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An indirect finite element method for variable-coefficient space-fractional diffusion equations and its optimal order error estimates

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Abstract We study an indirect finite element approximation for two-sided space-fractional diffusion equations in one space dimension. By the representation formula of the solutions $u(x)$ to the proposed variable coefficient models in terms of $v(x)$, the solutions to the constant coefficient analogues, we apply finite element methods for the constant coefficient fractional diffusion equations to solve for the approximations $v_h(x)$ to $v(x)$ and then obtain the approximations $u_h(x)$ of $u(x)$ by plugging $v_h(x)$ into the representation of $u(x)$. Optimal-order convergence estimates of $u(x) - u_h(x)$ are proved in both L^2 and $H^{\alpha/2}$ norms. Several numerical experiments are presented to demonstrate the sharpness of the derived error estimates.

Keywords Fractional diffusion equation · Finite element method · Convergence estimate

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1 Introduction

Fractional differential equations (FDEs) provide adequate descriptions of challenging phenomena such as the anomalously diffusive transport and nonlocal spatial interactions [2, 14]. However, FDEs present new mathematical issues that are not common in the context of integer-order diffusion equations. In this paper we take a relatively simple one-dimensional space-fractional differential equations (sFDEs) to address some of the issues and demonstrate how we develop an indirect finite element method to overcome these difficulties.

Ervin and Roop [5] first carried out a rigorous analysis of the the Dirichlet boundary-value problem of the constant coefficient sFDEs of order $1 < \alpha < 2$ in one space dimension

$$\begin{aligned} -D\left(\left(r\ {}_0I_x^{2-\alpha} \ + \ (1-r)\ {}_xI_1^{2-\alpha}\right) Du(x)\right) &= \ f(x), \quad x \in (0,1)\,, \\ u(0) \ = \ u(1) &= \ 0. \end{aligned} \tag{1}$$

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Here the parameter $0 \le r \le 1$ indicates the relative weight of forward versus backward transition probability, D refers to the first-order differential operator, and the left and right fractional integrals of order $0 < \sigma < 1$ are defined as [18,19]

$$
\begin{aligned}
{}_0I_x^{\sigma} g(x) &:= \frac{1}{\Gamma(\sigma)} \int_0^x \frac{g(s)}{(x-s)^{1-\sigma}} ds, \\
{}_xI_1^{\sigma} g(x) &:= \frac{1}{\Gamma(\sigma)} \int_x^1 \frac{g(s)}{(s-x)^{1-\sigma}} ds
\end{aligned}
$$

where $\Gamma(\cdot)$ is the Gamma function.

They derived the following Galerkin weak formulation for problem (1): Find $u \in H_0^{\alpha/2}(0,1)$ such that

$$
\begin{aligned}
B(u,v) &:= r\big({}_0I_x^{1-\alpha/2} Du, {}_xI_1^{1-\alpha/2} Dv\big)_{L^2(0,1)} + (1-r)\big({}_xI_1^{1-\alpha/2} Du, {}_0I_x^{1-\alpha/2} Dv\big)_{L^2(0,1)} \\
&= \langle f, v \rangle, \quad \forall v \in H_0^{\alpha/2}(0,1).
\end{aligned} \tag{2}
$$

They proved that the bilinear form $B(\cdot,\cdot)$ is coercive and continuous in $H_0^{\alpha/2}(0,1) \times H_0^{\alpha/2}(0,1)$ and so the wellposedness of the Galerkin weak formulation (2).

Furthermore, Ervin and Roop went on to define a family of finite element methods for problem (1) based on the Galerkin weak formulation (2) and proved the following optimal-order error estimate for the finite element approximation $u_h \in S_h^m(0,1)$ in the energy norm under the assumption that *the true solution u to problem* (1) *belongs to* $H^{m+1}(0,1) \cap H_0^{1-\frac{\beta}{2}}(0,1)$

$$
\|u_h - u\|_{H^{\frac{\alpha}{2}}(0,1)} \le Ch^{m+1-\alpha/2} \|u\|_{H^{m+1}(0,1)}. \tag{3}
$$

Here $S_h^m(0,1) \subset H_0^{\frac{\alpha}{2}}(0,1)$ consists of continuous and piecewise polynomials of degree up to m. In addition, under the assumption that *for each L^2 right-hand side, the true solution to the dual problem has full regularity* they proved the following optimal-order error estimates for the finite element approximation in the L^2 norm

$$
\|u_h - u\|_{L^2(0,1)} \le Ch^{m+1} \|u\|_{H^{m+1}(0,1)}. \tag{4}
$$

Subsequently, these techniques were extended to the mathematical analysis of multidimensional problems, time-dependent problems as well as other numerical methods. While these error estimates are in parallel to these integer-order analogues [3], it was realized subsequently that there are actually significant differences between FDEs and their integer-order analogues. Here we illustrate some of them.

The homogeneous Dirichlet boundary-value problems of one-dimensional linear elliptic sFDEs with smooth coefficients and source terms yield solutions with singularities. It was shown in [13,22,25] that the solutions to the homogeneous Dirichlet boundary-value problems of the one-dimensional linear elliptic sFDEs with constant coefficients and source terms exhibit boundary layer singularities. In other words, the smoothness (e.g., in Sobolev or Hölder spaces) of the coefficients and right-hand sides of linear elliptic sFDEs in one-space dimension cannot ensure the smoothness of their solutions in corresponding regularity spaces. This is in sharp contrast to integer-order linear elliptic differential equations in which the smoothness of their coefficients and right-hand sides (plus the smoothness the domain for multidimensional problems) ensure the smoothness of their true solutions [7]. Consequently, it is not clear *what conditions should be imposed on the coefficients and right-hand sides of sFDEs to ensure the smoothness of their solutions*, which were assumed in the proof of the optimal-order error estimates (3) of the finite element methods to problem (1) in the energy norm. Even worse, the optimal-order error estimates (4) of the finite element methods to problem (1) in the L^2 norm were proved via the Nitsche lifting technique or duality arguments, under the assumption that the true solutions to the dual problem of problem (1) have full regularity for *each* right-hand side function in L^2. While the counterexamples in [13,22,25] showed that the assumption does not hold. In other words, the Nitsche-lifting based optimal-order L^2 error estimates of the form (4) are inappropriate.

The variable-coefficient analogue of Galerkin weak formulation (2) *may lose its coercivity.* It was shown in [21] that the Galerkin weak formulation for a variable-diffusivity analogue of problem (1) with a variable diffusivity coefficient that has positive lower and upper bounds may lose its coercivity, and so the wellposedness of the problem cannot be guaranteed. In fact, it was shown in [24] that the finite element methods may diverge in this case. One may need to impose certain constraint on the magnitude of the diffusivity coefficient to obtain the coercivity the Galerkin weak formulation [9]. This is, again, in sharp contrast to integer-order linear elliptic differential equations, in which a variable-coefficient analogue of the homogeneous Dirichlet boundary-value problem of the Poisson equation is well posed. Moreover, their finite element approximations are guaranteed to converge with an optimal-order convergence rate.

Inhomogeneous Dirichlet boundary-value problems of linear elliptic sFDEs. In the context of the homogeneous Dirichlet boundary condition, the sFDE (1), which is in the conservative Caputo form, coincides with the sFDE in the Riemann-Liouville form [5]. However, the sFDEs in these two forms do not coincide in the context of inhomogeneous Dirichlet boundary conditions. In fact, it was shown in [22] that the inhomogeneous Dirichlet boundary-value analogue of problem (1) is well-posed, but the inhomogeneous Dirichlet boundary-value problem of the sFDE in the Riemann-Liouville form has no weak solution!

Extensive research activities have been carried out on the numerical approximations and associated mathematical analysis of sFDEs. Here we briefly outline some of the recent developments.

A Petrov-Galerkin weak formulation for variable-coefficient sFDEs. A Petrov-Galerkin weak formulation was derived for the homogeneous Dirichlet boundary-value problem of the one-sided linear sFDE with a variable diffusivity coefficient, in which the weak formulation was proved to be weakly coercive that ensures the wellposedness of the weak formulation [21]. A Petrov-Galerkin finite element method was developed for the problem subsequently [23]. Petrov-Galerkin formulations were developed for one-sided constant-coefficient sFDEs, in which the closed form analytical solutions were utilized to analyze the wellposedness of sFDEs [13].

Indirect finite element methods and spectral Galerkin methods for the inhomogeneous Dirichlet boundary-value problems of one-sided variable-coefficient sFDEs. Based on the results in [21], indirect finite element methods and spectral Galerkin methods were developed in [24] and [25], respectively, for the inhomogeneous Dirichlet boundary-value problems of the one-sided variable-coefficient sFDEs. This approach can be summarized as follows: (i) The true solutions to the inhomogeneous Dirichlet boundary-value problems of the one-sided variable-coefficient sFDEs are expressed in terms of the fractional differentiation of the solutions to integer-order differential equations. (ii) Solve the integer-order differential equations to obtain their numerical approximations (e.g., finite element solutions or spectral Galerkin solutions). (iii) Obtain the indirect numerical (e.g., finite element or spectral) solutions by postprocessing the corresponding numerical solutions of the integer-order differential equations using the formulation derived in (i).

The key advantage of the indirect approach is as follows. In this approach, the smoothness of coefficients and source terms of the second-order differential equations ensures the smoothness of the solutions to the second-order equations, which ensures the high-order convergence rates of the corresponding finite element methods and spectral Galerkin methods. Subsequently, the indirect solutions to the sFDEs, which are obtained by postprocessing the numerical solutions to the integer-order differential equations by the formulation derived in (i), retain high-order convergence rates to the true solutions to the sFDEs in the standard (not weighted) L^2 norm, under the regularity assumptions of the coefficients and source terms (but not the regularity of the true solutions). These results were proved mathematically and justified numerically in [24,25].

Spectral and spectral Galerkin methods for sFDEs. Spectral and spectral Galerkin methods provide an alternative potential approach in the numerical approximations of sFDEs. Some related works were conducted under the fully regularity assumptions of the solutions to the models, see, e.g., [11, 28]. In [15], a spectral Galerkin method using Jacobi polynomials (cf [8,20]) was developed for a two-sided variable-coefficient sFDE in which the regularity of the solutions were assumed to belong to some weighted Sobolev spaces in the convergence analysis.

The regularity of solutions to model (1) was throughly studied by Ervin, Heuer and Roop [6], in which, by means of weighted Jacobi spectral expansions, the solutions were proved to have a boundary layer in the form $(1-x)^{\alpha-\beta}x^{\beta}$ for some $\alpha - 1 \leq \beta \leq 1$ determined by α and r. Based on the regularity results, rigorous analysis for both the finite element approximations and spectral methods were conducted and (sub-) optimal-order convergence estimates of the error were derived in L^2 and $H^{\alpha/2}$ norms and in weighted L^2 norms, respectively. A Petrov-Galerkin method for model (1) was then studied in [16] with the (two-sided) Jacobi polyfractonomials as basis and test functions and the corresponding error estimates were derived in proper weighted Sobolev spaces with the exponential decay rate. In [27], the regularity of the fractional Laplacian diffusion-reaction equations with constant coefficients were analyzed in weighted Sobolev spaces, based on which the sub-optimal convergence rates of the Jacobi spectral Galerkin method were proved in the weighted L^2 norms. The proposed methods were further extended to the fractional Laplacian advection-diffusion-reaction equations with constant diffusivity coefficients [10], in which the optimal-order convergence rates of the spectral Galerkin method were proved in both the weighted L^2 and $H^{\alpha/2}$ norms.

In this paper we study an indirect finite element approximations for the variable-coefficient two-sided fractional diffusion equations. A key ingredient is to utilize the relation (11) between the solutions to the variable-constant model and that of the constant-coefficient analogues (1). Then after solving the constant-coefficient problems by the well-developed methods in references, we can obtain the approximations of the solutions to the variable-coefficient fractional diffusion equations by postprocessing. The advantages of the proposed indirect method are listed as follows: (i) By the relation (11) we only need to work on the constant-coefficient analogues of the variable-coefficient fractional diffusion equations, the numerical approximations of which have been extensively investigated and analyzed so the developed results can be directly applied in the analysis of the proposed method; (ii) As the main task of the proposed method is to solve the constant coefficient fractional diffusion equations, the Galerkin weak formulation of which is naturally coercive, we avoid any artificial assumption on the data and the numerical divergence; (iii) In the case of finite element methods, we use the convergence estimates of the approximations for the constant-coefficient models (cf. [6]) to prove that of the approximations for the variable-coefficient models in L^2 and $H^{\alpha/2}$ norms with the optimal-order rather than in the weighted Sobolev norms; (iv) The representation (11) can be extended to accommodate the inhomogeneous boundary conditions of the variable-coefficient models (cf (24)), based on which we are allowed to approximate the solutions to inhomogeneous boundary value problems with variable coefficients by only solving the constant-coefficient analogues with homogeneous boundary conditions.

The rest of the paper is organized as follows: In Section 2 we present the formulation of the model and introduce notations and key lemmas used in the analysis. The representation of the solutions $u(x)$ to the proposed variable-coefficient model is presented in Section 3, based on which a finite element approximation for the solutions $v(x)$ to the constant-coefficient analogues is introduced and the approximations $u_h(x)$ of $u(x)$ is obtained by plugging $v_h(x)$, the approximation of $v(x)$, into the representation of $u(x)$. Optimal-order convergence analysis of $u(x)-u_h(x)$ is proved based on that of $v(x)-v_h(x)$. Several numerical experiments are presented in Section 4 to demonstrate the sharpness of the derived error estimates and some concluding remarks are referred in the last section.

2 Problem formulation and preliminaries

We consider the following space-fractional diffusion equations for $1<\alpha<2$ [2,26]

$$-D\left((r\,{}_0I_x^{2-\alpha} \;+\; (1-r)\,{}_xI_1^{2-\alpha})\,K(x)Du(x)\right) = \;f(x),\quad x\in(0,1)\,, \tag{5}$$

$$u(0) \;=\; u(1) = \;0\,, \tag{6}$$

where $K(x)$ is the diffusivity coefficient with $0<K_m\le K(x)\le K_M\le\infty$, $0\le r\le 1$ and $f(x)$ is the source or sink term. For $m\in\mathbb{N}^+$ and $0<\sigma<1$, the left and right Riemann–Liouville fractional derivatives of order $m-\sigma$ are defined by [18,19]

$$\begin{aligned}{}_0D_x^{m-\sigma}g(x) &:= \frac{1}{\Gamma(\sigma)}\frac{d^m}{dx^m}\int_0^x\frac{g(s)}{(x-s)^{1-\sigma}}ds,\\ {}_xD_1^{m-\sigma}g(x) &:= \frac{(-1)^m}{\Gamma(\sigma)}\frac{d^m}{dx^m}\int_x^1\frac{g(s)}{(s-x)^{1-\sigma}}ds.\end{aligned}$$

Let $C^\infty(0,1)$ denote the space of continuous, infinitely differentiable, functions on the interval $(0,1)$, and $C_0^\infty(0,1)$ be the functions in $C^\infty(0,1)$ that have compact support within $(0,1)$. Define the weighted L^2 space, $L^2_\omega(0,1)$, and L^2 weighted inner product for any non-negative integrable function $\omega(x)$ on $x\in(0,1)$ as

$$\begin{aligned}L^2_\omega(0,1) \;&:=\; \left\{f(x)\;:\;\|f\|^2_{L^2_\omega(0,1)} := \int_0^1\omega(x)\,f(x)^2\,dx \;<\;\infty\right\},\\ (f\,,\,g)_\omega \;&:=\; \int_0^1\omega(x)\,f(x)\,g(x)\,dx\,.\end{aligned} \tag{7}$$

In particular, if $w\equiv 1$, the space and the inner product in (7) degenerate to the standard L^2 space equipped with the norm $\|\cdot\|:=\|\cdot\|_{L^2_w(0,1)}$ and the L^2 inner product, respectively. Another important weight function takes the form of

$$\omega^{(\kappa,\nu)}(x) := (1-x)^\kappa x^\nu,\quad \kappa,\nu>-1, \tag{8}$$

which could properly characterize the boundary layer of the solutions to the fractional diffusion equations. We also denote $L^\infty(0,1)$ by the space of essentially bounded functions, see [1].

Following [5], for $\mu>0$ define the semi-norm $|g|_{J_L^\mu(0,1)}$, and norm $\|g\|_{J_L^\mu(0,1)}$ as

$$|g|_{J_L^\mu(0,1)} := \|{}_0D_x^\mu g\|\,,\quad \|g\|_{J_L^\mu(0,1)} := (\|g\|^2+|g|^2_{J_L^\mu(0,1)})^{1/2}.$$

The left fractional derivative space $J^\mu_{L,0}(0,1)$ is then defined as the closure of $C_0^\infty(0,1)$ with respect to $\|\cdot\|_{J_L^\mu(0,1)}$.

For $0<\mu<1$, define the (Slobodetskii) semi-norm [4]

$$|g|_{H^\mu(0,1)} \;:=\; \left(\int_0^1\int_0^1\frac{(g(x)\;-\;g(y))^2}{|x\;-\;y|^{1\,+\,2\mu}}dx\,dy\right)^{1/2},$$

and for $m-1\;<\;\sigma\;<\;m$, $m\in\mathbb{N}^+$ the norm

$$\|g\|_{H^\sigma(0,1)} \;:=\; \left(\|g\|^2_{H^{m-1}(0,1)} \;+\; |D^{m-1}g|^2_{H^{\sigma-m+1}(0,1)}\right)^{1/2},$$

and let the fractional Sobolev spaces $H_0^\sigma(0,1)$ denote the closure of $C_0^\infty(0,1)$ with respect to $\|\cdot\|_{H^\sigma(0,1)}$. We also define $\tilde{H}^\sigma(0,1)$ the set of functions in $H^\sigma(0,1)$ whose extension by 0 are in $H^\sigma(\mathbb{R})$.

Based on these definitions, we introduce the following lemmas.

Lemma 1 *[5] For $1/2 < \mu < 1$, the spaces $J^\mu_{L,0}(0,1)$ and $H^\mu_0(0,1)$ are equal with equivalent norms and semi-norms.*

Lemma 2 *[5] For $\mu \geq 0$ and $g \in J^\mu_{L,0}(0,1)$, there exists a constant $C > 0$ such that*

$$\|g\| \leq C\, |g|_{J^\mu_L(0,1)}.$$

Lemma 3 *[17] Suppose $0 < \mu < 1$ and $\phi(x)$ is Lipschitz continuous on $x \in [0,1]$. Then there exists a constant $C > 0$ such that*

$$\|\phi g\|_{H^\mu(0,1)} \leq C \|\phi\|_{L^\infty(0,1)} \|g\|_{H^\mu(0,1)}.$$

We finally introduce the Gauss three-parameter hypergeometric function ${}_2F_1(a,b;c,x)$ defined by a series as follows:

$${}_2F_1(a,b;c,x) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n x^n}{(c)_n n!},$$

where $(s)_n$ is the rising Pochhammer symbol defined by $(s)_n = \Gamma(s+n)/\Gamma(s)$. Some properties of ${}_2F_1(a,b;c,x)$ are presented in the following lemma [19].

Lemma 4 *The Gauss three-parameter hypergeometric function ${}_2F_1(a,b;c,x)$ is absolutely convergent for $|x| < 1$ and for $|x| = 1$ if $Re(c-a-b) > 0$. When $Re(c) > Re(b) > 0$, it has an integral representation*

$${}_2F_1(a,b;c,x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 z^{b-1}(1-z)^{c-b-1}(1-zx)^{-a} dz.$$

Furthermore, it satisfies the Euler transformation formula

$${}_2F_1(a,b;c,x) = (1-x)^{c-a-b}\, {}_2F_1(c-a, c-b; c, x).$$

Remark: Throughout, we use C to denote a generic positive constant whose actual value may change from line to line in the analysis.

3 Representation of the solution $u(x)$ and its finite element approximations

Let $v(x)$ solve (5)–(6) with $K(x) \equiv 1$, i.e., $v(x)$ satisfies

$$-D\left(\left(r\ {}_0I_x^{2-\alpha} \ +\ (1-r)\ {}_xI_1^{2-\alpha}\right) Dv(x)\right) =\ f(x), \quad x \in (0,1)\,, \tag{9}$$

$$v(0) \ =\ v(1) =\ 0\,. \tag{10}$$

Next, let $u(x)$ be defined by

$$u(x) := \int_0^x \frac{Dv(s)}{K(s)} ds - \frac{\displaystyle\int_0^1 \frac{Dv(s)}{K(s)} ds}{\displaystyle\int_0^1 \frac{w^{(\alpha-\beta-1,\beta-1)}(s)}{K(s)} ds} \int_0^x \frac{w^{(\alpha-\beta-1,\beta-1)}(s)}{K(s)} ds, \tag{11}$$

where $w^{(\alpha-\beta-1,\beta-1)}(x)$ is defined by (8), $\alpha - 1 \le \alpha - \beta$, $\beta \le 1$ and β is determined by

$$r = \frac{\sin(\pi\beta)}{\sin(\pi(\alpha-\beta)) + \sin(\pi\beta)}. \tag{12}$$

By construction, $u(0) = u(1) = 0$. From (11), (9), and the fact that $w^{(\alpha-\beta-1,\beta-1)}(x)$ is a kernel function of the operator $-D\left(r\ {}_0I_x^{2-\alpha} + (1-r)\ {}_xI_1^{2-\alpha}\right)$ [6], we obtain

$$\begin{aligned}
&-D\left(r\ {}_0I_x^{2-\alpha} + (1-r)\ {}_xI_1^{2-\alpha}\right)\left(K(x)Du(x)\right) \\
&= -D\big(r\ {}_0I_x^{2-\alpha} + (1-r)\ {}_xI_1^{2-\alpha}\big) \\
&\quad \times \left(Dv(x) - \frac{\displaystyle\int_0^1 \frac{Dw(s)}{K(s)}ds}{\displaystyle\int_0^1 \frac{w^{(\alpha-\beta-1,\beta-1)}(s)}{K(s)}ds} w^{(\alpha-\beta-1,\beta-1)}(x) \right) \\
&= f(x).
\end{aligned}$$

Therefore we conclude that $u(x)$ defined by (11) is a solution of (5)–(6), which motivates us to investigate the solution $v(x)$ of (9)–(10), and then obtain $u(x)$ by (11). We refer the wellposedness of both the model (5)–(6) and model (9)–(10) in the following theorem.

Theorem 1 *[12] Let $f(x) \in L^2_{\omega^{(\beta,\alpha-\beta)}}(0,1)$. Then model (9)–(10) has a unique solution $v(x) \in L^2_{\omega^{(-(\alpha-\beta),-\beta)}}(0,1)$. In addition, there exists $C > 0$ such that*

$$\|v\|_{\omega^{(-(\alpha-\beta),-\beta)}} + \|Dv\|_{\omega^{(-(\alpha-\beta)+1\,,\,-\beta+1)}} \le C\,\|f\|_{\omega^{(\beta,\alpha-\beta)}}$$

Furthermore, there exists a unique solution $u(x) \in L^\infty(0,1)$ to (5)–(6), given by (11).

Additionally, for $\epsilon_1, \epsilon_2 > 0$ there exists $C > 0$ such that

$$\|u\|_{L^\infty} + \|u\|_{\omega^{(-1+\epsilon_1\,,\,-1+\epsilon_2)}} \le C\,\|f\|_{\omega^{(\beta\,,\,\alpha-\beta)}}\,.$$

3.1 Finite element approximations to $u(x)$

Let $X := \tilde{H}^{\alpha/2}(0,1)$. The weak formulation of (9)–(10) is: *Given $f(x) \in H^{-\alpha/2}(0,1) \cap L^2_{\omega^{(\beta,\alpha-\beta)}}(0,1)$, determine $v(x) \in X$ satisfying*

$$B(v,\hat{v}) = \langle f, \hat{v}\rangle, \quad \forall \hat{v} \in X,$$

where $B(\cdot,\cdot)$ is defined by (2) and $\langle\cdot,\cdot\rangle$ denotes the L^2 duality pairing between $H^{-\alpha/2}(0,1)$ and $\tilde{H}^{\alpha/2}(0,1)$.

For $0 = x_0 < x_1 < \ldots < x_N = 1$ denoting a quasi-uniform partition of $(0,1)$, $X_h \subset X$ denoting the space of continuous, piecewise polynomials of degree ≤ 1 on the partition, the finite element approximation $v_h \in X_h$ to v is given by

$$B(v_h,\hat{v}_h) = \langle f, \hat{v}_h\rangle, \quad \forall \hat{v}_h \in X_h.$$

From [6], the following error estimates of $v - v_h$ hold for any $0 < \varepsilon \ll 1$:

$$\|v - v_h\|_{\tilde{H}^{\alpha/2}} \le \begin{cases} C\,h^{1/2-\varepsilon}\|v\|_{H^{\alpha/2+1/2-\varepsilon}}, & r = 1/2, \\ C\,h^{\min\{\alpha-\beta\,,\,\beta\}+1/2-\alpha/2-\varepsilon}\|v\|_{H^{\min\{\alpha-\beta\,,\,\beta\}+1/2-\alpha/2-\varepsilon}}, & r \ne 1/2, \end{cases} \tag{13}$$

$$\|v - v_h\| \le \begin{cases} C\,h^{1-2\varepsilon}\|v\|_{H^{\alpha/2+1/2-\varepsilon}}, & r = 1/2, \\ C\,h^{2\min\{\alpha-\beta\,,\,\beta\}+1-\alpha-2\varepsilon}\|v\|_{H^{\min\{\alpha-\beta\,,\,\beta\}+1/2-\alpha/2-\varepsilon}}, & r \ne 1/2. \end{cases} \tag{14}$$

Using (11), the approximation $u_h(x)$ of $u(x)$ is

$$u_h(x) := \int_0^x \frac{Dv_h(s)}{K(s)} ds - \frac{\displaystyle\int_0^1 \frac{Dv_h(s)}{K(s)} ds}{\displaystyle\int_0^1 \frac{w^{(\alpha-\beta-1,\beta-1)}(s)}{K(s)} ds} \int_0^x \frac{w^{(\alpha-\beta-1,\beta-1)}(s)}{K(s)} ds. \tag{15}$$

3.2 Convergence estimates

We first prove the following lemma used in the convergence estimates of the finite element approximations.

Lemma 5 *Let $1 < \alpha < 2$, $0 \le r \le 1$ and β be determined by (12). Then*

$$\begin{cases} {}_0I_x^{1-\alpha/2} w^{(\alpha-\beta-1,\beta-1)}(x) \in L^2(0,1), & r \neq 1/2, \\ {}_0I_x^{1-\alpha/2+\sigma} w^{(\alpha-\beta-1,\beta-1)}(x) \in L^2(0,1), & r = 1/2, \quad 0 < \sigma \ll 1. \end{cases} \tag{16}$$

Proof We first consider the case $r \neq 1/2$, which implies $\beta \neq \alpha/2$ by (12). A direct calculation yields

$$\begin{aligned} &\left| {}_0I_x^{1-\alpha/2} w^{(\alpha-\beta-1,\beta-1)}(x) \right| \\ &\quad = \frac{1}{\Gamma(1-\alpha/2)} \int_0^x \frac{(1-s)^{\alpha-\beta-1} s^{\beta-1}}{(x-s)^{\alpha/2}} ds \\ &\quad = \frac{x^{\beta-\alpha/2}}{\Gamma(1-\alpha/2)} \int_0^1 y^{\beta-1} (1-y)^{-\alpha/2} (1-xy)^{\alpha-\beta-1} dy \quad (\text{using } y = s/x) \\ &\quad = \frac{\Gamma(\beta) x^{\beta-\alpha/2}}{\Gamma(\beta+1-\alpha/2)} {}_2F_1(1+\beta-\alpha, \beta; \beta+1-\alpha/2, x) \qquad (17) \\ &\quad = \frac{\Gamma(\beta) x^{\beta-\alpha/2} (1-x)^{\alpha/2-\beta}}{\Gamma(\beta+1-\alpha/2)} {}_2F_1(\alpha/2, 1-\alpha/2; \beta+1-\alpha/2, x), \qquad (18) \end{aligned}$$

where we used the Euler transformation formula (Lemma 4) in the last step. Also, from Lemma 4, we have that the hypergeometric functions in (17) and (18) are absolutely convergent on $x \in [0,1)$.

For $\beta < \alpha/2$, using Lemma 4 and the condition $Re(c-a-b) > 0$, i.e., $\beta + 1 - \alpha/2 - (1 + \beta - \alpha) - \beta = \alpha/2 - \beta > 0$ the hypergeometric function in (17) is absolutely convergent at $x = 1$, and hence on $[0,1]$. Thus it follows that hypergeometric function in (17) is bounded on $[0,1]$. Also, as $(\beta - \alpha/2) \ge (\alpha - 1 - \alpha/2) = \alpha/2 - 1 > -1/2$ then, from (17) it follows that ${}_0I_x^{1-\alpha/2}\big((1-x)^{\alpha-\beta-1} x^{\beta-1}\big) \in L^2(0,1)$ for $\beta < \alpha/2$.

For $\beta > \alpha/2$, using (18) it follows similarly that ${}_0I_x^{1-\alpha/2}\big((1-x)^{\alpha-\beta-1} x^{\beta-1}\big) \in L^2(0,1)$.

Finally, for the case $r = 1/2$, the relation (12) implies $\beta = \alpha/2$. A direct calculation yields

$$\begin{aligned} &\left| {}_0I_x^{1-\alpha/2+\sigma} w^{(\alpha/2-1,\alpha/2-1)}(x) \right| \\ &\quad = \frac{1}{\Gamma(1-\alpha/2+\sigma)} \int_0^x \frac{(1-s)^{\alpha/2-1} s^{\alpha/2-1}}{(x-s)^{\alpha/2-\sigma}} ds \\ &\quad = \frac{x^{\sigma}}{\Gamma(1-\alpha/2+\sigma)} \int_0^1 y^{\alpha/2-1} (1-y)^{-\alpha/2+\sigma} (1-xy)^{\alpha/2-1} dy \quad (\text{using } y = s/x) \\ &\quad = \frac{\Gamma(\alpha/2) x^{\sigma}}{\Gamma(1+\sigma)} {}_2F_1(1-\alpha/2, \alpha/2; 1+\sigma, x) \end{aligned}$$

As $1 + \sigma - (1-\alpha/2) - \alpha/2 = \sigma > 0$, we apply a similar argument as above to finish the proof.

The error estimates of $u - u_h$ is given by the following theorem.

Theorem 2 *Suppose $K(x) \in C^1[0,1]$. Then,*

$$\|u - u_h\|_{H^{\alpha/2-\sigma}} \leq \begin{cases} C\, h^{1/2-\varepsilon} \|v\|_{H^{\alpha/2+1/2-\varepsilon}}, & r = 1/2,\ 0 < \sigma \ll 1 \\ C\, h^{\min\{\alpha-\beta\,,\,\beta\}+1/2-\alpha/2-\varepsilon} \|v\|_{H^{\min\{\alpha-\beta\,,\,\beta\}+1/2-\alpha/2-\varepsilon}}, & r \neq 1/2,\ 0 \leq \sigma \ll 1, \end{cases}$$

$$\|u - u_h\| \leq \begin{cases} C\, h^{1-2\varepsilon} \|v\|_{H^{\alpha/2+1/2-\varepsilon}}, & r = 1/2, \\ C\, h^{2\min\{\alpha-\beta\,,\,\beta\}+1-\alpha-2\varepsilon} \|v\|_{H^{\min\{\alpha-\beta\,,\,\beta\}+1/2-\alpha/2-\varepsilon}}, & r \neq 1/2, \end{cases} \tag{19}$$

for any $0 < \varepsilon \ll 1$.

Proof Let $e_u := u - u_h$ and $e_v := v - v_h$. We subtract (11) from (15) to obtain

$$e_u(x) = \int_0^x \frac{De_v(s)}{K(s)} ds - \frac{\displaystyle\int_0^1 \frac{De_v(s)}{K(s)} ds}{\displaystyle\int_0^1 \frac{w^{(\alpha-\beta-1,\beta-1)}(s)}{K(s)} ds} \int_0^x \frac{w^{(\alpha-\beta-1,\beta-1)}(s)}{K(s)} ds. \tag{20}$$

Using the integration by parts and the homogeneous boundary conditions of $e_v(x)$, (20) can be rewritten as

$$e_u(x) = \frac{e_v(x)}{K(x)} - \int_0^x e_v(s) D\Big(\frac{1}{K(s)}\Big) ds \tag{21}$$

$$+ \frac{\displaystyle\int_0^1 e_v(s) D\Big(\frac{1}{K(s)}\Big) ds}{\displaystyle\int_0^1 \frac{w^{(\alpha-\beta-1,\beta-1)}(s)}{K(s)} ds} \int_0^x \frac{w^{(\alpha-\beta-1,\beta-1)}(s)}{K(s)} ds. \tag{22}$$

Taking the L^2 norm of both sides of (21), we obtain

$$\begin{aligned} \|e_u\| \leq & \left\|\frac{e_v}{K}\right\| + \left\| \int_0^x e_v(s) D\Big(\frac{1}{K(s)}\Big) ds \right\| \\ & + \left\| \frac{\displaystyle\int_0^1 e_v(s) D\Big(\frac{1}{K(s)}\Big) ds}{\displaystyle\int_0^1 \frac{w^{(\alpha-\beta-1,\beta-1)}(s)}{K(s)} ds} \int_0^x \frac{w^{(\alpha-\beta-1,\beta-1)}(s)}{K(s)} ds \right\| \\ \leq & \left\|\frac{e_v}{K_m}\right\| + C \left\| \int_0^1 |e_v(s)| ds \right\| + C \left\| \int_0^1 |e_v(x)| dx \int_0^1 \frac{w^{(\alpha-\beta-1,\beta-1)}(s)}{K_m} ds \right\| \\ \leq & \ C\, \|e_v\|, \end{aligned}$$

in which we have used the boundedness of $D(1/K(x))$ derived based on $K(x) \in C^1[0,1]$ as follows

$$\Big| D\Big(\frac{1}{K(x)}\Big) \Big| = \frac{|K'(x)|}{K^2(x)} \leq \frac{\|K\|_{C^1[0,1]}}{K_m^2}.$$

Then an application of (14) leads to (19).

We turn to prove the estimate of $\|u-u_h\|_{H^{\alpha/2}}$. We firstly consider the case $r \neq 0.5$. Applying the operator ${}_0D_x^{\alpha/2}$ to both sides of (20) we obtain

$$
\begin{aligned}
\left|{}_0D_x^{\alpha/2}e_u\right| &= \left|{}_0D_x^{\alpha/2}\int_0^x \frac{De_v(s)}{K(s)}ds \right.\\
&\qquad \left. - \frac{\displaystyle\int_0^1 e_v(s)D\Big(\frac{1}{K(s)}\Big)ds}{\displaystyle\int_0^1 \frac{w^{(\alpha-\beta-1,\beta-1)}(s)}{K(s)}ds}\,{}_0D_x^{\alpha/2}\int_0^x \frac{w^{(\alpha-\beta-1,\beta-1)}(s)}{K(s)}ds\right| \\
&= \left|{}_0I_x^{1-\alpha/2}\Big(\frac{De_v(x)}{K(x)}\Big)\right.\\
&\qquad \left. - \frac{\displaystyle\int_0^1 e_v(s)D\Big(\frac{1}{K(s)}\Big)ds}{\displaystyle\int_0^1 \frac{w^{(\alpha-\beta-1,\beta-1)}(s)}{K(s)}ds}\,{}_0I_x^{1-\alpha/2}\Big(\frac{w^{(\alpha-\beta-1,\beta-1)}(x)}{K(x)}\Big)\right| \\
&\le \left|{}_0I_x^{1-\alpha/2}\Big(\frac{De_v(x)}{K(x)}\Big)\right| \\
&\qquad + C\left|\int_0^1 e_v(s)D\Big(\frac{1}{K(s)}\Big)ds\right|\left|{}_0I_x^{1-\alpha/2}\Big(\frac{w^{(\alpha-\beta-1,\beta-1)}(x)}{K_m}\Big)\right| \\
&\le \left|{}_0I_x^{1-\alpha/2}\Big[D\Big(\frac{e_v(x)}{K(x)}\Big) - e_v(x)D\Big(\frac{1}{K(x)}\Big)\Big]\right| \\
&\qquad + C\|e_v\|\left|{}_0I_x^{1-\alpha/2}w^{(\alpha-\beta-1,\beta-1)}(x)\right|.
\end{aligned} \tag{23}
$$

Taking the L^2 norm of both sides of (23) and using Lemma 1, Lemma 3, Lemma 5, and that $K \in C^1[0,1]$, we obtain

$$
\begin{aligned}
|e_u|_{H^{\alpha/2}} &\le C\,\|{}_0D_x^{\alpha/2}e_u\| \\
&\le \left\|{}_0I_x^{1-\alpha/2}\Big[D\Big(\frac{e_v(x)}{K(x)}\Big) - e_v(x)D\Big(\frac{1}{K(x)}\Big)\Big]\right\| + C\|e_v\| \\
&\le \left\|{}_0D_x^{\alpha/2}\Big(\frac{e_v}{K}\Big)\right\| + C\left\|\int_0^x \frac{|e_v(s)|}{(x-s)^{\alpha/2}}ds\right\| + C\|e_v\| \\
&\le C\left|\frac{e_v}{K}\right|_{H^{\alpha/2}} + C\,\|e_v\|\cdot\|x^{-\alpha/2}\|_{L^1} + C\,\|e_v\| \quad \text{(using Young's inequality)} \\
&\le C\,|e_v|_{H^{\alpha/2}} + C\,\|e_v\|.
\end{aligned}
$$

Then an application of Lemma 2 and (13) yields the error estimate of $u-u_h$ under the norm $\|\cdot\|_{H^{\alpha/2}}$ for the case $r \neq 0.5$.

For $r = 0.5$, we follow exactly the same procedure as above where we apply ${}_0D_x^{\alpha/2-\sigma}$ on both sides of (20) instead of ${}_0D_x^{\alpha/2}$ for any $0 < \sigma \ll 1$ to derive the error estimate of $u-u_h$ under the norm $\|\cdot\|_{H^{\alpha/2-\sigma}}$.

4 Numerical experiments

In this section we present some numerical examples to support the error analysis. We observe that for $r \neq 1/2$, the numerical results are in strong agreement with the error estimates proved in Theorem 2. For the case $r = 1/2$, the estimate of $\|u-u_h\|_{H^{\alpha/2-\sigma}}$ for any $0 < \sigma \ll 1$ was

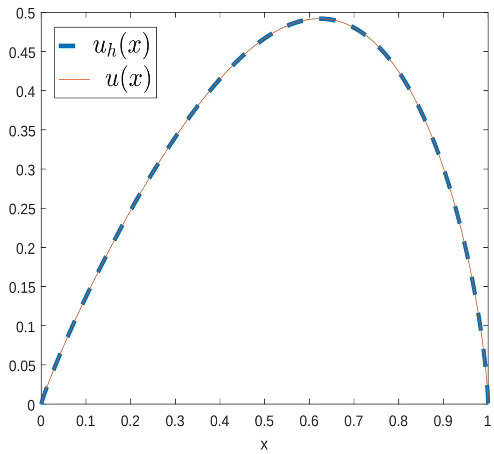

Fig. 1: Plots of true solution $u(x)$ and numerical solution $u_h(x)$ for Experiment 1 with $\alpha = 1.6$, $\beta = 0.9$ and $r = 0.2764$

Table 1: Convergence rates for Experiment 1 when $\alpha = 1.6$, $\beta = 0.8$ and $r = 0.5$.

h	$\|e\|_{H^{\alpha/2}}$	Cvg. Rate	$\|e\|_{L^2}$	Cvg. Rate
1/128	3.36E-03		3.52E-04	
1/256	2.38E-03	0.50	1.85E-04	0.92
1/512	1.68E-03	0.50	9.58E-05	0.95
1/1024	1.19E-03	0.50	4.89E-05	0.97
Pred.		0.50		1.00

proved in Theorem 2 while we measured $\|u - u_h\|_{H^{\alpha/2}}$ in numerical experiments as it seems that the requirement of $\sigma > 0$ is caused by the limitation of analysis techniques and is not consistent with the estimates of $v - v_h$ in (13)–(14). Numerical experiments show that the $H^{\alpha/2}$ norm of $u - u_h$ converges for $r = 1/2$ with the same convergence rate as $\|u - u_h\|_{H^{\alpha/2-\sigma}}$ proved in Theorem 2, which numerically justifies the proceeding discussions. Further study is needed to improve this estimate.

Experiment 1. Let $K(x) = 1/(1 + x)$ and β be determined by (12). For the right-hand side term in (5) given by

$$f(x) = -(1 - r)\Gamma(1 + \alpha)\frac{\sin(\pi\alpha)}{\sin(\pi(\alpha - \beta))},$$

the solution $u(x)$ is given by (11) where $v(x) = (1 - x)^{\alpha-\beta}\, x^{\beta}$.

Experiment 2. Let $K(x) = 1/(1 + x)$ and β be determined by (12). For the right-hand side term in (5) given by

$$f(x) = -r\frac{x^{1-\alpha}}{\Gamma(2 - \alpha)} + (1 - r)\frac{(1 - x)^{1-\alpha}}{\Gamma(2 - \alpha)},$$

the solution $u(x)$ is given by (11) where

$$v(x) = x - Cx^{\beta}\,{}_2F_1(-\alpha + \beta + 1, \beta; \beta + 1, x), \quad C = {}_2F_1(-\alpha + \beta + 1, \beta; \beta + 1, 1)^{-1}.$$

Table 2: Convergence rates for Experiment 1 when $\alpha = 1.6$, $\beta = 0.9$ and $r = 0.2764$.

h	$\|e\|_{H^{\alpha/2}}$	Cvg. Rate	$\|e\|_{L^2}$	Cvg. Rate
1/128	1.04E-02		3.28E-04	
1/256	7.91E-03	0.40	1.65E-04	1.00
1/512	6.00E-03	0.40	8.26E-05	0.99
1/1024	4.55E-03	0.40	4.15E-05	0.99
Pred.		0.40		0.80

Table 3: Convergence rates for Experiment 1 when $\alpha = 1.2$, $\beta = 0.6$ and $r = 0.5$.

h	$\|e\|_{H^{\alpha/2}}$	Cvg. Rate	$\|e\|_{L^2}$	Cvg. Rate
1/128	9.94E-03		1.24E-03	
1/256	7.08E-03	0.49	6.65E-04	0.90
1/512	5.03E-03	0.49	3.51E-04	0.92
1/1024	3.57E-03	0.50	1.83E-04	0.94
Pred.		0.50		1.00

Table 4: Convergence rates for Experiment 1 when $\alpha = 1.8$, $\beta = 0.95$ and $r = 0.2563$.

h	$\|e\|_{H^{\alpha/2}}$	Cvg. Rate	$\|e\|_{L^2}$	Cvg. Rate
1/128	2.08E-03		6.97E-05	
1/256	1.54E-03	0.44	3.74E-05	0.90
1/512	1.13E-03	0.44	1.95E-05	0.94
1/1024	8.28E-04	0.45	1.01E-05	0.96
Pred.		0.45		0.90

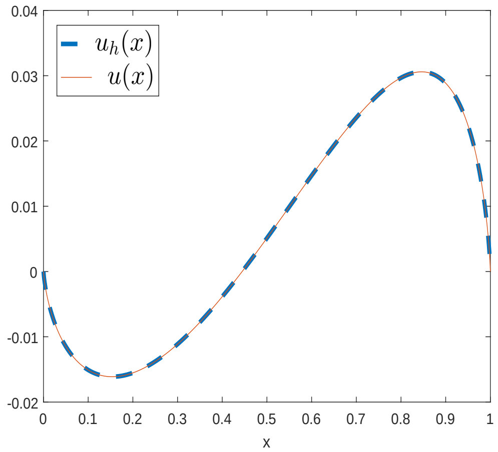

Fig. 2: Plots of true solution $u(x)$ and numerical solution $u_h(x)$ for Experiment 2 with $\alpha = 1.8$, $\beta = 0.9$ and $r = 0.5$

Table 5: Convergence rates for Experiment 2 when $\alpha = 1.4$, $\beta = 0.7$ and $r = 0.5$.

h	$\|e\|_{H^{\alpha/2}}$	Cvg. Rate	$\|e\|_{L^2}$	Cvg. Rate
1/128	5.09E-03		3.98E-04	
1/256	3.54E-03	0.52	2.17E-04	0.87
1/512	2.47E-03	0.52	1.16E-04	0.91
1/1024	1.72E-03	0.52	6.05E-05	0.94
Pred.		0.50		1.00

Table 6: Convergence rates for Experiment 2 when $\alpha = 1.4$, $\beta = 0.85$ and $r = 0.3149$.

h	$\|e\|_{H^{\alpha/2}}$	Cvg. Rate	$\|e\|_{L^2}$	Cvg. Rate
1/128	2.77E-02		1.01E-03	
1/256	2.18E-02	0.34	4.98E-04	1.01
1/512	1.72E-02	0.35	2.47E-04	1.01
1/1024	1.35E-02	0.35	1.23E-04	1.01
Pred.		0.35		0.70

Table 7: Convergence rates for Experiment 2 when $\alpha = 1.8$, $\beta = 0.9$ and $r = 0.5$.

h	$\|e\|_{H^{\alpha/2}}$	Cvg. Rate	$\|e\|_{L^2}$	Cvg. Rate
1/128	1.05E-03		5.66E-05	
1/256	7.36E-04	0.52	3.01E-05	0.91
1/512	5.12E-04	0.52	1.57E-05	0.94
1/1024	3.58E-04	0.52	8.06E-06	0.96
Pred.		0.50		1.00

Table 8: Convergence rates for Experiment 2 when $\alpha = 1.2$, $\beta = 0.5$ and $r = 0.5528$.

h	$\|e\|_{H^{\alpha/2}}$	Cvg. Rate	$\|e\|_{L^2}$	Cvg. Rate
1/128	1.07E-02		6.92E-04	
1/256	8.02E-03	0.41	3.66E-04	0.92
1/512	6.03E-03	0.41	1.92E-04	0.93
1/1024	4.55E-03	0.41	1.00E-04	0.94
Pred.		0.40		0.80

5 Concluding remarks

In this paper we established an indirect finite element approximation for the two-sided space-fractional diffusion equations in one space dimension. By the representation of the solutions $u(x)$ to the proposed variable coefficient model in terms of $v(x)$, the solutions to the constant coefficient analogues, we apply finite element methods for the constant coefficient fractional diffusion equations to solve for the approximations $v_h(x)$ to $v(x)$ and then obtain the approximations $u_h(x)$ of $u(x)$ by plugging $v_h(x)$ into the representation of $u(x)$. We also proved the optimal-order convergence estimates of $u(x) - u_h(x)$ in L^2 and $H^{\alpha/2}$ norms by those of $v(x) - v_h(x)$ proved in [5]. Numerical results are in agreement with the derived theoretical estimates.

If the solutions $u(x)$ to model (5)–(6) have inhomogeneous boundary conditions, i.e., $u(0) = a$ and $u(1) = b$, the representation (11) can be modified to accommodate the impact

$$u(x) := \int_0^x \frac{Dv(s)}{K(s)} ds + \frac{b - a - \int_0^1 \frac{Dv(s)}{K(s)} ds}{\int_0^1 \frac{w^{(\alpha-\beta-1,\beta-1)}(s)}{K(s)} ds} \int_0^x \frac{w^{(\alpha-\beta-1,\beta-1)}(s)}{K(s)} ds + a, \tag{24}$$

where $v(x)$ is still the solutions to (9)–(10). All the derivations in this paper can be extended to this case without any difficulty. Other developed methods for the constant-coefficient fractional diffusion equations, which can recover the convergence rates of the numerical approximations, can also be applied to the variable-coefficient models by the relation (24) and we will carry on these studies in the near future.

6 Statement of conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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