Optimal Petrov-Galerkin spectral approximation method for the fractional diffusion, advection, reaction equation on a bounded interval

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Abstract

In this paper we investigate the numerical approximation of the fractional diffusion, advection, reaction equation on a bounded interval. Recently the explicit form of the solution to this equation was obtained. Using the explicit form of the boundary behavior of the solution and Jacobi polynomials, a Petrov-Galerkin approximation scheme is proposed and analyzed. Numerical experiments are presented which support the theoretical results, and demonstrate the accuracy and optimal convergence of the approximation method.

Key words. Fractional diffusion equation, Petrov-Galerkin, Jacobi polynomials, spectral method, weighted Sobolev spaces

AMS Mathematics subject classifications. 65N30, 35B65, 41A10, 33C45

1 Introduction

Of interest in this paper is the approximation of the solution to the fractional diffusion, advection, reaction equation

$$\mathcal{L}_r^\alpha u(x) \ + \ b(x)Du(x) \ + \ c(x)u(x) = \ f(x)\,, \quad x \in \mathrm{I}\,, \tag{1.1}$$

$$\text{subject to } u(0) \,=\, u(1) = \, 0\,, \tag{1.2}$$

$$\text{where } \ \mathcal{L}_r^\alpha u(x) \ := \ -D\big(rD^{-(2-\alpha)} + \ (1-r)D^{-(2-\alpha)*}\big)Du(x), \tag{1.3}$$

and I $:= (0,1)$, $1 < \alpha < 2$, $0 \le r \le 1$, $c(x) - \frac{1}{2}Db(x) \ge 0$, D denotes the usual derivative operator, D^α the α-order left fractional derivative operator, and $D^{\alpha *}$ the α-order right fractional derivative

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operator, defined by:

$$D^{\alpha}u(x) := D\, {}_0D_x^{-(2-\alpha)}\, Du(x) \ = \ D\, \frac{1}{\Gamma(2-\alpha)} \int_0^x \frac{1}{(x-s)^{\alpha-1}}\, Du(s)\, ds\,, \tag{1.4}$$

$$D^{\alpha *}u(x) := D\, {}_xD_1^{-(2-\alpha)}\, Du(x) \ = \ D\, \frac{1}{\Gamma(2-\alpha)} \int_x^1 \frac{1}{(s-x)^{\alpha-1}}\, Du(s)\, ds\,. \tag{1.5}$$

In recent years fractional differential equations have received increased attention as they have been used in modeling a number of physical phenomena such as contaminant transport in ground water flow [4], viscoelasticity [29], image processing [7, 15], turbulent flow [29, 35], and chaotic dynamics [41].

There are two important properties that distinguish a fractional order differential equation from its integer order counterpart. Firstly, as can be noted from (1.3), fractional differential equations are nonlocal in nature. Secondly, the solution of fractional differential equations (typically) have a lack of regularity at the boundary of the domain. Finite difference methods [10, 27, 34, 37, 38], finite element methods [14, 23, 28, 39], discontinuous Galerkin methods [40], and mixed methods [8, 26], have all been developed for fractional differential equations. These methods typically exhibit slow convergence due to the lack of regularity of the solution at the boundary. In [22, 24] an enriched subspace was given for one sided fractional differential equations, where the boundary behavior of the solution was included in the finite element trial space. Mao and Shen in [33] extended the work of Gui and Babuška in [17] to establish that, for an assumed boundary behavior of the solution, a geometrically spaced mesh with increasing polynomial degree trial function on the subintervals resulted in an exponential rate of converge for the approximation. For a special class of self-adjoint fractional differential equations a spectral approximation scheme was presented in [42] using a special class of functions, polyfractonomials. Spectral methods, exploiting a special property satisfied by fractional diffusion operator applied to Jacobi polynomials (see (2.16)) has been particularly effective for the approximation of the solution to fractional diffusion equations [9, 13, 25, 30, 32, 31, 43, 44].

Three recent papers have established the explicit form of the solution to fractional diffusion, advection, reaction equation on a bounded domain in $\mathbb{R}^1$. In [20], Hao and Zhang studied the case for $r = 1/2$, for which $\mathcal{L}_r^{\alpha}$ is a symmetric operator. The general fractional diffusion reaction, equation was investigated by Hao, Lin and Zhang in [19]. (As commented by the authors in their summary, the regularity results obtained in [19] are not optimal.) The work in these papers was extended in [12] to the general fractional diffusion, advection, reaction equation. The solution was shown to have the form $u(x) \ = \ (1-x)^{\alpha-\beta}x^{\beta}\phi(x)$, where ϕ is contained in the weighted Sobolev space $H^{\alpha+\widetilde{s}}_{(\alpha-\beta\,,\,\beta)}(\mathrm{I})$ (defined in Section 2), where β and $\widetilde{s}$ are explicit functions of α, r, and the regularity of the right hand side function, f (see Theorems 2.2 and 2.3 below). Of particular note is that for the fractional diffusion, reaction problem, and the fractional diffusion, advection, reaction problem, the regularity of the solution u is bounded, regardless of the regularity of f. This boundedness in the regularity of u is not the case for the fractional diffusion, advection, reaction equation on $\mathbb{R}$, as was recently established by Ginting and Li in [16].

The numerical approximation scheme presented below is accurate as, using [12], the precise boundary behavior of the solution is incorporated into the approximate solution. Additionally, using the special property of the fractional diffusion operator applied to Jacobi polynomials (see (2.16))

$$\mathcal{L}_r^{\alpha}\omega(x)\, \widehat{G}_k^{(\alpha-\beta\,,\,\beta)}(x) \ = \ \lambda_k\, \widehat{G}_k^{(\beta\,,\,\alpha-\beta)}(x)\,,$$

and that $\{\widehat{G}_k^{(\alpha-\beta\,,\,\beta)}\}_{k=0}^{\infty}$ is a basis for $H^r_{(\alpha-\beta\,,\,\beta)}(\mathrm{I})$, the approximation scheme using Jacobi polynomial is efficient in that if the solution is $C^{\infty}(\mathrm{I})$ (very rarely the case) the approximation converges exponentially. If the solution has bounded regularity (typically the case) the approximation converges optimally at an algebraic rate of convergence.

This paper is organized as follows. In the following section definitions, notation, and several known results are summarized. Section 3 contains the Petrov-Galerkin weak formulation for (1.1),(1.2), and establishes the existence and uniqueness of its solution. The analysis follows the work of Jin, Lazarov and Zhou in [24], wherein the lower order terms are handled using the Petree-Tartar Lemma. The approximation scheme is given in Section 4, and associated error estimates derived. Numerical experiments are presented in Section 5.

2 Notation and Properties

Jacobi polynomials have an important connection with fractional order diffusion equations [2, 13, 31, 30]. We briefly review their definition and some of their important properties [1, 36].

Usual Jacobi Polynomials, $P_n^{(a,b)}(t)$, on $(-1\,,\,1)$.
Definition: $P_n^{(a,b)}(t) \ := \ \sum_{m=0}^{n} p_{n,m}\,(t-1)^{(n-m)}(t+1)^m$, where

$$p_{n,m} \ := \ \frac{1}{2^n}\begin{pmatrix} n+a \\ m \end{pmatrix}\begin{pmatrix} n+b \\ n-m \end{pmatrix}. \tag{2.1}$$

Orthogonality:

$$\int_{-1}^{1}(1-t)^a(1+t)^b\,P_j^{(a,b)}(t)\,P_k^{(a,b)}(t)\,dt \ = \ \begin{cases} 0, & k \neq j \\ |||P_j^{(a,b)}|||^2\,, & k=j \end{cases}\ ,$$

$$\text{where}\quad |||P_j^{(a,b)}||| \ = \ \left(\frac{2^{(a+b+1)}}{(2j\,+\,a\,+\,b\,+1)}\,\frac{\Gamma(j+a+1)\,\Gamma(j+b+1)}{\Gamma(j+1)\,\Gamma(j+a+b+1)}\right)^{1/2}. \tag{2.2}$$

In order to transform the domain of the family of Jacobi polynomials to $[0,1]$, let $t \to 2x-1$ and introduce $G_n^{(a,b)}(x) \ = \ P_n^{(a,b)}(t(x))$. From (2.2),

$$\begin{aligned}
\int_{-1}^{1}(1-t)^a(1+t)^b\,P_j^{(a,b)}(t)\,P_k^{(a,b)}(t)\,dt &= \int_0^1 2^a\,(1-x)^a\,2^b\,x^b\,P_j^{(a,b)}(2x-1)\,P_k^{(a,b)}(2x-1)\,2\,dx \\
&= 2^{a+b+1}\int_0^1 (1-x)^a\,x^b\,G_j^{(a,b)}(x)\,G_k^{(a,b)}(x)\,dx \\
&= \begin{cases} 0, & k \neq j\,, \\ 2^{a+b+1}\,|||G_j^{(a,b)}|||^2\,, & k=j\,. \end{cases} \\
\text{where}\quad |||G_j^{(a,b)}||| &= \left(\frac{1}{(2j\,+\,a\,+\,b\,+1)}\,\frac{\Gamma(j+a+1)\,\Gamma(j+b+1)}{\Gamma(j+1)\,\Gamma(j+a+b+1)}\right)^{1/2}.
\end{aligned} \tag{2.3}$$

From [30, equation (2.19)] we have that

$$\frac{d^k}{dt^k} P_n^{(a,b)}(t) \;=\; \frac{\Gamma(n+k+a+b+1)}{2^k\,\Gamma(n+a+b+1)} P_{n-k}^{(a+k\,,\,b+k)}(t)\,. \tag{2.4}$$

Hence,

$$\frac{d^k}{dx^k} G_n^{(a,b)}(x) = \frac{\Gamma(n+k+a+b+1)}{\Gamma(n+a+b+1)} G_{n-k}^{(a+k\,,\,b+k)}(x)\,. \tag{2.5}$$

Note that, from Stirling's formula, we have that

$$\lim_{n\to\infty} \frac{\Gamma(n+\sigma)}{\Gamma(n)\,n^{\sigma}} \;=\; 1\,, \text{ for } \sigma\in\mathbb{R}. \tag{2.6}$$

For compactness of notation, let

$$\omega^{(a,b)} \;=\; \omega^{(a,b)}(x) \;:=\; (1-x)^a\,x^b\,. \tag{2.7}$$

We let $\mathbb{N}_0 := \mathbb{N}\cup 0$ and use $y_n \sim n^p$ to denote that there exists constants c and $C>0$ such that, as $n\to\infty$, $c\,n^p \le |y_n| \le C\,n^p$. Additionally, we use $a \lesssim b$ to denote that there exists a constant C such that $a \le C\,b$.

For $t\in\mathbb{R}$, $\lfloor t\rfloor$ is used to denote the largest integer that is less than or equal to t, and $\lceil t\rceil$ is used to denote the smallest integer that is greater than or equal to t.

Function space $L^2_\sigma(\mathrm{I})$.
For $\sigma(x)>0,\ x\in(0,1)$, let

$$L^2_\sigma(\mathrm{I}) \;:=\; \{f(x) \,:\, \int_0^1 \sigma(x)\,f(x)^2\,dx \;<\; \infty\}\,. \tag{2.8}$$

Associated with $L^2_\sigma(0,1)$ is the inner product, $(\cdot,\cdot)_\sigma$, and norm, $\|\cdot\|_\sigma$, defined by

$$(f\,,\,g)_\sigma \;:=\; \int_0^1 \sigma(x)\,f(x)\,g(x)\,dx\,, \quad \text{and} \quad \|f\|_\sigma \;:=\; \left(\langle f\,,\,f\rangle_\sigma\right)^{1/2}\,.$$

The set of orthogonal polynomials $\{G_j^{(a,b)}\}_{j=0}^\infty$ form an orthogonal basis for $L^2_{\omega^{(a,b)}}(\mathrm{I})$, and for $\widehat{G}_j^{(a,b)} := G_j^{(a,b)}/\||G_j^{(a,b)}|\|$, $\{\widehat{G}_j^{(a,b)}\}_{j=0}^\infty$ form an orthonormal basis for $L^2_{\omega^{(a,b)}}(\mathrm{I})$.

Without a subscript, $(\cdot,\cdot)$ denotes the usual $L^2(\mathrm{I})$ inner product.

Function space $H^s_{(a,b)}(\mathrm{I})$.
The weighted Sobolev spaces $H^s_{(a,b)}(\mathrm{I})$ differ from the usual $H^s(\mathrm{I})$ spaces in that the associated norms apply a polynomial weight at each endpoint of I, namely, x^b and $(1-x)^a$. These weights increase with the order of the derivative. We give two equivalent definitions for the $H^s_{(a,b)}(\mathrm{I})$ spaces. In the first definition the spaces $H^s_{(a,b)}(\mathrm{I})$, for $0<s\notin\mathbb{N}$, are defined by the K- method of interpolation. The second definition is based on the decay rate of the coefficients of a function expanded in terms

of the Jacobi polynomials $\widehat{G}_j^{(a,b)}(x)$. Both definitions are useful, and used in the analysis below. The equivalence of the spaces is discussed in [12].

Definition: [Based on the K- method of interpolation.]
Following Babuška and Guo [3], and Guo and Wang [18], we introduce the weighted Sobolev spaces $H^s_{\omega^{(a,b)}}(\mathrm{I})$.

Definition 2.1 *Let $s, a, b \in \mathbb{R}$, $s \geq 0$, $a, b > -1$. Then*

$$H^s_{\omega^{(a,b)}}(\mathrm{I}) := \left\{ v \,:\, \|v\|^2_{s,\omega^{(a,b)}} := \sum_{j=0}^{s} \left\| D^j v \right\|^2_{\omega^{(a+j,b+j)}} < \infty \right\}. \tag{2.9}$$

Definition (2.9) is extended to $s \in \mathbb{R}^+$ using the K- method of interpolation. For $s < 0$ the spaces are defined by (weighted) L^2 duality.

Definition: [Based on the decay rate of Jacobi polynomial coefficients.]
Next we define function spaces in terms of the decay rate of the Jacobi coefficients of their member functions.

Given v, let

$$v_j \;=\; \int_0^1 \omega^{(a,b)}(x)\, v(x)\, \widehat{G}_j^{(a,b)}(x)\, dx\,. \tag{2.10}$$

Note that for $v \in L^2_{\omega^{(a,b)}}(\mathrm{I})$,

$$v(x) \;=\; \sum_{j=0}^{\infty} v_j\, \widehat{G}_j^{(a,b)}(x)\,. \tag{2.11}$$

Definition 2.2 *Let $s, a, b \in \mathbb{R}$, $a, b > -1$, $L^2_{(a,b)}(\mathrm{I}) := L^2_{\omega^{(a,b)}}(\mathrm{I})$, and v_j be given by* (2.10). *Then, define*

$$H^s_{(a,b)}(\mathrm{I}) \;:=\; \{ v \,:\, \sum_{j=0}^{\infty} (1+j^2)^s\, v_j^2 \;<\; \infty \} \tag{2.12}$$

as the (a,b)-weighted Sobolev space of order s.

Theorem 2.1 *[12, Theorem 4.1] The spaces $H^s_{(a,b)}(\mathrm{I})$ and $H^s_{\omega^{(a,b)}}(\mathrm{I})$ coincide, and their corresponding norms are equivalent.*

With the structure of the $H^s_{(a,b)}(\mathrm{I})$ spaces, and properties (2.5) and (2.3), it is straight forward to show that D is a bounded mapping from $H^s_{(a,b)}(\mathrm{I})$ onto $H^{s-1}_{(a+1\,,\,b+1)}(\mathrm{I})$.

Lemma 2.1 *[12, Lemma 4.5] For $s, a, b \in \mathbb{R}$, $a, b > -1$, the differential operator D is a bounded mapping from $H^s_{(a,b)}(\mathrm{I})$ onto $H^{s-1}_{(a+1\,,\,b+1)}(\mathrm{I})$.*

For convenience, from hereon we use $H^s_{(a,b)}(\mathrm{I})$ to represent the spaces $H^s_{\omega^{(a,b)}}(\mathrm{I})$ and $H^s_{(a,b)}(\mathrm{I})$.

Definition: **Condition A**
The parameters a, b, and r and constant c_*^* satisfy: $1 < \alpha < 2$, $\alpha - 1 \le \beta$, $\alpha - \beta \le 1$, $0 \le r \le 1$

$$c_*^* = \frac{\sin(\pi\alpha)}{\sin(\pi(\alpha-\beta)) + \sin(\pi\beta)}, \tag{2.13}$$

where β is determined by

$$r = \frac{\sin(\pi\,\beta)}{\sin(\pi(\alpha-\beta)) + \sin(\pi\,\beta)}. \tag{2.14}$$

For compactness of notation, for α and r defined in (1.1) and β defined in (2.14) we introduce

$$\omega(x) := \omega^{(\alpha-\beta,\beta)}(x) = (1-x)^{\alpha-\beta}\,x^{\beta}, \quad \text{and} \quad \omega^*(x) := \omega^{(\beta,\alpha-\beta)}(x) = (1-x)^{\beta}\,x^{\alpha-\beta}. \tag{2.15}$$

Additionally, we use $\langle\cdot,\cdot\rangle_\omega$ to denote the weighted L^2 duality pairing between functions if $H^{-s}_{(\alpha-\beta\,,\,\beta)}(\mathrm{I})$ and $H^{s}_{(\alpha-\beta\,,\,\beta)}(\mathrm{I})$.

From [13, 21],

$$\mathcal{L}_r^\alpha \omega(x)\,\widehat{G}_k^{(\alpha-\beta\,,\,\beta)}(x) = \lambda_k\,\widehat{G}_k^{(\beta\,,\,\alpha-\beta)}(x), \quad \text{where} \quad \lambda_k = -c_*^*\frac{\Gamma(k+1+\alpha)}{\Gamma(k+1)},\ k = 0,1,2,\ldots, \tag{2.16}$$

and c_*^* given by (2.13). Also, using (2.6), $\lambda_k \sim k^\alpha$.

Let $\mathcal{S}_N$ denote the space of polynomials of degree less than or equal to N. We define the weighted L^2 orthogonal projection $P_N : L^2_\omega(\mathrm{I}) \to \mathcal{S}_N$ by the condition

$$\big(v - P_N v\,,\,\phi_N\big)_\omega = 0, \quad \forall \phi_N \in \mathcal{S}_N. \tag{2.17}$$

Note that $P_N v = \sum_{j=0}^{N} v_j\,\widehat{G}_j^{(a,b)}(x)$, where $v_j = \int_0^1 \omega(x)\,v(x)\,\widehat{G}_j^{(a,b)}(x)\,dx$.

Lemma 2.2 *[18, Theorem 2.1] For $\mu \in \mathbb{N}_0$ and $v \in H^t_\omega(\mathrm{I})$, with $0 \le \mu \le t$, there exists a constant C, independent of N, α and β such that*

$$\|v - P_N v\|_{H^\mu_\omega(\mathrm{I})} \le C\,N^{\mu-t}\,\|v\|_{H^t_\omega(\mathrm{I})}. \tag{2.18}$$

Remark: In [18] (2.18) is stated for $t \in \mathbb{N}_0$. The result extends to $t \in \mathbb{R}^+$ using interpolation.

The regularity of the solution to (1.1) can be influenced by the regularity of $b(x)$ and $c(x)$. The following lemma enables us to insulate the influence of these terms.

Introduce the space $W^{k,\infty}_w(\mathrm{I})$ and its associated norm, defined for $k \in \mathbb{N}_0$, as

$$W^{k,\infty}_w(\mathrm{I}) := \Big\{ f\ :\ (1-x)^{j/2}x^{j/2}D^j f(x) \in L^\infty(\mathrm{I}),\ j = 0,1,\ldots,k \Big\}, \tag{2.19}$$

$$\|f\|_{W^{k,\infty}_w} := \max_{0\le j\le k} \|(1-x)^{j/2}x^{j/2}D^j f(x)\|_{L^\infty(\mathrm{I})}. \tag{2.20}$$

The subscript w denotes the fact that $W^{k,\infty}_w(\mathrm{I})$ is a weaker space than $W^{k,\infty}(\mathrm{I})$ in that the derivative of functions in $W^{k,\infty}_w(\mathrm{I})$ may be unbounded at the endpoints of the interval.

Lemma 2.3 *[12, Lemma 7.1] Let $s \geq 0$, α, $\beta > -1$, $k \geq s$, and $f \in W_w^{k,\infty}(\mathcal{I})$. For*

$$(i)\ g \in H^s_{(\alpha,\beta)}(\mathrm{I}) \text{ then } fg \in H^s_{(\alpha,\beta)}(\mathrm{I}), \text{ and for} \tag{2.21}$$

$$(ii)\ g \in H^{-s}_{(\alpha,\beta)}(\mathrm{I}) \text{ then } fg \in H^{-s}_{(\alpha,\beta)}(\mathrm{I}). \tag{2.22}$$

Theorem 2.2 *[12, Theorem 7.1] Let $s \geq -\alpha$, β be determined by* ***Condition A****, $c \in W_w^{\lceil \min\{s\,,\,\alpha+(\alpha-\beta)+1\,,\,\alpha+\beta+1\}\rceil,\infty}(\mathrm{I})$ satisfying $c(x) \geq 0$ and*

$$f \in H^{-\alpha/2}(\mathrm{I}) \cap H^s_{(\beta\,,\,\alpha-\beta)}(\mathrm{I}). \tag{2.23}$$

Then there exists a unique solution $u(x) = (1-x)^{\alpha-\beta}\, x^{\beta}\, \phi(x)$, with $\phi(x) \in H^{\alpha+\min\{s\,,\,\alpha+(\alpha-\beta)+1\,,\,\alpha+\beta+1\}}_{(\alpha-\beta\,,\,\beta)}(\mathrm{I})$, to

$$\mathcal{L}_r^{\alpha} u(x) \ + \ c(x)\, u(x) \ = \ f(x)\,, \ x \in \mathrm{I}, \quad \text{subject to } u(0) = u(1) = 0\,. \tag{2.24}$$

The inclusion of an advection term can significantly reduced the regularity of the solution.

Theorem 2.3 *[12, Theorem 7.2] Let $s \geq -\alpha$, β be determined by* ***Condition A****, b, $c \in W_w^{\lceil \min\{s\,,\,\alpha+(\alpha-\beta)-1\,,\,\alpha+\beta-1\}\rceil,\infty}(\mathrm{I})$ satisfying $c(x) - 1/2Db(x) \geq 0$, and*

$$f \in H^{-\alpha/2}(\mathrm{I}) \cap H^s_{(\beta\,,\,\alpha-\beta)}(\mathrm{I}). \tag{2.25}$$

Then there exists a unique solution $u(x) = (1-x)^{\alpha-\beta}\, x^{\beta}\, \phi(x)$, with $\phi(x) \in H^{\alpha+\min\{s\,,\,\alpha+(\alpha-\beta)-1\,,\,\alpha+\beta-1\}}_{(\alpha-\beta\,,\,\beta)}(\mathrm{I})$, to

$$\mathcal{L}_r^{\alpha} u(x) \ + \ b(x)\, Du(x) \ + \ c(x)\, u(x) \ = \ f(x)\,, \ x \in \mathrm{I}, \quad \text{subject to } u(0) = u(1) = 0\,. \tag{2.26}$$

Introduce $\widetilde{s}$ defined by

$$\widetilde{s} := \begin{cases} \min\{s,\, \alpha+(\alpha-\beta)+1,\, \alpha+\beta+1\}, & \text{if } b = 0 \ \ (\text{see Theorem 2.2}) \\ \min\{s,\, \alpha+(\alpha-\beta)-1,\, \alpha+\beta-1\}, & \text{if } b \neq 0 \ \ (\text{see Theorem 2.3})\,. \end{cases} \tag{2.27}$$

3 Weak Formulation

In place of (1.1), (1.2), we consider the following problem.

Given $H^{-\alpha/2}_{\omega^*}(\mathrm{I})$, and b and c satisfying

$$\left.\begin{array}{c} b \in W_w^{\lceil \min\{s\,,\,\alpha+(\alpha-\beta)-1\,,\,\alpha+\beta-1\}\rceil+1,\infty}(\mathrm{I}), \quad c \in W_w^{\lceil \min\{s\,,\,\alpha+(\alpha-\beta)-1\,,\,\alpha+\beta-1\}\rceil,\infty}(\mathrm{I}), \\ c(x) \ - \ 1/2Db(x) \ \geq 0, \ x \in \mathrm{I}\,, \end{array}\right\} \tag{3.1}$$

determine $\phi \in H^{\alpha/2}_{\omega}(\mathrm{I})$ such that $u(x) \ = \ \omega(x)\,\phi(x)$ satisfies

$$\langle \mathcal{L}_r^{\alpha} u \ + \ b\, Du \ + \ c\, u\,,\, \psi \rangle_{\omega^*} \ = \ \langle f\,,\, \psi \rangle_{\omega^*}\,, \quad \forall\, \psi \in H^{\alpha/2}_{\omega^*}(\mathrm{I})\,. \tag{3.2}$$

Remark: The assumption on $b(\cdot)$ is stronger than that required for Theorem 2.3, and that of $f(\cdot)$ is weaker. This extra regularity for $b(\cdot)$ is needed in the proof of Lemma 3.5, where Theorem 2.3 is applied to the adjoint of equation (2.26) (see (3.22)).

Note that the formulation (3.2) has different test and trial spaces. With this in mind we recall the Banach-Nečas-Babuška theorem.

Theorem 3.1 *[11, Pg. 85, Theorem 2.6] Let H_1 and H_2 denote two real Hilbert spaces, $B(\cdot,\cdot) : H_1 \times H_2 \to \mathbb{R}$ a bilinear form, and $F : H_2 \to \mathbb{R}$ a bounded linear functional on H_2. Suppose there are constants $C_1 < \infty$ and $C_2 > 0$ such that*

$$(i)\ |B(w,v)| \le C_1 \|w\|_{H_1} \|v\|_{H_2}, \quad \text{for all } w \in H_1,\ v \in H_2, \tag{3.3}$$

$$(ii)\ \sup_{0 \ne v \in H_2} \frac{|B(w,v)|}{\|v\|_{H_2}} \ \ge\ C_2 \|w\|_{H_1}, \ \text{for all } w \in H_1, \tag{3.4}$$

$$(iii)\ \sup_{w \in H_1} |B(w,v)| \ >\ 0, \ \text{for all } v \in H_2,\ v \ne 0. \tag{3.5}$$

Then there exists a unique solution $w_0 \in H_1$ satisfying $B(w_0, v) = F(v)$ for all $v \in H_2$. Further, $\|w_0\|_{H_1} \le C_2 \|F\|_{H_2}$.

For $f \in H^{-\alpha/2}_{\omega^*}(\mathrm{I})$, and b and c satisfying (3.1), let $B : H^{\alpha/2}_{\omega} \times H^{\alpha/2}_{\omega^*} \to \mathbb{R}$, and $F : H^{\alpha/2}_{\omega^*} \to \mathbb{R}$ be defined by

$$B(\phi,\psi) := \ \langle \mathcal{L}^{\alpha}_{r} \omega\phi \ + \ b\, D\omega\phi \ + \ c\,\omega\phi \,,\, \psi \rangle_{\omega^*}\,, \tag{3.6}$$

$$F(\psi) := \ \langle f \,,\, \psi \rangle_{\omega^*}\,. \tag{3.7}$$

3.1 Continuity of $B(\cdot,\cdot)$

In order to establish that $B(\cdot,\cdot)$ is well defined and continuous we need to determine which $H^{t}_{(a,b)}(\mathrm{I})$ space $\omega\,\phi$ lies in.

The $H^{s}_{(a,b)}(\mathrm{I})$ space a function f lies in is determined by its behavior at: (i) the left endpoint ($x = 0$), (ii) the right endpoint ($x = 1$), and (iii) away from the endpoints. In order to separate the consideration of the endpoint behaviors, following [6], we introduce the following function space $H^{s}_{(\gamma)}(\mathrm{J})$. Let $\mathrm{J} := (0,\, 3/4)$, and

$$\begin{aligned}\Lambda^* :=& \left\{(x,y) \,:\, \frac{2}{3}x < y < \frac{3}{2}x,\, 0 < x < \frac{1}{2}\right\} \cup \left\{(x,y) \,:\, \frac{3}{2}x - \frac{1}{2} < y < \frac{2}{3}x + \frac{1}{3},\, 1/2 \le x < 3/4\right\} \\ :=& \ \Lambda \cup \Lambda_1 \ \ \text{(see Figure 3.1)}\,.\end{aligned}$$

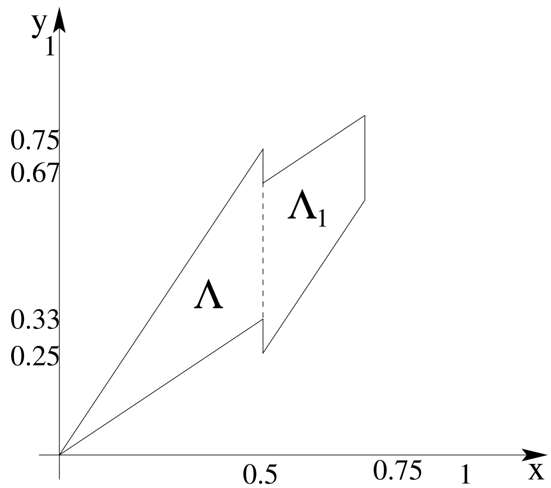

Figure 3.1: Domain $\Lambda^* \, = \, \Lambda \cup \Lambda_1$.

Introduce the semi-norm and norm

$$|f|^2_{H^s_{(\gamma)}(\mathrm{J})} := \iint_\Lambda x^{\gamma+s} \frac{|D^{\lfloor s\rfloor} f(x) \, - \, D^{\lfloor s\rfloor} f(y)|^2}{|x \, - \, y|^{1\,+\,2(s-\lfloor s\rfloor)}} dy\, dx \; + \; \iint_{\Lambda_1} x^{\gamma+s} \frac{|D^{\lfloor s\rfloor} f(x) \, - \, D^{\lfloor s\rfloor} f(y)|^2}{|x \, - \, y|^{1\,+\,2(s-\lfloor s\rfloor)}} dy\, dx$$
$$:= |f|^2_{H^s_{(\gamma)}(\Lambda)} \; + \; |f|^2_{H^s_{(\gamma)}(\Lambda_1)} \, ,$$

$$\text{and} \quad \|f\|^2_{H^s_{(\gamma)}(\mathrm{J})} := \begin{cases} \sum_{j=0}^{s} \|D^j f\|^2_{L^2_{(\gamma+j)}(\mathrm{J})} \, , & \text{for } s \in \mathbb{N}_0 \\ \sum_{j=0}^{\lfloor s\rfloor} \|D^j f\|^2_{L^2_{(\gamma+j)}(\mathrm{J})} \; + \; |f|^2_{H^s_{(\gamma)}(\mathrm{J})} \, , & \text{for } s \in \mathbb{R}^+ \backslash \mathbb{N}_0 \end{cases} ,$$

$$\text{where} \quad \|g\|^2_{L^2_{(\gamma)}(\mathrm{J})} := \int_{\mathrm{J}} x^\gamma \, g^2(x) \, dx \, .$$

Then, $H^s_{(\gamma)}(\mathrm{J}) \, := \, \{f \, : \, f$ is measurable and $\|f\|_{H^s_{(\gamma)}(\mathrm{J})} \, < \, \infty\}$.

Note: A function $f(x)$ is in $H^s_{(a,b)}(\mathrm{I})$ if and only if $f(\frac{3}{4}x) \in H^s_{(b)}(\mathrm{J})$ and $f(\frac{3}{4}(1-x)) \in H^s_{(a)}(\mathrm{J})$.

From [12] we have the following theorem.

Theorem 3.2 *[12, Theorem 6.4] Let* $n \, \leq \, s < n+1$, $n \in \mathbb{N}_0$, $p \geq n$, $\mu > -1$, *and* $\psi \in H^s_{(\mu)}(\mathrm{J})$. *Then* $x^p \, \psi \in H^t_{(\sigma)}(\mathrm{J})$ *provided*

$$0 \leq t \leq s \, , \quad \sigma \, + \, 2p \, \geq \, \mu \, , \quad \sigma \, + \, 2p - t \, > \, -1 \, , \quad \text{and} \quad \sigma \, + \, 2p + t \, \geq \, \mu \, + \, s \, . \tag{3.8}$$

Additionally, when (3.8) *is satisfied, there exists* $C > 0$ *(independent of* ψ*) such that* $\|x^p \, \psi\|_{H^t_{(\sigma)}(\mathrm{J})} \leq C \, \|\psi\|_{H^s_{(\mu)}(\mathrm{J})}$.

Lemma 3.1 *The terms* $\langle \mathcal{L}_r^\alpha \omega\phi,\, \psi\rangle_{\omega^*}$, $\langle b\, D\omega\phi\,,\, \psi\rangle_{\omega^*}$ *and* $\langle c\,\omega\phi\,,\, \psi\rangle_{\omega^*}$ *are well defined. Additionally, there exists* $C > 0$ *such that for* $\phi(x) \in H_\omega^{\alpha/2}(\mathrm{I})$ *and* $\psi(x) \in H_{\omega^*}^{\alpha/2}(\mathrm{I})$

$$|B(\phi,\psi)| \ = \ |\langle \mathcal{L}_r^\alpha \omega\phi \, + \, b\, D\omega\phi \, + \, c\,\omega\phi\,,\, \psi\rangle_{\omega^*}| \ \le \ C\, \|\phi\|_{H_\omega^{\alpha/2}(\mathrm{I})}\, \|\psi\|_{H_{\omega^*}^{\alpha/2}(\mathrm{I})}\,. \tag{3.9}$$

Proof: We begin by considering the $\langle b\, D\omega\phi\,,\, \psi\rangle_{\omega^*}$ term.

From Theorem 3.2, with $s = \alpha/2$, $\mu = \beta$, $p = \beta$, and choosing $\sigma = \alpha-\beta-1$ we have that $t \le \alpha/2$. Hence for $\phi_0 \in H_\beta^{\alpha/2}(\mathrm{J})$, $x^\beta \phi_0(x) \in H_{(\alpha-\beta-1)}^{\alpha/2}(\mathrm{I})$, with $\|x^\beta \phi_0(x)\|_{H_{(\alpha-\beta-1)}^{\alpha/2}} \lesssim \|\phi_0(x)\|_{H_{(\beta)}^{\alpha/2}}$.

Again, using Theorem 3.2, with $s = \alpha/2$, $\mu = \alpha-\beta$, $p = \alpha-\beta$, and choosing $\sigma = \beta$ we have that $t \le \alpha/2$. Hence for $\phi_1 \in H_{\alpha-\beta}^{\alpha/2}(\mathrm{J})$, $x^{\alpha-\beta}\phi_1(x) \in H_{(\beta-1)}^{\alpha/2}(\mathrm{I})$ with $\|x^{\alpha-\beta}\phi_1(x)\|_{H_{(\beta-1)}^{\alpha/2}} \lesssim \|\phi_1(x)\|_{H_{(\alpha-\beta)}^{\alpha/2}}$.

Combining the above two applications of Theorem 3.2 we have that for $\phi \in H_\omega^{\alpha/2}(\mathrm{I})$, $\omega\phi \in H_{(\beta-1\,,\,\alpha-\beta-1)}^{\alpha/2}(\mathrm{I})$ with

$$\|\omega\phi\|_{H_{(\beta-1\,,\,\alpha-\beta-1)}^{\alpha/2}(\mathrm{I})} \ \lesssim \ \|\phi\|_{H_\omega^{\alpha/2}(\mathrm{I})}\,. \tag{3.10}$$

A similar application of Theorem 3.2 establishes that for $\phi \in H_\omega^{\alpha/2}(\mathrm{I})$, $\omega\phi \in H_{\omega^*}^{\alpha/2}(\mathrm{I})$ with

$$\|\omega\phi\|_{H_{\omega^*}^{\alpha/2}(\mathrm{I})} \ \lesssim \ \|\phi\|_{H_\omega^{\alpha/2}(\mathrm{I})}\,. \tag{3.11}$$

From (3.10) and Lemma 2.1 we have that $D\omega\phi \in H_{(\beta\,,\,\alpha-\beta)}^{\alpha/2-1}(\mathrm{I})$ with $\|D\omega\phi\|_{H_{(\beta\,,\,\alpha-\beta)}^{\alpha/2-1}} \lesssim \|\phi\|_{H_\omega^{\alpha/2}}$. Thus, with the assumption on b and using Lemma 2.3,

$$\begin{aligned}
\langle b\, D\omega\phi\,,\, \psi\rangle_{\omega^*} &\le \ \|D\omega\phi\|_{H_{(\beta\,,\,\alpha-\beta)}^{\alpha/2-1}}\, \|b\,\psi\|_{H_{(\beta\,,\,\alpha-\beta)}^{1-\alpha/2}} \\
&\lesssim \ \|\phi\|_{H_\omega^{\alpha/2}}\, \|\psi\|_{H_{\omega^*}^{1-\alpha/2}}\,, \qquad (3.12)\\
&\lesssim \ \|\phi\|_{H_\omega^{\alpha/2}}\, \|\psi\|_{H_{\omega^*}^{\alpha/2}}\,, \qquad (3.13)
\end{aligned}$$

where in the last step we have used that $1-\alpha/2 \le \alpha/2$.

For $\|\phi\| \in H_\omega^{\alpha/2}(\mathrm{I})$ and $\|\psi\| \in H_{\omega^*}^{\alpha/2}(\mathrm{I})$, using (3.11) and the assumption on c,

$$\begin{aligned}
\langle c\,\omega\phi\,,\, \psi\rangle_{\omega^*} &= \ \int_I \omega^*(x)\, c(x)\, \omega(x)\phi(x)\, \psi(x)\, dx \\
&\le \ \|\omega^{1/2}\, \omega^{*1/2}\, \|_{L^\infty} \int_I \omega^{1/2}(x)\phi(x)\, \omega^{*1/2}(x)\, c(x)\, \psi(x)\, dx \\
&\le \ \|\phi\|_{L_\omega^2}\, \|c\,\psi\|_{L_{\omega^*}^2} \\
&\lesssim \|\phi\|_{H_\omega^{\alpha/2}}\, \|\psi\|_{H_{\omega^*}^{1-\alpha/2}}. \qquad (3.14)\\
&\lesssim \|\phi\|_{H_\omega^{\alpha/2}}\, \|\psi\|_{H_{\omega^*}^{\alpha/2}}\,. \qquad (3.15)
\end{aligned}$$

For $\phi(x) = \sum_{i=0}^{\infty} \phi_i \, \widehat{G}_i^{(\alpha-\beta\,,\beta)}(x) \in H_{\omega}^{\alpha/2}(\mathrm{I})$ and $\psi(x) = \sum_{j=0}^{\infty} \psi_j \, \widehat{G}_j^{(\beta\,,\alpha-\beta)}(x) \in H_{\omega^*}^{\alpha/2}(\mathrm{I})$, using (2.16)

$$\langle \mathcal{L}_r^{\alpha} \omega \phi, \, \psi \rangle_{\omega^*} = \left(\sum_{i=0}^{\infty} -c_*^* \, \lambda_i \, \phi_i \, \widehat{G}_i^{(\beta\,,\alpha-\beta)}(x) \, , \, \sum_{j=0}^{\infty} \psi_j \, \widehat{G}_j^{(\beta\,,\alpha-\beta)}(x) \right)_{\omega^*} \tag{3.16}$$

$$= -c_*^* \sum_{k=0}^{\infty} \lambda_k \, \phi_k \, \psi_k \; \sim \; \sum_{k=0}^{\infty} k^{\alpha} \, \phi_k \, \psi_k$$

$$\lesssim \left(\sum_{k=0}^{\infty} k^{\alpha} \, \phi_k \right)^{1/2} \left(\sum_{k=0}^{\infty} k^{\alpha} \, \psi_k \right)^{1/2} \lesssim \left(\sum_{k=0}^{\infty} (1+k^2)^{\alpha/2} \, \phi_k \right)^{1/2} \left(\sum_{k=0}^{\infty} (1+k^2)^{\alpha/2} \, \psi_k \right)^{1/2}$$

$$\lesssim \|\phi\|_{H_{\omega}^{\alpha/2}} \, \|\psi\|_{H_{\omega^*}^{\alpha/2}} \, , \quad \text{using (2.12)} \, . \tag{3.17}$$

Combining (3.13), (3.15) and (3.17) we obtain (3.9).

■

3.2 Conditions (3.4) and (3.5)

For the case $r = 1/2$ we have $\alpha - \beta = \beta = \alpha/2$ and, consequently, $\omega = \omega^*$. In this case for $\psi = \phi$

$$\begin{aligned} \langle b \, D(\omega \, \phi) \; + \; c \, \omega \, \phi \; , \; \psi \rangle_{\omega} &= \int_0^1 \omega \, (b \, D(\omega \, \phi) \; + \; c \, \omega \, \phi) \, \phi \; dx \\ &= \int_0^1 b \, \frac{1}{2} D(\omega \, \phi)^2 \; + \; c \, (\omega \, \phi)^2 \; dx \\ &= \int_0^1 \left(c \, - \, \frac{1}{2} Db \right) \, (\omega \, \phi)^2 \; dx \, . \end{aligned} \tag{3.18}$$

Proceeding as in (3.16), for $\psi = \phi$ and $\omega^* = \omega$,

$$\begin{aligned} \langle \mathcal{L}_{1/2}^{\alpha} (\omega \, \phi), \, \phi \rangle_{\omega} &\sim \sum_{k=0}^{\infty} k^{\alpha} \, \phi_k^2 \; \sim \; \sum_{k=0}^{\infty} (1+k^2)^{\alpha/2} \, \phi_k^2 \\ &\sim \; \|\phi\|_{H_{(\alpha/2,\alpha/2)}^{\alpha/2}}^2 \, . \end{aligned} \tag{3.19}$$

Hence for $(c - \frac{1}{2}Db) \geq 0$, combining (3.18) and (3.19) we have that $B(\cdot, \cdot)$ is coercive on $H_{(\alpha/2,\alpha/2)}^{\alpha/2} \times H_{(\alpha/2,\alpha/2)}^{\alpha/2}$. Then, from the Lax-Milgram, we have the following lemma.

Lemma 3.2 *For $1 < \alpha < 2$ and $r = 1/2$, given $f \in H_{(\alpha/2,\alpha/2)}^{-\alpha/2}(\mathrm{I})$ and $b(x)$ and $c(x)$ satisfying (3.1), there exists a unique solution $u(x) = (1-x)^{\alpha/2} x^{\alpha/2} \, \phi(x)$ to (3.2), with $\phi \in H_{(\alpha/2,\alpha/2)}^{\alpha/2}(\mathrm{I})$ satisfying $\|\phi\|_{H_{(\alpha/2,\alpha/2)}^{\alpha/2}(\mathrm{I})} \lesssim \|f\|_{H_{(\alpha/2,\alpha/2)}^{-\alpha/2}(\mathrm{I})}$.*

This special case of (3.2) corresponding to $r = 1/2$ has been thoroughly investigated by Hao and Zhang in [20].

For the general case, ($r \neq \frac{1}{2}$), to show (3.4) and (3.5), and hence establish the well posedness of the formulation, following an approach by Jin, Lazarov and Zhou in [24], we use the Petree-Tartar Lemma.

Lemma 3.3 *[11, Pg. 469] (Petree-Tartar). Let X, Y, Z be three Banach spaces. Let $A \in \mathcal{L}(X;Y)$ be an injective operator and let $T \in \mathcal{L}(X;Z)$ be a compact operator. If there exists $c_1 > 0$ such that $c_1 \|x\|_X \leq \|Ax\|_Y + \|Tx\|_Z$, then $\mathrm{Im}(A)$ is closed; equivalently, there is $c_2 > 0$ such that*

$$\forall x \in X, \quad c_2\|x\|_X \leq \|Ax\|_Y . \tag{3.20}$$

To relate the Petree-Tartar Lemma to the formulation (3.2), with b and c satisfying (3.1), let $X = H^{\alpha/2}_{\omega}(\mathrm{I})$, $Y = Z = H^{-\alpha/2}_{\omega^*}(\mathrm{I})$,

$$\begin{aligned} &A : X \to Y \quad \text{be defined by} \quad A\phi := \mathcal{L}^{\alpha}_r \omega\phi + b\,D\omega\phi + c\,\omega\phi\,, \quad \text{and} \\ &T : X \to Z \quad \text{be defined by} \quad T\phi := -(\,b\,D\omega\phi + c\,\omega\phi)\,. \end{aligned}$$

That $A \in \mathcal{L}(X;Y)$ follows from its definition and the continuity of $B(\cdot,\cdot)$. To establish the injectivity of A, consider $k \in Y$ and assume ϕ_1 and ϕ_2 satisfy $A\phi_1 = k$ and $A\phi_2 = k$. Then, correspondingly, $u_1 = \omega\phi_1$ and $u_2 = \omega\phi_2$ would satisfy

$$\mathcal{L}^{\alpha}_r(u_1 - u_2)(x) + b(x)\,D(u_1 - u_2)(x) + c(x)\,(u_1 - u_2)(x) = 0 \in H^{-\alpha/2}(\mathrm{I}) \cap H^{-\alpha/2}_{(\beta,\alpha-\beta)}(\mathrm{I})\,,$$

with $(u_1 - u_2)(0) = (u_1 - u_2)(1) = 0$. Theorem 2.3 would then implies $(u_1 - u_2)(x) = 0$, i.e., $u_1 = u_2 \iff \phi_1 = \phi_2$. Hence A is injective on Y.

The fact that $T \in \mathcal{L}(X;Z)$ follows from its definition and (3.13) and (3.15). Also, from (3.12) and (3.14) we have that $T : H^{\alpha/2}_{\omega}(\mathrm{I}) \to H^{1-\alpha/2}_{\omega^*}(\mathrm{I})$ is bounded. As $H^{s}_{\omega^*}(\mathrm{I})$ is compactly embedded in $H^{t}_{\omega^*}(\mathrm{I})$ for $s > t$, [12, pg. 10, Remark 2], since $1 - \alpha/2 > -\alpha/2$, it follows that $T \in \mathcal{L}(X;Z)$ is a compact operator.

Let $\phi(x) = \sum_{i=1}^{\infty} \phi_i \widehat{G}^{(\alpha-\beta,\beta)}_i(x) \in H^{\alpha/2}_{\omega}(\mathrm{I})$ and $\psi(x) = \sum_{i=1}^{\infty} \phi_i \widehat{G}^{(\beta,\alpha-\beta)}_i(x) \in H^{\alpha/2}_{\omega^*}(\mathrm{I})$. Note that $\|\phi\|_{H^{\alpha/2}_{\omega}(\mathrm{I})} = \|\psi\|_{H^{\alpha/2}_{\omega^*}(\mathrm{I})}$. Then,

$$\begin{aligned} \|\phi\|^2_{H^{\alpha/2}_{\omega}(\mathrm{I})} &= \sum_{i=0}^{\infty} \left(1 + i^2\right)^{\alpha/2} \phi_i^2 \\ &\lesssim \sum_{i=0}^{\infty} \lambda_i\,\phi_i^2 = \langle \mathcal{L}^{\alpha}_r \omega\phi\,,\,\psi\rangle_{\omega^*} \\ &= \langle \mathcal{L}^{\alpha}_r \omega\phi + b\,D\omega\phi + c\,\omega\phi\,,\,\psi\rangle_{\omega^*} + \langle -(\,b\,D\omega\phi + c\,\omega\phi)\,,\,\psi\rangle_{\omega^*}. \\ &= \langle A\phi\,,\,\psi\rangle_{\omega^*} + \langle T\phi\,,\,\psi\rangle_{\omega^*} \\ &\leq \|A\phi\|_{H^{-\alpha/2}_{\omega^*}(\mathrm{I})}\,\|\psi\|_{H^{\alpha/2}_{\omega^*}(\mathrm{I})} + \|T\phi\|_{H^{-\alpha/2}_{\omega^*}(\mathrm{I})}\,\|\psi\|_{H^{\alpha/2}_{\omega^*}(\mathrm{I})}\,. \end{aligned}$$

Using $\|\phi\|_{H^{\alpha/2}_{\omega}(\mathrm{I})} = \|\psi\|_{H^{\alpha/2}_{\omega^*}(\mathrm{I})}$, we obtain that there exists $c_1 > 0$ such that

$$c_1\,\|\phi\|_X \leq \|A\phi\|_Y + \|T\phi\|_Z\,.$$

Then, applying the Petree-Tartar Lemma, it follows that there exists $C_2 > 0$ such that

$$C_2\|\phi\|_X \leq \|A\phi\|_Y\,, \quad \text{i.e.,} \quad C_2\|\phi\|_{H^{\alpha/2}_{\omega}(\mathrm{I})} \leq \|A\phi\|_{H^{-\alpha/2}_{\omega^*}(\mathrm{I})}\,. \tag{3.21}$$

Lemma 3.4 *For $B(\cdot, \cdot)$ defined by (3.6), the condition (ii) given by (3.4) is satisfied.*

Proof: Noting that

$$\sup_{0 \neq v \in H^{\alpha/2}_{\omega^*}(\mathrm{I})} \frac{|B(w,v)|}{\|v\|_{H^{\alpha/2}_{\omega^*}(\mathrm{I})}} \ \geq \ C_2 \, \|w\|_{H^{\alpha/2}_{\omega}(\mathrm{I})} \quad \text{is equivalent to} \quad \|Aw\|_{H^{-\alpha/2}_{\omega^*}(\mathrm{I})} \ \geq \ C_2 \, \|w\|_{H^{\alpha/2}_{\omega}(\mathrm{I})} \, ,$$

the condition (ii) follows from (3.21).

■

Lemma 3.5 *For $B(\cdot, \cdot)$ defined by (3.6), the condition (iii) given by (3.5) is satisfied.*

Proof: The adjoint problem to (3.2) is: *Given $g \in H^{-\alpha/2}_{\omega}(\mathrm{I})$, determine $\psi \in H^{\alpha/2}_{\omega^*}(\mathrm{I})$ such that $v(x) \ = \ \omega^*(x) \, \psi(x)$ satisfies*

$$\langle \mathcal{L}^{\alpha}_{(1-r)} v \ - \ b \, Dv \ + \ (c \ - \ Db) v \, , \, \phi \rangle_{\omega} \ = \ \langle g \, , \, \phi \rangle_{\omega} \, , \quad \forall \, \phi \in H^{\alpha/2}_{\omega}(\mathrm{I}) \, . \tag{3.22}$$

Observe that the advection coefficient $(-b)$, and the reaction coefficient $(c \ - \ Db)$, satisfy the assumptions of (3.2).

In relation to Theorem 2.3, the weak form corresponds to the fractional diffusion, advection, reaction equation: *Given $\tilde{g} \in H^{-\alpha/2}(\mathrm{I}) \cap H^{-\alpha/2}_{\omega}(\mathrm{I})$ determine $v(x)$ satisfying*

$$\mathcal{L}^{\alpha}_{(1-r)} v(x) - b(x) \, Dv(x) + \big(c(x) - Db(x) \big) v(x) \ = \ \tilde{g}(x) \, , \ x \in \mathrm{I}, \ \text{ subject to } v(0) = v(1) = 0 \, . \tag{3.23}$$

Note that for the weak formulation (3.22), g may be chosen in $H^{-\alpha/2}_{\omega}(\mathrm{I})$, whereas Theorem 2.3 requires the RHS, $\tilde{g}$, to be in $H^{-\alpha/2}(\mathrm{I}) \cap H^{-\alpha/2}_{\omega}(\mathrm{I})$. Also, note that properties (ii) and (iii) of Theorem 3.1 are similar (property (ii) a stronger condition), where the supremum is taken over one function space with the element in the other function space fixed.

$$\begin{aligned} \text{For } \ B^*(\psi, \phi) := \ & \langle \mathcal{L}^{\alpha}_{1-r} \omega^* \psi \ + \ b \, D \omega^* \psi \ + \ c \, \omega^* \psi \, , \, \phi \rangle_{\omega} \\ = \ & \langle \mathcal{L}^{\alpha}_{r} \omega \phi \ + \ b \, D \omega \phi \ + \ c \, \omega \phi \, , \, \psi \rangle_{\omega^*} \ = \ B(\phi, \psi) \, . \end{aligned}$$

An analogous argument as used to establish condition (ii) can be applied to $B^*(\cdot, \cdot)$ to obtain

$$\begin{aligned} \sup_{0 \neq w \in H^{\alpha/2}_{\omega}(\mathrm{I})} \frac{|B^*(v,w)|}{\|w\|_{H^{\alpha/2}_{\omega}(\mathrm{I})}} \ \geq \ \tilde{C}_2 \, \|v\|_{H^{\alpha/2}_{\omega^*}(\mathrm{I})} \ \Longleftrightarrow \ & \sup_{0 \neq w \in H^{\alpha/2}_{\omega}(\mathrm{I})} \frac{|B(w,v)|}{\|w\|_{H^{\alpha/2}_{\omega}(\mathrm{I})}} \ \geq \ \tilde{C}_2 \, \|v\|_{H^{\alpha/2}_{\omega^*}(\mathrm{I})} \ \text{ for all } v \in H^{\alpha/2}_{\omega^*}(\mathrm{I}) \, , \\ \Longrightarrow \ & \sup_{w \in H^{\alpha/2}_{\omega}(\mathrm{I})} |B(w,v)| \ > \ 0 \ \text{ for all } v \in H^{\alpha/2}_{\omega^*}(\mathrm{I}) \, , \ v \neq 0 \, . \end{aligned}$$

(Recall that in establishing condition (ii) Theorem 2.3 is only used with RHS function equal to 0 in establishing the injectivity of the operator A.)

■

Combining Lemmas 3.1, 3.4 and 3.5 with Theorem 3.1 we obtain the following.

Theorem 3.3 *There exists a unique solution ϕ to* (3.2)*, satisfying* $\|\phi\|_{H_{\omega}^{\alpha/2}(\mathrm{I})} \leq \frac{1}{C_2}\|f\|_{H_{\omega^*}^{-\alpha/2}(\mathrm{I})}$.

Proof: First, note that F defined by (3.7) satisfies

$$\begin{aligned}\|F\| = \sup_{0\neq\psi\in H_{\omega^*}^{\alpha/2}(\mathrm{I})} \frac{|F(\psi)|}{\|\psi\|_{H_{\omega^*}^{\alpha/2}(\mathrm{I})}} &= \sup_{0\neq\psi\in H_{\omega^*}^{\alpha/2}(\mathrm{I})} \frac{|\langle f,\psi\rangle_{\omega^*}|}{\|\psi\|_{H_{\omega^*}^{\alpha/2}(\mathrm{I})}} \\ \leq \sup_{0\neq\psi\in H_{\omega^*}^{\alpha/2}(\mathrm{I})} \frac{\|f\|_{H_{\omega^*}^{\alpha/2}(\mathrm{I})}\|\psi\|_{H_{\omega^*}^{\alpha/2}(\mathrm{I})}}{\|\psi\|_{H_{\omega^*}^{\alpha/2}(\mathrm{I})}} &= \|f\|_{H_{\omega^*}^{\alpha/2}(\mathrm{I})}\,.\end{aligned}$$

Hence, F defines a bounded linear functional. The existence and uniqueness of ϕ then follows from combining Lemmas 3.1, 3.4 and 3.5 with Theorem 3.1. To obtain the bound for $\|\phi\|_{H_{\omega}^{\alpha/2}(\mathrm{I})}$, from Lemma 3.4

$$\begin{aligned}\|\phi\|_{H_{\omega}^{\alpha/2}(\mathrm{I})} \leq \frac{1}{C_2}\sup_{0\neq\psi\in H_{\omega^*}^{\alpha/2}} \frac{|B(\phi,\psi)|}{\|\psi\|_{H_{\omega^*}^{\alpha/2}(\mathrm{I})}} &= \frac{1}{C_2}\sup_{0\neq\psi\in H_{\omega^*}^{\alpha/2}} \frac{|\langle f\,,\,\psi\rangle_{\omega^*}|}{\|\psi\|_{H_{\omega^*}^{\alpha/2}(\mathrm{I})}} \\ \leq \frac{1}{C_2}\sup_{0\neq\psi\in H_{\omega^*}^{\alpha/2}} \frac{\|f\|_{H_{\omega^*}^{-\alpha/2}(\mathrm{I})}\,\|\psi\|_{H_{\omega^*}^{\alpha/2}(\mathrm{I})}}{\|\psi\|_{H_{\omega^*}^{\alpha/2}(\mathrm{I})}} &= \frac{1}{C_2}\|f\|_{H_{\omega^*}^{-\alpha/2}(\mathrm{I})}\,. \qquad (3.24)\end{aligned}$$

■

Corollary 3.1 *For $f \in H_{\omega^*}^{s}(\mathrm{I})$, $s \geq -\alpha/2$, and b and c satisfying* (3.1)*, there exists $C > 0$ such that with ϕ given by* (3.2) *satisfies*

$$\|\phi\|_{H_{\omega}^{\widetilde{s}+\alpha}(\mathrm{I})} \;\leq\; C\,\|f\|_{H_{\omega^*}^{\widetilde{s}}(\mathrm{I})}\,. \qquad (3.25)$$

Proof: The proofs of Theorems 2.2 and 2.3 use a boot strapping argument. The first part of the proof establishes that, for $f \in H^{-\alpha/2}(\mathrm{I})$, the existence and uniqueness of a solution $u(x) = \omega(x)\,\phi(x)$, where $\phi \in L^2(\mathrm{I})$, which then implies that $u \in H_{\omega^*}^{0}(\mathrm{I})$. The subsequent (finite) steps in the proofs iteratively improve the regularity of ϕ (boot strapping argument), until the optimum regularity of ϕ is obtained.

In view of Theorem 3.3, for $f \in H_{\omega^*}^{s}(\mathrm{I})$, $s \geq -\alpha/2$, there exists $\phi \in H_{\omega}^{\alpha/2}(\mathrm{I})$ satisfying (3.2). Consequently, for $u(x) = \omega(x)\,\phi(x)$, using Theorem 3.2, $u \in H_{\omega^*}^{\alpha/2}(\mathrm{I})$. Repeating the boot strapping argument used in the proofs of Theorems 2.2 and 2.3 results in $u(x) = \omega(x)\,\phi(x)$ satisfying (2.26), with $\phi \in H_{\omega}^{\widetilde{s}+\alpha}(\mathrm{I})$, where $\widetilde{s}$ is defined in (2.27). The norm estimate (3.25) follows from that at each of the (finite number of) steps in the boot strapping argument the terms on the right hand side are bounded by a constant times $\|f\|_{H_{\omega^*}^{\widetilde{s}}(\mathrm{I})}$.

■

Remark: Comparing Corollary 3.1 with Theorems 2.2 and 2.3, for Corollary 3.1: (i) the regularity condition for b is stronger, (ii) the condition on f is weaker, and (iii) a bound for ϕ is not given in Theorems 2.2 and 2.3.

Remark: A corresponding weak formulation to (3.2) can be given for u, and subsequent analysis performed. As the unknown in our computational algorithm is ϕ_N we have chosen to present the analysis in terms of ϕ.

4 Approximation Scheme

As $\{\widehat{G}_j^{(a,b)}\}_{j=0}^{\infty}$ is a basis for $H^{\alpha/2}_{(a,b)}(\mathrm{I})$, let $X_N := \mathrm{span}\{\widehat{G}_j^{(\alpha-\beta\,,\,\beta)}\}_{j=0}^{N} \subset H^{\alpha/2}_{(\alpha-\beta\,,\,\beta)}(\mathrm{I})$, and $Y_N := \mathrm{span}\{\widehat{G}_j^{(\beta\,,\,\alpha-\beta)}\}_{j=0}^{N} \subset H^{\alpha/2}_{(\beta\,,\,\alpha-\beta)}(\mathrm{I})$. Corresponding to (3.2) we have the following approximation scheme.

Given $f \in H^{-\alpha/2}_{\omega^*}(\mathrm{I})$, and b and c satisfying (3.1), determine $\phi_N \in X_N$ such that $u_N(x) = \omega(x)\,\phi_N(x)$ satisfies

$$\langle \mathcal{L}_r^{\alpha} \omega(x)\,\phi_N(x) + \ b(x)\,D\omega(x)\,\phi_N(x) + \ c(x)\,\omega(x)\,\phi_N(x)\,,\,\psi_N\rangle_{\omega^*} \ = \ \langle f\,,\,\psi_N\rangle_{\omega^*}\,, \quad \forall\,\psi_N \in Y_N\,. \tag{4.1}$$

The following lemma is used to establish the well posedness of (4.1).

Lemma 4.1 *There exists $C_3 > 0$, such that for N sufficiently large,*

$$\sup_{0\neq\psi_N\in Y_N} \frac{|B(\phi_N,\psi_N)|}{\|\psi_N\|_{H^{\alpha/2}_{\omega^*}(\mathrm{I})}} \ \ge \ C_3\,\|\phi_N\|_{H^{\alpha/2}_{\omega}(\mathrm{I})}\,, \quad \forall\,\phi_N \in X_N\,. \tag{4.2}$$

Proof: Let $\phi_N \in X_N$. For $\psi \in H^{\alpha/2}_{\omega^*}(\mathrm{I})$, let $\psi_N \ = \ \sum_{i=0}^{N} \psi_i G_i^{(\beta,\alpha-\beta)}(x)$. Using Lemma 3.4,

$$\begin{aligned}
C_2\,\|\phi_N\|_{H^{\alpha/2}_{\omega}(\mathrm{I})} \ &\le \ \sup_{0\neq\psi\in H^{\alpha/2}_{\omega^*}(\mathrm{I})} \frac{B(\phi_N,\psi)}{\|\psi\|_{H^{\alpha/2}_{\omega^*}(\mathrm{I})}} \ \le \ \sup_{0\neq\psi\in H^{\alpha/2}_{\omega^*}(\mathrm{I})} \frac{B(\phi_N,\psi_N) \ + \ B(\phi_N\,,\,\psi-\psi_N)}{\|\psi\|_{H^{\alpha/2}_{\omega^*}(\mathrm{I})}} \\
&\le \ \sup_{0\neq\psi\in H^{\alpha/2}_{\omega^*}(\mathrm{I})} \frac{B(\phi_N,\psi_N)}{\|\psi\|_{H^{\alpha/2}_{\omega^*}(\mathrm{I})}} \ + \ \sup_{0\neq\psi\in H^{\alpha/2}_{\omega^*}(\mathrm{I})} \frac{\langle -\big(b(x)\,D\omega(x)\,\phi_N(x) + \ c(x)\,\omega(x)\,\phi_N(x)\big)\,,\,\psi-\psi_N\rangle_{\omega^*}}{\|\psi\|_{H^{\alpha/2}_{\omega^*}(\mathrm{I})}}\,,
\end{aligned} \tag{4.3}$$

where in the last step we have used $\langle \mathcal{L}_r^{\alpha}\phi_N\,,\,\psi-\psi_N\rangle_{\omega^*} \ = \ 0$.

From (3.12) and (3.14), and using (2.18),

$$\begin{aligned}
\big|\langle\big(b(x)\,D\omega(x)\,\phi_N(x) + \ c(x)\,\omega(x)\,\phi_N(x)\big)\,,\,\psi-\psi_N\rangle_{\omega^*}\big| \ &\le \ C\,\|\phi_N\|_{H^{\alpha/2}_{\omega}}\,\|\psi-\psi_N\|_{H^{1-\alpha/2}_{\omega^*}} \\
&\le \ C\,\|\phi_N\|_{H^{\alpha/2}_{\omega}}\,N^{1-\alpha}\,\|\psi\|_{H^{\alpha/2}_{\omega^*}}\,.
\end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4), for N sufficiently large we obtain (4.2).

■

Theorem 4.1 *There exists a unique $\phi_N \in H^{\alpha/2}_{\omega}(\mathrm{I})$ satisfying (4.1). In addition, for C_3 given in (4.2), $\|\phi_N\|_{H^{\alpha/2}_{\omega}(\mathrm{I})} \ \le \ \frac{1}{C_3}\|f\|_{H^{-\alpha/2}_{\omega^*}(\mathrm{I})}$.*

Proof: For $\phi_N \ = \ \sum j = 0^N c_j\,\widehat{G}_j^{(\alpha-\beta,\beta)}(x)$, from (4.1), the constants c_j are determined from

$$\mathbb{A}\mathbf{c} \ = \ \mathbf{b}\,, \quad \text{where} \quad \mathbb{A}_{i+1\,j+1} \ = \ B(\widehat{G}_j^{(\alpha-\beta,\beta)}\,,\,\widehat{G}_i^{(\beta,\alpha-\beta)})\,, \quad \text{and } \mathbf{b}_i \ = \ \langle f(x), \widehat{G}_i^{(\beta,\alpha-\beta)}(x)\rangle_{\omega^*}\,,$$

for $0 \le i, j \le N$. Condition (4.2) implies the invertible of the square matrix $\mathbb{A}$, and hence the uniqueness of ϕ_N satisfying (4.1). The bound for ϕ_N is obtained in an analogous manner to the bound for ϕ in (3.24).

■

For ϕ_N given by (4.1) we have the following error bound.

Lemma 4.2 *There exists $C > 0$ such that for ϕ satisfying (3.2) and ϕ_N satisfying (4.1)*

$$\|\phi - \phi_N\|_{H_\omega^{\alpha/2}(\mathrm{I})} \le C \inf_{\zeta_N \in X_N} \|\phi - \zeta_N\|_{H_\omega^{\alpha/2}(\mathrm{I})} . \tag{4.5}$$

Proof: Note that for $\zeta_N \in X_N$, using (4.2),

$$\begin{aligned} C_3 \|\phi_N - \zeta_N\|_{H_\omega^{\alpha/2}(\mathrm{I})} &\le \sup_{\substack{\psi_N \in Y_N \\ \psi_N \neq 0}} \frac{|B(\phi_N - \zeta_N , \psi_N)|}{\|\psi_N\|_{H_{\omega^*}^{\alpha/2}(\mathrm{I})}} = \sup_{\substack{\psi_N \in Y_N \\ \psi_N \neq 0}} \frac{|\langle f , \psi_N \rangle_{\omega^*} - B(\zeta_N , \psi_N)|}{\|\psi_N\|_{H_{\omega^*}^{\alpha/2}(\mathrm{I})}} \\ &= \sup_{\substack{\psi_N \in Y_N \\ \psi_N \neq 0}} \frac{|B(\phi - \zeta_N , \psi_N)|}{\|\psi_N\|_{H_{\omega^*}^{\alpha/2}(\mathrm{I})}} \quad (\text{using } (3.2)) \\ &\le \sup_{\substack{\psi_N \in Y_N \\ \psi_N \neq 0}} \frac{C_1 \|\phi - \zeta_N\|_{H_\omega^{\alpha/2}(\mathrm{I})} \|\psi_N\|_{H_{\omega^*}^{\alpha/2}(\mathrm{I})}}{\|\psi_N\|_{H_{\omega^*}^{\alpha/2}(\mathrm{I})}} = C_1 \|\phi - \zeta_N\|_{H_\omega^{\alpha/2}(\mathrm{I})} . \end{aligned} \tag{4.6}$$

With the triangle inequality and (4.6), we obtain

$$\|\phi - \phi_N\|_{H_\omega^{\alpha/2}(\mathrm{I})} \le \|\phi - \zeta_N\|_{H_\omega^{\alpha/2}(\mathrm{I})} + \|\zeta_N - \phi_N\|_{H_\omega^{\alpha/2}(\mathrm{I})} \le (1 + C_1) \|\phi - \zeta_N\|_{H_\omega^{\alpha/2}(\mathrm{I})} .$$

As $\zeta_N \in X_N$ is arbitrary, then (4.5) follows.

■

Combining Lemma 4.2 with Lemma 2.2 and Theorems 2.2 and 2.3 we obtain the following error estimate.

Corollary 4.1 *For $f \in H_{\omega^*}^s(\mathrm{I})$, $s \ge -\alpha/2$, and b and c satisfying (3.1), there exists $C > 0$ such that for ϕ satisfying (3.2) and ϕ_N satisfying (4.1)*

$$\|\phi - \phi_N\|_{H_\omega^{\alpha/2}(\mathrm{I})} \le C N^{-(\widetilde{s} + \alpha/2)} \|\phi\|_{H_\omega^{\widetilde{s} + \alpha}(\mathrm{I})} \le C N^{-(\widetilde{s} + \alpha/2)} \|f\|_{H_{\omega^*}^{\widetilde{s}}(\mathrm{I})} . \tag{4.7}$$

Proof: From Corollary 3.1 we have that ϕ satisfies $\phi \in H_\omega^{\widetilde{s} + \alpha}(\mathrm{I})$. Then, applying Lemma 2.2, with $\mu = \alpha/2$ and $t = \widetilde{s} + \alpha$, and using Corollary 3.1, we obtain (4.7).

■

An estimate for $\|\phi - \phi_N\|_{L_\omega^2(\mathrm{I})}$ can be obtained using a Aubin-Nitsche type argument.

Corollary 4.2 *For $f \in H_{\omega^*}^s(\mathrm{I})$, $s \ge -\alpha/2$, and b and c satisfying (3.1), there exists $C > 0$ such that for ϕ satisfying (3.2) and ϕ_N satisfying (4.1)*

$$\|\phi - \phi_N\|_{L_\omega^2(\mathrm{I})} \le C N^{-(\widetilde{s} + \alpha)} \|\phi\|_{H_\omega^{\widetilde{s} + \alpha}(\mathrm{I})} \le C N^{-(\widetilde{s} + \alpha)} \|f\|_{H_{\omega^*}^{\widetilde{s}}(\mathrm{I})} . \tag{4.8}$$

Proof: Introduce $\psi \in H^{\alpha/2}_{\omega^*}(\mathrm{I})$ satisfying

$$\mathcal{L}^{\alpha}_{(1-r)}\omega^* \psi \; - \; b\, D\omega^* \psi \; + \; \big(c \, - \, Db\big)\, \omega^* \psi \; = \; \phi \, - \, \phi_N \, .$$

As $(\phi \, - \, \phi_N) \in L^2_{\omega}(\mathrm{I})$, analogous to (3.25), we have that

$$\|\psi\|_{H^{\alpha}_{\omega^*}(\mathrm{I})} \; \le \; C \, \|\phi \, - \, \phi_N\|_{L^2_{\omega}(\mathrm{I})} \, . \tag{4.9}$$

Then,

$$\begin{aligned}
\|\phi \, - \, \phi_N\|_{L^2_{\omega}} \; &= \; \big((\phi \, - \, \phi_N)\, , \, (\phi \, - \, \phi_N)\big)_{\omega} \; = \; \big((\phi \, - \, \phi_N)\, , \, \mathcal{L}^{\alpha}_{(1-r)}\omega^* \psi \; - \; b\, D\omega^* \psi \; + \; \big(c \, - \, Db\big)\, \omega^* \psi\big)_{\omega} \\
&= \; \big(\mathcal{L}^{\alpha}_{r}\omega\, (\phi \, - \, \phi_N) \; + \; b\, D\omega\, (\phi \, - \, \phi_N) \; + \; c\, \omega\, (\phi \, - \, \phi_N)\, , \, \psi\big)_{\omega^*} \; = \; B((\phi \, - \, \phi_N)\, , \, \psi) \\
&= \; B((\phi \, - \, \phi_N)\, , \, \psi \, - \, \eta_N)\, , \quad \text{for } \eta_N \in Y_N \, , \text{ (using Galerkin orthogonality)} \\
&\le \; C_1 \, \|\phi \, - \, \phi_N\|_{H^{\alpha/2}_{\omega}} \, \|\psi \, - \, \eta_N\|_{H^{\alpha/2}_{\omega^*}} \, , \quad \text{using (3.9)}\, , \\
&\le \; C \, N^{-(\widetilde{s}+\alpha/2)} \, \|\phi\|_{H^{\widetilde{s}+\alpha}_{\omega}} \, N^{-\alpha/2} \, \|\psi\|_{H^{\alpha}_{\omega^*}} \, , \quad \text{using (2.18)}\, , \\
&\le \; C \, N^{-(\widetilde{s}+\alpha)} \, \|\phi\|_{H^{\widetilde{s}+\alpha}_{\omega}} \, \|\phi \, - \, \phi_N\|_{L^2_{\omega}} \, , \quad \text{using (4.9)}\, .
\end{aligned}$$

Finally, dividing through by $\|\phi \, - \, \phi_N\|_{L^2_{\omega}}$ and using (3.25) we obtain (4.8).

■

Error estimate for $u - u_N$.
The weighted $L^2_{\omega^{-1}}$ error estimate for $u - u_N$, where $u_N \, := \, \omega\, \phi_N$, follows easily from the definitions of u_N and the $L^2_{\omega^{-1}}$ norm, and the estimate (4.8). The proof of the estimate for $u - u_N$ in the $H^{\alpha/2}_{\omega^{-1}}$ norm is not so straight forward. The following lemma is helpful in establishing the $H^{\alpha/2}_{\omega^{-1}}$ error estimate.

Lemma 4.3 *Let $0 \le \mu \le 1$. For $\zeta \in H^{\mu}_{\omega}(\mathrm{I})$, then $z \, := \, \omega\, \zeta \in H^{\mu}_{\omega^{-1}}(\mathrm{I})$, with, for some $C > 0$,*

$$\|z\|_{H^{\mu}_{\omega^{-1}}(\mathrm{I})} \; \le \; C \, \|\zeta\|_{H^{\mu}_{\omega}(\mathrm{I})} \, . \tag{4.10}$$

Proof: For this proof it is convenient to use the definition of the $H^{s}_{(a,b)}(\mathrm{I})$ spaces given by (2.9).
Let $\mu = 0$, and $\zeta \in H^0_{\omega}(\mathrm{I}) \, = \, L^2_{\omega}(\mathrm{I})$. Then, for $z \, = \, \omega\, \zeta$

$$\|z\|^2_{H^0_{\omega^{-1}}(\mathrm{I})} \; = \; \|z\|^2_{L^2_{\omega^{-1}}(\mathrm{I})} \; = \; \int_{\mathrm{I}} (1-x)^{-(\alpha-\beta)} x^{-\beta} \left(\omega\, \zeta\right)^2 dx \; = \; \|\zeta\|_{L^2_{\omega}(\mathrm{I})} \; = \; \|\zeta\|_{H^0_{\omega}(\mathrm{I})} \, . \tag{4.11}$$

Next, for $\mu = 1$, let $\zeta \in C^{\infty}(\mathrm{I}) \subset H^1_{\omega}(\mathrm{I})$, and let $z \, = \, \omega\, \zeta$. Note that $Dz \, \sim \, (1-x)^{\alpha-\beta} x^{\beta-1} \zeta(x) \, + \, (1-x)^{\alpha-\beta-1} x^{\beta} \zeta(x) \, + \, (1-x)^{\alpha-\beta} x^{\beta} D\zeta(x)$, and

$$\begin{aligned}
\int_{\mathrm{I}} (1-x)^{-(\alpha-\beta)+1} x^{-\beta+1} \big(Dz\big)^2 \, dx \sim \; & \int_{\mathrm{I}} (1-x)^{(\alpha-\beta)+1} x^{\beta-1} \zeta(x)^2 \, dx \; + \; \int_{\mathrm{I}} (1-x)^{(\alpha-\beta)-1} x^{\beta+1} \zeta(x)^2 \, dx \\
& + \; \int_{\mathrm{I}} (1-x)^{(\alpha-\beta)+1} x^{\beta+1} \, D\zeta(x)^2 \, dx \\
:= \; & \mathcal{I}_1 \; + \mathcal{I}_2 \; + \; \|D\zeta\|^2_{L^2_{(1-x)^{(\alpha-\beta)+1} x^{\beta+1}}(\mathrm{I})} \, .
\end{aligned} \tag{4.12}$$

To bound $\mathcal{I}_1$ and $\mathcal{I}_2$ in terms of $\|\zeta\|_{L^2_\omega(\mathrm{I})}$ and $\|D\zeta\|_{L^2_{(1-x)^{(\alpha-\beta)+1}x^{\beta+1}}(\mathrm{I})}$ we use Hardy's inequality [5, Lemma 3.2].

$$\begin{aligned}
\mathcal{I}_1 &= \int_0^{1/2}(1-x)^{(\alpha-\beta)+1}x^{\beta-1}\zeta(x)^2\,dx \;+\; \int_{1/2}^{1}(1-x)^{(\alpha-\beta)+1}x^{\beta-1}\zeta(x)^2\,dx \\
&\lesssim \int_0^{1/2} x^{\beta-1}\zeta(x)^2\,dx \;+\; \int_{1/2}^{1}(1-x)^{(\alpha-\beta)+1}x^{\beta+1}\zeta(x)^2\,dx \\
&\lesssim \int_0^{1/2} x^{\beta+1}(D\zeta(x))^2\,dx \;+\; \int_0^{1/2} x^{\beta+1}\zeta(x)^2\,dx \;+\; \int_{1/2}^{1}(1-x)^{(\alpha-\beta)+1}x^{\beta+1}\zeta(x)^2\,dx \qquad (4.13)\\
&\qquad \text{(using Hardy's inequality)} \\
&\lesssim \int_0^{1/2}(1-x)^{(\alpha-\beta)+1}x^{\beta+1}(D\zeta(x))^2\,dx \;+\; \int_0^{1/2}\omega\,\zeta(x)^2\,dx \;+\; \int_{1/2}^{1}\omega\,\zeta(x)^2\,dx \\
&\lesssim \|\zeta\|^2_{L^2_\omega(\mathrm{I})} \;+\; \|D\zeta\|^2_{L^2_{(1-x)^{(\alpha-\beta)+1}x^{\beta+1}}(\mathrm{I})}\,. \qquad (4.14)
\end{aligned}$$

An analogous argument yields

$$\mathcal{I}_2 \;\lesssim\; \|\zeta\|^2_{L^2_\omega(\mathrm{I})} \;+\; \|D\zeta\|^2_{L^2_{(1-x)^{(\alpha-\beta)+1}x^{\beta+1}}(\mathrm{I})}\,. \qquad (4.15)$$

Combining (4.11), (4.12), (4.14), and (4.15), we obtain

$$\|z\|_{H^1_{\omega^{-1}}(\mathrm{I})} \;\le\; C\,\|\zeta\|_{H^1_\omega(\mathrm{I})}\,. \qquad (4.16)$$

Estimate (4.16) extends to $\zeta \in H^1_\omega(\mathrm{I})$, using the density of $C^\infty(\mathrm{I})$ in $H^1_\omega(\mathrm{I})$.

Finally, estimate (4.10) then follows from (4.11) and (4.16) using interpolation.

■

Corollary 4.3 *For $H^s_{\omega^*}(\mathrm{I})$, $s \ge -\alpha/2$, and b and c satisfying* (3.1)*, there exists $C > 0$ such that for u determined from* (3.2) *and u_N determined from* (4.1)

$$\|u \,-\, u_N\|_{L^2_{\omega^{-1}}(\mathrm{I})} \;\le\; C\,N^{-(\widetilde{s}+\alpha)}\,\|f\|_{H^{\widetilde{s}}_{\omega^*}(\mathrm{I})}\,, \qquad (4.17)$$

$$\|u \,-\, u_N\|_{H^{\alpha/2}_{\omega^{-1}}(\mathrm{I})} \;\le\; C\,N^{-(\widetilde{s}+\alpha/2)}\,\|f\|_{H^{\widetilde{s}}_{\omega^*}(\mathrm{I})}\,. \qquad (4.18)$$

Proof: As commented above, (4.17) follows from the definition of u_N and (4.2). The estimate (4.18) follows from (4.10) (with $z = u - u_N$, $\zeta = \phi - \phi_N$) and (4.7).

■

5 Numerical Experiments

In this section we present three numerical experiments to investigate the approximation of (1.1),(1.2) using (4.1). We compare the approximation errors with those predicted by Corollary 4.2.

For the numerical experiments we use $f(x) = 1$ and $f(x) = \begin{cases} 0, & 0 < x \le 1/2\,, \\ 1, & 1/2 < x < 1 \end{cases}$. For these choices of f the true solution is unknown. In order to be able to compute a convergence rate for the approximation a very accurate approximation (using $N = 40$) is used as the reference solution. For the computational experiments the entries of the coefficient matrices, which require the evaluation of integrals of weighted products of Jacobi polynomials on I, are evaluated using the Legendre-Gauss quadrature rule with 200 nodes. This ensures sufficient accuracy in order to accurately measure the error associated with the approximation scheme (4.1). We evaluate the norms of the error using the norms associated with Definition 2.2.

The numerical convergence rate, κ, corresponding to $\|u_{40} - u_N\|_{\text{norm}} \lesssim N^{-\kappa}$, is presented in the tables together with the errors. Also included are plots of the reference solution u_{40}, and the error $u_{40} - u_N$.

In Experiment 1 the data is symmetric about $x = 1/2$. However the operator is not symmetric ($r = 0.2$), corresponding to a preferred diffusion toward $x = 1$ over diffusion toward $x = 0$. This is reflected in the solution being slightly skewed toward $x = 1$ (see Figure 5.1). In Experiment 2 the larger value of r ($r = 0.3$), together with a left-to-right drift (advection) term results in a solution highly skewed to the right (see Figure 5.2). For Experiment 3, with the diffusion and drift parameters as used in Experiment 2, the source term is taken to be zero for $x \in (0, 1/2)$ and one for $x \in (1/2, 1)$. This data results again in a solution highly skewed to the right (see Figure 5.3).

Typically when approximating a function which is itself, or its derivative, singular at a point x_s, the error in the approximation will be significantly larger in a neighborhood of x_s. In the approximation scheme studied herein the correct endpoint behavior of the solution is built into the approximation. Figures 5.1-5.3 contain plots of the error for the approximations. In Experiments 1 and 2 the largest errors occur at the right hand endpoint, $x = 1$. Notable is that the errors in a neighborhood of $x = 1$ are the same order of magnitude as the errors across the interval. For Experiment 3 the largest errors occur in a neighborhood of the discontinuity in the source term, around $x = 1/2$.

Experiment 1. Fractional diffusion, reaction equation with $C^{\infty}(\mathrm{I})$ data.
For this experiment we use $\alpha = 1.60$, $r = 0.20$, $b(x) = 0$, $c(x) = 5$, and $f(x) = 1$. Theorem 2.2 states that even with $C^{\infty}(\mathrm{I})$ data the regularity of the solution is bounded. For this data $\beta = 0.93$, and $\widetilde{s} = \min\{\infty,\, \alpha+(\alpha-\beta)+1,\, \alpha+\beta+1\} = 3.27$. Corollary 4.3 predicts that $\|u - u_N\|_{L^2_{\omega^{-1}}(\mathrm{I})} \sim N^{-4.87}$ and $\|u - u_N\|_{H^{\alpha/2}_{\omega^{-1}}(\mathrm{I})} \sim N^{-4.07}$. The numerical convergence rates for the errors are presented in Table 5.1, and are in good agreement with the predicted rates. A plot of the reference solution and plots of the errors are given in Figure 5.1.

Table 5.1: Experiment 1: $\alpha = 1.60$, $r = 0.20$, $b(x) = 0$, $c(x) = 5$, and $f(x) = 1$.

N	$\|u - u_N\|_{L^2_{\omega^{-1}}}$	κ	$\|u - u_N\|_{H^{\alpha/2}_{\omega^{-1}}}$	
6	1.05E-04		5.36E-04	
8	2.52E-05	4.97	1.56E-04	4.30
10	8.62E-06	4.81	6.22E-05	4.11
12	3.61E-06	4.77	2.97E-05	4.06
14	1.74E-06	4.76	1.59E-05	4.05
Pred.		4.87		4.07

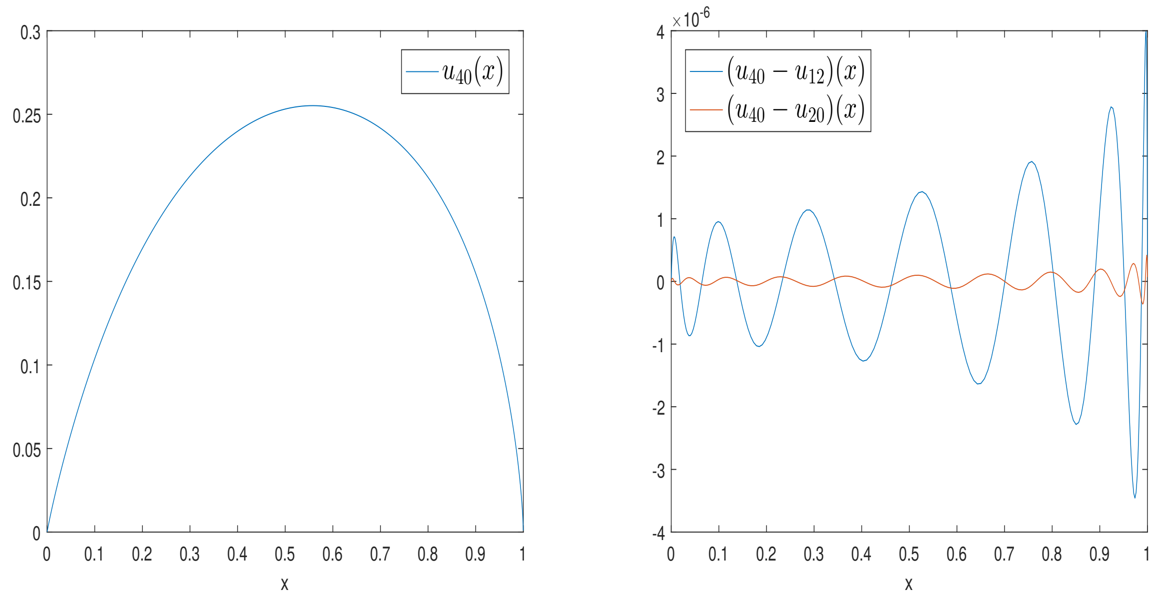

Figure 5.1: The plot of the reference solution $u_{40}(x)$ (left), and the plot of the errors for Experiment 1.

Experiment 2. Fractional diffusion, advection, reaction equation with $C^\infty(\mathrm{I})$ data. For this experiment we use $\alpha = 1.40$, $r = 0.40$, $b(x) = e^x$, $c(x) = 5 + \sin(x)$, and $f(x) = 1$. As previously commented, even with $C^\infty(\mathrm{I})$ data the regularity of the solution is bounded. In addition, comparing Theorems 2.2 and 2.3, the presence of an advection term results in reduced regularity of the solution of the fractional diffusion, advection, reaction equation to that of the fractional diffusion, reaction equation. For this data $\beta = 0.93$, and $\widetilde{s} = \min\{\infty, \alpha + (\alpha - \beta) - 1, \alpha + \beta - 1\} = 1.01$. Corollary 4.3 predicts that $\|u - u_N\|_{L^2_{\omega^{-1}}(\mathrm{I})} \sim N^{-2.41}$ and $\|u - u_N\|_{H^{\alpha/2}_{\omega^{-1}}(\mathrm{I})} \sim N^{-1.71}$. The numerical convergence rates for the errors are presented in Table 5.2, and are in good agreement with the predicted rates. A plot of the reference solution and plots of the errors are given in Figure 5.2

Table 5.2: Experiment 2: $\alpha = 1.40$, $r = 0.40$, $b(x) = e^x$, $c(x) = 5 + \sin(x)$, and $f(x) = 1$

N	$\|u - u_N\|_{L^2_{\omega^{-1}}}$	κ	$\|u - u_N\|_{H^{\alpha/2}_{\omega^{-1}}}$	κ
12	5.67E-03		3.57E-02	
14	4.11E-03	2.08	2.82E-02	1.53
16	3.09E-03	2.15	2.27E-02	1.62
18	2.38E-03	2.21	1.86E-02	1.71
20	1.87E-03	2.26	1.54E-02	1.80
Pred.		2.41		1.71

Experiment 3. Fractional diffusion, advection, reaction equation with $f \in H^{1/2-\epsilon}_{\omega^*}(\mathrm{I})$.
For this experiment we use $\alpha = 1.70$, $r = 0.30$, $b(x) = 2$, $c(x) = 5$, and $f(x) = \begin{cases} 0, & 0 < x \le 1/2\,, \\ 1, & 1/2 < x < 1 \end{cases}$.
In this case the regularity of the solution is limited by the the regularity of f. For this data $\beta = 0.91$, and $\widetilde{s} = \min\{1/2 - \epsilon, \alpha + (\alpha - \beta) - 1, \alpha + \beta - 1\} = 1/2 - \epsilon$. Corollary 4.3 predicts that

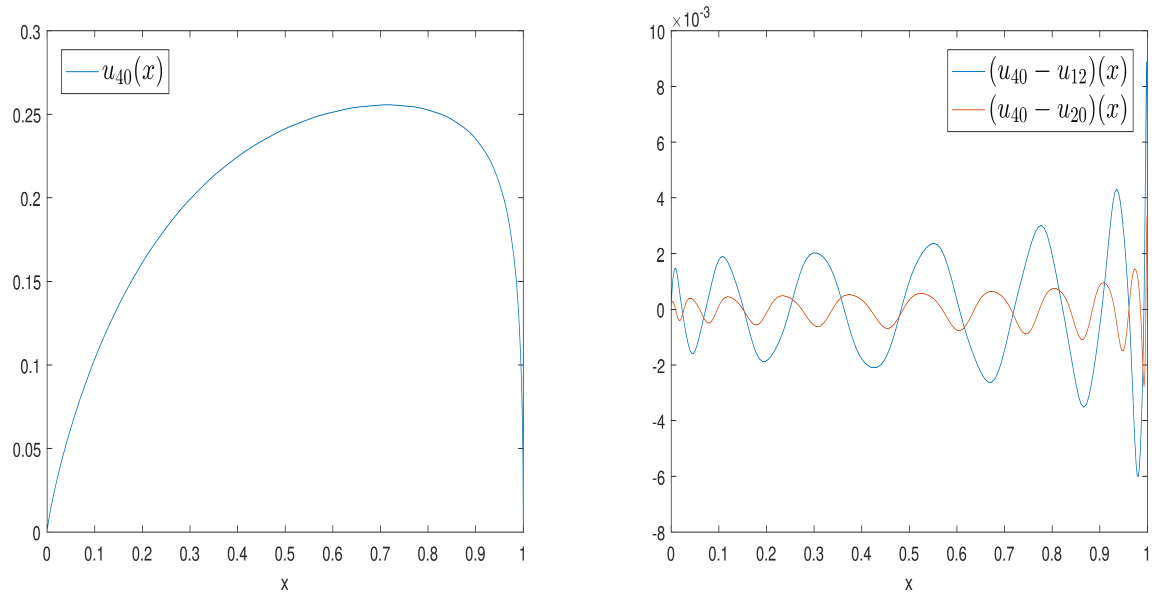

Figure 5.2: The plot of the reference solution $u_{40}(x)$ (left), and the plot of the errors for Experiment 2.

$\|u - u_N\|_{L^2_{\omega^{-1}}(\mathrm{I})} \sim N^{-2.2}$ and $\|u - u_N\|_{H^{\alpha/2}_{\omega^{-1}}(\mathrm{I})} \sim N^{-1.35}$. The numerical convergence rates for the errors are presented in Table 5.3, and are in good agreement with the predicted rates. A plot of the reference solution and plots of the errors are given in Figure 5.3.

Table 5.3: Experiment 3: $\alpha = 1.70$, $r = 0.30$, $b(x) = 2$, $c(x) = 5$

N	$\|u - u_N\|_{L^2_{\omega^{-1}}}$	κ	$\|u - u_N\|_{H^{\alpha/2}_{\omega^{-1}}}$	κ
12	3.71E-04		4.27E-03	
14	2.69E-04	2.10	3.45E-03	1.38
16	2.08E-04	1.91	2.92E-03	1.26
18	1.61E-04	2.18	2.44E-03	1.50
20	1.30E-04	2.02	2.11E-03	1.40
Pred.		2.20		1.35

Declarations

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All data generated or analyzed during this study are included in this published article.

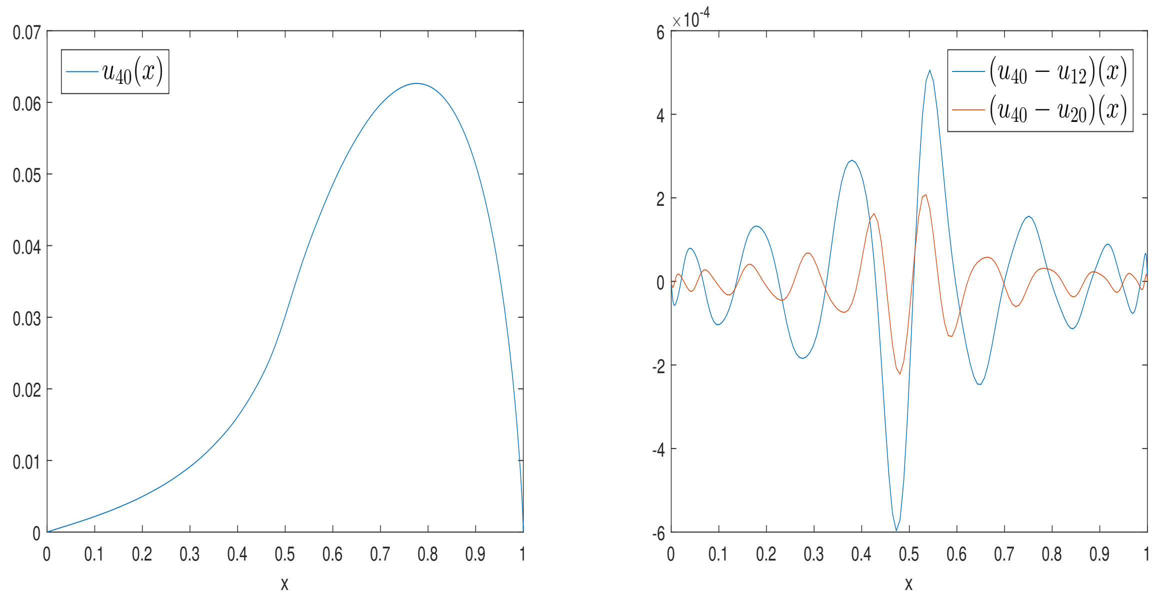

Figure 5.3: The plot of the reference solution $u_{40}(x)$ (left), and the plot of the errors for Experiment 3.

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