9.96E-01	1.271E-05	2.615E+04
1/8	2.285E-03	2.00E+00	3.801E-02	9.98E-01	1.281E-05	2.440E+04
1/10	1.463E-03	2.00E+00	3.043E-02	9.98E-01	1.287E-05	2.334E+04
1/12	1.015E-03	2.00E+00	2.536E-02	9.99E-01	1.291E-05	2.263E+04
1/14	7.456E-04	2.00E+00	2.174E-02	9.99E-01	1.294E-05	2.564E+04
1/16	5.705E-04		1.903E-02		1.297E-05	2.879E+04
Exptd.		2.0		1.0		

Table 6.3
Example 1. Convergence rates for a linear approximation with $\delta = 0.005$ and $q = 2$.

h	$\|c^n - c_p^n\|_{n1}$	Cvg. rate	$\|\nabla(c^n - c_p^n)\|_{n2}$	Cvg. rate	$\|c_f - c_p\|_{m2}$	Cond. Num.
1/4	2.922E-04	2.98E+00	5.942E-03	1.99E+00	1.287E-05	3.676E+04
1/6	8.713E-05	2.97E+00	2.649E-03	2.00E+00	1.295E-05	3.577E+04
1/8	3.707E-05	2.92E+00	1.492E-03	2.00E+00	1.300E-05	4.053E+04
1/10	1.933E-05	2.78E+00	9.554E-04	2.00E+00	1.302E-05	4.847E+04
1/12	1.164E-05	2.50E+00	6.638E-04	2.00E+00	1.304E-05	5.653E+04
1/14	7.920E-06	2.06E+00	4.879E-04	2.00E+00	1.305E-05	6.471E+04
1/16	6.012E-06		3.737E-04		1.306E-05	7.292E+04
Exptd.		3.0		2.0		

TABLE 6.4

Example 1. Convergence rates for a quadratic approximation with $\delta = 0.005$ and $q = 2$.

h	$\|c^n - c_p^n\|_{n1}$	Cvg. rate	$\|\nabla(c^n - c_p^n)\|_{n2}$	Cvg. rate	$\|c_f - c_p\|_{m2}$	Cond. Num.
1/4	2.919E-04	2.99E+00	5.943E-03	1.99E+00	1.288E-07	3.675E+06
1/6	8.678E-05	2.99E+00	2.650E-03	2.00E+00	1.295E-07	3.575E+06
1/8	3.667E-05	3.00E+00	1.492E-03	2.00E+00	1.300E-07	4.049E+06
1/10	1.879E-05	3.00E+00	9.556E-04	2.00E+00	1.302E-07	4.842E+06
1/12	1.088E-05	3.00E+00	6.639E-04	2.00E+00	1.304E-07	5.647E+06
1/14	6.854E-06	3.00E+00	4.879E-04	2.00E+00	1.305E-07	6.462E+06
1/16	4.593E-06		3.736E-04		1.306E-07	7.280E+06
Exptd.		3.0		2.0		

TABLE 6.5

Example 1. Convergence rates for a quadratic approximation with $\delta = 0.0005$ and $q = 2$.

Remark: In order to avoid the stagnation of the errors under spatial mesh refinement, computations were also performed using $\delta = c\, h^s$ (see [EKL17]). Such a choice for δ is not covered by the theory developed herein. The numerical results indicate that if s is chosen sufficiently large, ($s \geq 1$ for a linear approximation, $s \geq 1.5$ for a quadratic approximation), the approximations converge optimally. However, using $\delta = 0.5\, h^s$ results in a significant growth in the condition number under mesh refinement.

6.1.2. Investigation of δ for $q = 4$. As noted for the computations with $q = 2$, the accuracy of the approximation of c in Ω is controlled by the value of $[c]$ along I. For δ fixed, under mesh refinement, the accuracy of the approximations in Ω stagnate once the order of magnitude of the error of c_f and c_p in Ω_f and Ω_p, respectively, are equal to the order of magnitude of their errors on I. The same observations hold for the nonlinear case, $q = 4$. For the nonlinear case, at each time step an iteration was used to solve the nonlinear system of equations. Specifically, given c^n, c^{n+1} was computed using the damped iteration. The functions ckn and $cknold$, the same type as c^n, were used in the iteration.

$$
\begin{aligned}
& c^{n+1,0} \;=\; c^n \\
& cknold \;=\; c^n \\
& ckn \;=\; 0 \\
& k = 1 \\
& \text{while } \|ukn - uknold\|_{l^\infty} \;>\; 10^{-8} \\
& \quad cknold \;=\; ckn \\
& \quad \beta\left(\frac{ckn - c^n}{\Delta t}\right) \;+\; \frac{1}{2}(u^{n+1}\cdot\nabla ckn\,,\,v) \;-\; \frac{1}{2}(u^{n+1}\cdot\nabla v\,,\,ckn) \\
& \quad\; + \; \frac{1}{2}\int_{\Omega_p}(\nabla\cdot u^{n+1})\,c^n\,v\,dx \;+\; \delta^{-q}\int_I |[c^{n+1,k-1}]|^{q-2}\,[ckn]\,[v]\,ds \;=\; (s^{n+1}\,,\,v)\,, \forall v \in X \\
& \quad c^{n+1,k} \;=\; 0.5 * c^{n+1,k-1} \;+\; 0.5 * ckn \\
& \quad k \;=\; k+1 \\
& \text{end} \\
& c^{n+1} = ckn
\end{aligned}
$$

For all of the computations, the number of iterations required to solve the nonlinear system of equations was less than 20.

As the iteration converges $[c^{n+1,k-1}] << 1$, thus the multiplier in the integrand over I acts to damp the coefficients of the unknowns on the interface. For $\delta << 1$, the δ^{-q} multiplier outside the integral over I amplifies the coefficients of the unknowns on the interface. For example, with $\delta = 0.05$, if $[c^{n+1,k-1}] \approx 10^{-4}$ the coefficients in the integrand are initially multiplied by $\approx 10^{-8}$ and then post multiplied by $(0.05)^{-4} = 4*10^6$. The result of this pre and post scaling is that it is difficult to control the accuracy of the error in the interface. This was most noticeable in the case of a continuous, piecewise quadratic approximation for c_f and c_p. From Tables 6.8-6.9 we note accurate computations for c_p and c_f, however we do not observe the expected asymptotic convergence rates as the domain error is of the same order of magnitude as the error on the interface.

h	$\|c^n - c_p^n\|_{n1}$	Cvg. rate	$\|\nabla(c^n - c_p^n)\|_{n2}$	Cvg. rate	$\|c_f - c_p\|_{m2}$
1/4	8.213E-03	1.90E+00	7.615E-02	9.44E-01	1.212E-02
1/6	3.801E-03	1.08E+00	5.194E-02	8.99E-01	1.258E-02
1/8	2.787E-03	3.06E-01	4.011E-02	8.40E-01	1.280E-02
1/10	2.603E-03	-7.37E-03	3.325E-02	7.74E-01	1.292E-02
1/12	2.607E-03	-9.41E-02	2.887E-02	7.04E-01	1.300E-02
1/14	2.645E-03	-1.09E-01	2.590E-02	6.35E-01	1.305E-02
1/16	2.684E-03		2.380E-02		1.309E-02
Exptd.		2.0		1.0	

TABLE 6.6
Example 1 (Nonlinear Iteration). Convergence rates for a linear approximation with $\delta = 0.05$ and $q = 4$.

h	$\|c^n - c_p^n\|_{n1}$	Cvg. rate	$\|\nabla(c^n - c_p^n)\|_{n2}$	Cvg. rate	$\|c_f - c_p\|_{m2}$
1/4	9.061E-03	2.01E+00	7.565E-02	9.91E-01	5.751E-04
1/6	4.007E-03	2.03E+00	5.061E-02	9.96E-01	5.975E-04
1/8	2.234E-03	2.05E+00	3.801E-02	9.97E-01	6.085E-04
1/10	1.413E-03	2.07E+00	3.042E-02	9.98E-01	6.150E-04
1/12	9.684E-04	2.09E+00	2.536E-02	9.98E-01	6.193E-04
1/14	7.018E-04	2.10E+00	2.175E-02	9.98E-01	6.223E-04
1/16	5.303E-04		1.904E-02		6.246E-04
Exptd.		2.0		1.0	

TABLE 6.7
Example 1 (Nonlinear Iteration). Convergence rates for a linear approximation with $\delta = 0.005$ and $q = 4$.

h	$\|c^n - c_p^n\|_{n1}$	Cvg. rate	$\|\nabla(c^n - c_p^n)\|_{n2}$	Cvg. rate	$\|c_f - c_p\|_{m2}$
1/4	3.332E-04	1.74E+00	5.940E-03	1.93E+00	6.126E-04
1/6	1.644E-04	5.46E-01	2.713E-03	1.76E+00	6.219E-04
1/8	1.405E-04	1.51E-01	1.636E-03	1.43E+00	6.265E-04
1/10	1.358E-04	5.03E-02	1.190E-03	1.02E+00	6.292E-04
1/12	1.346E-04	1.98E-02	9.878E-04	6.58E-01	6.310E-04
1/14	1.342E-04	8.38E-03	8.925E-04	3.97E-01	6.323E-04
1/16	1.340E-04		8.464E-04		6.333E-04
Exptd.		3.0		2.0	

TABLE 6.8
Example 1 (Nonlinear Iteration). Convergence rates for a quadratic approximation with $\delta = 0.005$ and $q = 4$.

h	$\|c^n - c_p^n\|_{n1}$	Cvg. rate	$\|\nabla(c^n - c_p^n)\|_{n2}$	Cvg. rate	$\|c_f - c_p\|_{m2}$
1/4	3.019E-04	1.98E-01	5.947E-03	1.98E+00	3.497E-05
1/6	2.786E-04	6.24E-04	2.660E-03	1.96E+00	3.544E-05
1/8	2.786E-04	3.22E-04	1.513E-03	1.91E+00	3.564E-05
1/10	2.786E-04	0.00E+00	9.877E-04	1.81E+00	3.576E-05
1/12	2.786E-04	2.33E-04	7.097E-04	1.67E+00	3.583E-05
1/14	2.785E-04	0.00E+00	5.489E-04	1.47E+00	3.588E-05
1/16	2.785E-04		4.511E-04		3.592E-05
Exptd.		3.0		2.0	

TABLE 6.9
Example 1 (Nonlinear Iteration). Convergence rates for a quadratic approximation with $\delta = 0.0005$ and $q = 4$.

In summary, for the values of δ used, the nonlinear approach with $q = 4$ took considerably longer to compute the approximations compared to the linear approach ($q = 2$). The accuracy of the approximations in both cases were similar, except for the case of a continuous piecewise quadratic approximation with a fixed value for δ. For this case the accuracy of the nonlinear scheme was approximately one order of magnitude less accurate.

6.2. Partitioned algorithm. In this section we investigate the partitioned algorithm (5.1), (5.2) for the linear penalty method (i.e., $q = 2$). We note, as established by Theorem 5.1, that the approximation scheme is stable. The computations show that for a spatial approximation using linear elements (quadratic elements) with $\Delta t \approx h^2$ ($\Delta t \approx h^3$) the scheme is convergent (up to an accuracy predetermined by penalty parameter δ). (See Tables 6.10 and 6.12.) In other tests (not herein but reported in [EKL17]), choosing $\delta \approx h^s$ was not observed to be effective for the decoupled algorithm.

In Tables 6.10 - 6.12, the given expected convergence rate assumes that the approximation error is $O(\Delta t + \Delta t\, h^l)$, where l is the polynomial order of the spatial discretization.

h	$\|c^n - c_p^n\|_{n1}$	Cvg. rate	$\|\nabla(c^n - c_p^n)\|_{n2}$	Cvg. rate	$\|c_f - c_p\|_{m2}$	
1/4	1.894E-02	1.19E+00	1.684E-01	1.03E+00	1.258E-03	
1/6	1.171E-02	1.55E+00	1.108E-01	1.37E+00	1.263E-03	
1/8	7.507E-03	1.74E+00	7.481E-02	1.49E+00	1.269E-03	
1/10	5.092E-03	1.84E+00	5.361E-02	1.52E+00	1.273E-03	
1/12	3.639E-03	1.90E+00	4.065E-02	1.50E+00	1.276E-03	
1/14	2.713E-03	1.94E+00	3.225E-02	1.47E+00	1.278E-03	
1/16	2.092E-03		2.651E-02		1.280E-03	
Exptd.		2		2		

TABLE 6.10
Example 1 (Partitioned algorithm). Approximation using linear elements with $q = 2$, $\delta = 0.05$, $\Delta t = h^2$.

h	$\|c^n - c_p^n\|_{n1}$	Cvg. rate	$\|\nabla(c^n - c_p^n)\|_{n2}$	Cvg. rate	$\|c_f - c_p\|_{m2}$	
1/4	2.991E-02	3.75E-01	3.119E-01	3.68E-01	1.388E-03	
1/6	2.569E-02	5.68E-01	2.686E-01	6.38E-01	1.330E-03	
1/8	2.182E-02	8.03E-01	2.235E-01	8.90E-01	1.311E-03	
1/10	1.824E-02	1.04E+00	1.833E-01	1.12E+00	1.302E-03	
1/12	1.510E-02	1.25E+00	1.495E-01	1.31E+00	1.298E-03	
1/14	1.246E-02	1.42E+00	1.222E-01	1.47E+00	1.296E-03	
1/16	1.031E-02		1.004E-01		1.294E-03	
Exptd.		2		2		

TABLE 6.11
Example 1 (Partitioned algorithm). Approximation using quadratic elements with $q = 2$, $\delta = 0.05$, $\Delta t = h^2$.

h	$\|c^n - c_p^n\|_{n1}$	Cvg. rate	$\|\nabla(c^n - c_p^n)\|_{n2}$	Cvg. rate	$\|c_f - c_p\|_{m2}$	
1/4	2.196E-02	1.56E+00	2.173E-01	1.61E+00	1.299E-03	
1/6	1.167E-02	2.42E+00	1.130E-01	2.46E+00	1.286E-03	
1/8	5.813E-03	2.79E+00	5.578E-02	2.78E+00	1.285E-03	
1/10	3.121E-03	2.92E+00	2.997E-02	2.89E+00	1.286E-03	
1/12	1.831E-03	2.98E+00	1.768E-02	2.93E+00	1.287E-03	
1/14	1.157E-03	2.98E+00	1.125E-02	2.93E+00	1.288E-03	
1/16	7.771E-04		7.612E-03		1.288E-03	
Exptd.		3		3		

TABLE 6.12
Example 1 (Partitioned algorithm). Approximation using quadratic elements with $q = 2$, $\delta = 0.05$, $\Delta t = h^3$.

7. Conclusions. A number of effective decoupled (partitioned) schemes for the Stokes-Darcy problem have been developed. The velocity these calculate is important because it transports contaminants. Thus, partitioned methods for the associated transport problem have equal importance but are much less developed. Much of the previous work has either assumed a static flow field and then solved an evolutionary equation for the transport across the entire domain, or computed the time dependent Stokes-Darcy flow using a partitioned algorithm and then used a monolithic approach across the entire domain for the transport equation. There are several natural methods to uncouple this transport problem into subdomain transport problems. We have given a numerical analysis and presented tests of a natural first step in their development, a partitioned penalty approach. Like penalty methods in general, this approach is easily implemented (for $q = 2$). The analysis presented in this paper establishes an error bound between the original modeling equations and the modified equations that use the penalty parameter to enforce continuity of the concentration across the Stokes-Darcy interface. For the numerical approximation schemes, the analysis proves stability of the monolithic and partitioned algorithms, and gives an error analysis of the monolithic scheme. The presented numerical computations investigate the appropriate choice of the penalty parameter, and demonstrate that the presented partitioned algorithm is a viable method for the approximation of the concentration equation in a coupled flow with transport problem. Naturally, other approaches to partitioning the transport problem are needed followed by a comparison of their advantages and disadvantages.

REFERENCES

[AZ11] C. D'Angelo and P. Zunino *Robust numerical approximation of coupled Stokes' and Darcy's flows applied to vascular hemodynamics and biochemical transport.* ESAIM: Mathematical Modelling and Numerical Analysis 45.03 (2011): 447-476.

[A99] P. Angot, *Analysis of singular perturbations on the Brinkman problem for fictitious domain models of viscous flows*, Math. Methods Appl. Sci. 22 (1999), no. 16, 1395-1412

[AB07] T. Arbogast and D.S. Brunson, *A computational method for approximating a Darcy-Stokes system governing a vuggy porous medium*, Comput. Geosciences 11(2007) 207-218.

[BC10] J. Bear and A.H.-D. Cheng. *Modeling Groundwater flow and contaminant transport.* Springer, Berlin 2010.

[CGHW11] Y. Cao, M. Gunzburger, X.-M. He, and X. Wang, *Robin-Robin domain decomposition methods for the steady Stokes-Darcy model with Beaver-Joseph interface condition*, Numer. Math. 117 (2011), no. 4, 601-629.

[CGHW11b] Y. Cao, M. Gunzburger, X.-M. He, and X. Wang, *Parallel, non-iterative multi-physics domain decomposition methods for the time-dependent Stokes-Darcy problem*, Math. Comp. 83 (2014), no. 288, 1617-1644.

[CGHW08] Y. Cao, M. Gunzburger and F. Hua, X. Wang, *Coupled Stokes-Darcy Model with Beavers-Joseph Interface Boundary Condition*, Comm. Math. Sci., 8, (2010), 1-25.

[CR09] A. Cesmelioglu and B. Riviere, *Existence of a weak solution for the fully coupled Navier–Stokes/Darcy-transport problem*, J. Differential Equations 252 (2012), no. 7, 4138-4175.

[CHL09] J. Connors, J. Howell and W. Layton, *Partitioned timestepping for a parabolic two domain problem,* SIAM J. Numer. Anal. 47(2009) 3526-3549.

[CHL12] J. Connors, J. Howell and W. Layton, *Decoupled time stepping for a fluid-fluid interaction problem,* SIAM J. Numer. Anal. 50 (2012), pp. 1297-1319.

[CMX09] M. Cai, M. Mu and J. Xu, *Preconditioning techniques for a mixed Stokes/Darcy model in porous media applications*, J. Comput. Appl. Math. 233 (2009), no. 2, 346-355.

[DDD91] C.N. Dawson, Q. Du, and T.F. Dupont, *A finite difference domain decomposition algorithm for numerical solution of the heat equation*, Math. Comp., 1991, 57(195): 63-71.

[D04] M. Discacciati, *Domain decomposition methods for the coupling of surface and groundwater flows*, Ph.D. dissertation, École Polytechnique Fédérale de Lausanne, 2004.

[DMQ02] M. Discacciati, E. Miglio and A. Quarteroni, *Mathematical and numerical models for coupling surface and groundwater flows*, Appl. Numer. Math., 43 (2002), 57-74.

[DQ] M. Discacciati and A. Quarteroni, *Convergence analysis of a subdomain iterative method for the finite element approximation of the coupling of Stokes and Darcy equations*, Computing and Visualization in Science, 6,(2004) 93-103, DOI: 10.1007/s00791-003-0113-0

[DQ09] M. Discacciati and A. Quarteroni, *Navier-Stokes/Darcy Coupling: Modeling, Analysis, and Numerical Approximation*, Revista Matemática Complutense, 2009

[EJS09] V. Ervin, E. Jenkins, and S. Sun, *Coupled generalized non-linear Stokes flow with flow through a porous media*, SIAM J. Numer. Anal, 47(2009) 929-952.

[EKL15] V. Ervin, M. Kubacki, W. Layton, M. Moraiti, Z. Si and C. Trenchea, *On limiting behavior of contaminant transport models in coupled surface and groundwater flows*, Axioms, 4 (2015), 518-529.

[EKL17] V. Ervin, M. Kubacki, W. Layton, M. Moraiti, Z. Si and C. Trenchea, *Partitioned penalty methods for the evolutionary Stokes-Darcy-Transport problem*, University of Pittsburgh, Math. Dept., technical report, 2017, `http://www.mathematics.pitt.edu/sites/default/files/StokesDarcyTransportPenalty21Aug17.pdf`

[GS07] J. Galvis and M. Sarkis, *Non-matching mortar discretization analysis for the coupling Stokes- Darcy equations,* Electron. Trans. Numer. Anal. 26 (2007), 350 - 384.

[GR79] V. Girault, P. A. Raviart, *Finite Element Approximation of the Navier-Stokes Equations*, LNM vol. 749, Springer, Berlin, 1979.

[J09] B. Jiang, *A parallel domain decomposition method for coupling of surface and groundwater flows*, Comput. Methods Appl. Mech. Engrg. 198 (2009), no. 9-12, 947 - 957.

[JLM] V. John, A. Linke, C. Merdon, M. Neilan and L.G. Rebholz, *On the divergence constraint in mixed finite element methods for incompressible flows*, SIAM Rev., 59 (2017), 492–544.

[HWN06] N. Hanspal, A. Waghode, V. Nassehi, and R. Wakeman, *Numerical analysis of coupled Stokes/Darcy flows in industrial filtrations. Transport in Porous Media*, 64(1) (2006), 73-101.

[LSY] W. Layton, F. Schieweck and I. Yotov, *Coupling fluid flow with porous media flow*, SIAM J. Numer. Anal. 40 (2003) 2195-2218.

[LTT13] W. Layton, H. Tran and C. Trenchea, *Analysis of long time stability and errors of two partitioned methods for uncoupling evolutionary groundwater-surface water flows*, SIAM J. Numer. Anal. 51 (2013), no. 1, 248-272.

[LY08] C.F. Li and Y.R. Yuan, *Explicit-implicit domain decomposition method with modified upwind differences for convection-diffusion equations*, Computers and Mathematics with Applications, 2008, 55(11): 2565-2573.

[M12] M. Moraiti, *On the quasistatic approximation in the Stokes-Darcy model of groundwater-surfacewater flows*, *Journal of Mathematical Analysis and Applications* (2012), 394(2):796-808.

[MZ10] M. Mu and X. H. Zhu, *Decoupled schemes for a non-stationary mixed Stokes-Darcy model*, Math. Comp., 79 (2010), 707-731.

[PC06] G. Pinder and M. Celia, *Subsurface hydrology*, John Wiley and Sons, Hoboken, New Jersey, 2006.

[QVZ01] A. Quarteroni, A. Veneziani and P. Zunino, *Mathematical and numerical modeling of solute dynamics in blood flow and arterial walls*, SIAM J. Numer. Anal., Vol.39, No. 5, pp. 1488-1511, 2001

[R05] B. Riviere, *Analysis of a discontinuous finite element method for the coupled Stokes and Darcy problems*, J. Sci. Comput. 22/23 (2005), 479 - 500.

[R14] B. Riviere, *Discontinuous Finite Element Methods for Coupled Surface–Subsurface Flow and Transport Problems*, in: Recent Developments in Discontinuous Galerkin Finite Element Methods for Partial Differential Equations, V 157 in The IMA Volumes in Mathematics and its Applications, editors: Feng, Xiaobing, Karakashian, Ohannes and Xing, Yulong, Springer, 2014.

[RST96] H.-G. Roos, M. Stynes and L. Tobiska, *Numerical Method for Singularly Perturbed Differential Equations*, Springer, Berlin, 1996.

[SS12] P. Song and S. Sun, *Contaminant flow and transport simulation in cracked porous media using locally conservative schemes*, Adv. Appl. Math. Mech. 4 (2012), no. 4, 389-421.

[VY09] D. Vassilev and I. Yotov, *Coupling Stokes–Darcy flow with transport*, SIAM J. Sci. Comput., 31(2009), 3661–3684.

[ZYD09] L. Zhu, G. Yuan, and Q. Du, *An explicit-implicit predictor-corrector domain decomposition method for time dependent multi-dimensional convection diffusion equations.* Numer. Math. Theory Methods Appl., 2(2009), 301-325.