Approximation of the Axisymmetric Elasticity Equations

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Abstract

In this article we consider the linear elasticity problem in an axisymmetric three dimensional domain, with data which are axisymmetric and have zero angular component. The weak formulation of the three dimensional problem reduces to a two dimensional problem on the meridian domain, involving weighted integrals. The problem is formulated in a mixed method framework with both the stress and displacement treated as unknowns. The symmetry condition for the stress tensor is weakly imposed. Well posedness of the continuous weak formulation and its discretization are shown. Two approximation spaces are discussed and corresponding numerical computations are presented.

Key words. axisymmetric elasticity problem, well posedness, mixed finite element method

AMS Mathematics subject classifications. 35Q72, 65N30, 65N12

1 Introduction

During the past twenty years, a number of papers have emerged in the numerical analysis literature investigating three-dimensional axisymmetric problems. This class of problem has attracted attention because a three-dimensional axisymmetric problem can be reduced to a two-dimensional problem when cylindrical coordinates are used (see Figure 1.1). It is well recognized that the computational effort required to solve a two-dimensional problem is significantly less that the computational effort needed to solve a three-dimensional problem.

For axisymmetric problems, Mercier and Raugel [24] undertook one of the first finite element analyses of these problems. In [11], Bernardi, Dauge and Maday studied the axisymmetric formulation of a number of standard problems (including Laplace, Stokes and Maxwell equations), and introduced tools for analyzing axisymmetric spectral methods. Assous, Ciarlet, et al. investigated the numerical approximation of the axisymmetric solution of the static and time dependent Maxwell equations in [6, 7]. Following these papers, a number of studies analyzing different axisymmmetric problems appeared. Notably, a computational framework for the axisymmetric Poisson equation

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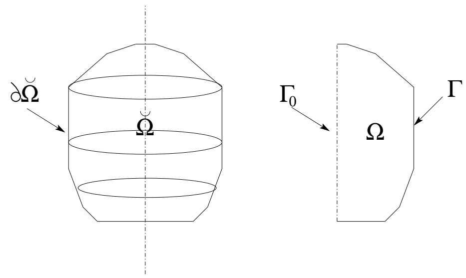

Figure 1.1: Axisymmetric Domain

was developed by Ciarlet, Jung et al. in [14], and a computational framework for div-curl systems was presented by Copeland, Gopalakrishnan, and Pasciak in [15]. More recently, [26] Oh used finite element exterior calculus techniques to study the axisymmetric Hodge Laplacian problem. For axisymmetric fluid dynamics problems, in [10], Bermúdez, Reales, et al. used axisymmetry to reduce the dimension of an eddy current model, and in [1] Anaya, Mora et al. developed a computational framework for axisymmetric Brinkman flows. The axisymmetric Stokes and Darcy problems have been studied in [8, 18, 19, 23, 30]. A coupled axisymmetric Stokes-Darcy problem was investigated by Ervin in [16].

The finite element approximation of the linear elasticity problem has been extensively studied (see [12] for a detailed discussion). For many years, the only known stable finite elements for the mixed method formulation, involving the stress and displacement, used macro-elements in which the stress tensor was approximated on a finer mesh than the displacement vector [3, 22, 29]. In [5] Arnold and Winther developed a stable pair of piecewise polynomials with respect to a single triangulation. These elements, however, carry a significant computational cost since the lowest order representation uses 24 degrees of freedom per triangle.

The major difficulty to creating a stable finite element scheme for the mixed formulation of the linear elasticity problem is in enforcing the symmetry of the stress tensor, which represents the law of conservation of angular momentum. To avoid enforcing symmetry in the stress tensor strongly, a Lagrangian multiplier can be used to weakly enforces symmetry in the stress tensor [2, 4, 20, 25, 27, 28].

The form of differential operators expressed in cylindrical coordinates (e.g. the addition of a $\frac{1}{r}$ term) is an important reason why the numerical analysis for the finite element approximation to the axisymmetric linear elasticity problem is challenging. A consequence of this radial scaling is that the gradient and divergence operators do not map polynomial spaces to polynomial spaces. This feature makes the construction of suitable inf-sup stable finite element approximation spaces more difficult than in the Cartesian setting.

Following, in Sections 2-4 notation and needed preliminary results are introduced. A continuous weak formulation for the axisymmetric linear elasticity problem is presented in Section 5 and shown to be well posed. Then, in Section 6, the corresponding discrete weak formulation is analyzed, and sufficient conditions for its well posedness established in terms of the existence of a suitable bounded projection operator. Shown in Section 7 is the existence of projection operators for two well known approximation spaces which, together with the assumption of boundedness of the projection, establishes the approximation spaces are inf-sup stable. An error analysis is given in Section 8.

The numerical computations presented in Section 9 support the derived theoretical results. Some concluding remarks are given in Section 10

2 Notation

In this section we introduce the notation used below. Bold Greek letters (e.g. $\boldsymbol{\sigma}$) represent vectors, while bold Greek letters with an underline (e.g. $\underline{\boldsymbol{\sigma}}$) denote tensors. For English letters, bold lowercase letters (e.g. $\mathbf{p}$) denote vectors, while bold uppercase letters (e.g. $\mathbf{P}$) denote tensors. Matrices are represented with capital, non-bold letters (e.g. A). Additionally, $\mathbb{M}^n$ denote the space of $n \times n$ dimensional real matrices, $\mathbb{S}^n$ denote the space of $n \times n$ dimensional real symmetric matrices and $\mathbb{K}^n$ denote the space of $n \times n$ dimensional real skew-symmetric matrices.

The space of piecewise polynomials of degree less than or equal to k on a partition, $\mathcal{T}_h$, of a domain is denoted as $P_k(\mathcal{T}_h)$. The polynomials of degree less than or equal to k on a specific domain T, or on an element $T \in \mathcal{T}_h$, are notated by $P_k(T)$. When referencing a vector or tensor space of polynomials, the notation $(P_k(T))^n$ and $(P_k(T))^{n\times n}$ is used, respectively.

The symmetric gradient operator, $\underline{\boldsymbol{\epsilon}}$ applied to a vector $\mathbf{u}$, is given by

$$\underline{\boldsymbol{\epsilon}}(\mathbf{u})_{ij} = \frac{1}{2}\left(\frac{\partial \mathbf{u}_i}{\partial x_j} + \frac{\partial \mathbf{u}_j}{\partial x_i}\right).$$

For $\underline{\boldsymbol{\sigma}}_i$ denoting the i row of $\underline{\boldsymbol{\sigma}}$, the vector $\nabla \cdot \underline{\boldsymbol{\sigma}}$ is given by

$$\left(\nabla \cdot \underline{\boldsymbol{\sigma}}\right)_i = \nabla \cdot \underline{\boldsymbol{\sigma}}_i\,.$$

The trace operator, tr, is defined as

$$\mathrm{tr}(\underline{\boldsymbol{\sigma}}) = \sum_{i=1}^{n} \sigma_{ii}.$$

The skew-symmetric part of a tensor $\underline{\boldsymbol{\sigma}}$ is defined as

$$as(\underline{\boldsymbol{\sigma}}) = \frac{1}{2}(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}^t)$$

where $\underline{\boldsymbol{\sigma}}^t$ is the transpose of $\underline{\boldsymbol{\sigma}}$.

For $q \in \mathbb{R}$, $\mathcal{S}^2(q)$ is defined as $\mathcal{S}^2(q) = \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix}$, and in two dimensions $as(\underline{\boldsymbol{\sigma}})$ can be identified as $as(\underline{\boldsymbol{\sigma}}) = \mathcal{S}^2(q)$ where $q = \frac{1}{2}(\sigma_{12} - \sigma_{21})$.

For vectors $\mathbf{a} = (a_1, a_2)^t$ and $\mathbf{b} = (b_1, b_2)^t$,

$$\nabla_{\text{curl}}\mathbf{a} = \begin{pmatrix} -\partial_y a_1 & \partial_x a_1 \\ -\partial_y a_2 & \partial_x a_2 \end{pmatrix}, \qquad \mathbf{a} \otimes \mathbf{b} = \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix}.$$

If $\mathbf{w} = (w_1, w_2)^t$ and $\mathbf{v} = (v_1, v_2)^t$ are vectors, then the two-dimensional wedge product is

$$\mathbf{w} \wedge \mathbf{v} = w_1 v_2 - w_2 v_1.$$

For a tensor $\underline{\boldsymbol{\tau}}$ and vector $\mathbf{v}$, the wedge product is

$$(\underline{\boldsymbol{\tau}} \wedge \mathbf{v}) = \begin{pmatrix} \tau_{11}v_2 - \tau_{21}v_1 \\ \tau_{12}v_2 - \tau_{22}v_1 \end{pmatrix}.$$

For $\mathbf{x} = (x_1, x_2)^t$, $\mathbf{x}^\perp$ is defined as, $\mathbf{x}^\perp = (x_2, -x_1)^t$.

To distinguish between inner product and bilinear forms defined in Cartesian coordinates from those defined in cylindrical coordinates, a c subscript is attached to all Cartesian inner products and bilinear forms.

3 Variational Formulation

As a starting point for the derivation of our weak formulation for the axisymmetric problem, we begin with the weak (mixed) formulation for the elasticity problem, subject to a weakly enforced symmetry condition for the stress.

For $\underline{\boldsymbol{\sigma}}$ denoting the stress tensor, $\mathbf{u}$ is the displacement, $\breve{\Omega} \subset \mathbb{R}^3$ a convex (axisymmetric) domain with Lipschitz continuous boundary, μ and λ Lamé constants, the modeling equations of linear elasticity, subject to a fixed boundary, are given by

$$\mathcal{A}\underline{\boldsymbol{\sigma}} = \underline{\boldsymbol{\epsilon}}(\mathbf{u}), \quad \nabla \cdot \underline{\boldsymbol{\sigma}} = \boldsymbol{f} \text{ in } \breve{\Omega}\,, \tag{3.1}$$

$$\text{subject to } \ \mathbf{u} = \mathbf{0} \quad \text{on } \partial\breve{\Omega}\,. \tag{3.2}$$

In (3.1) the compliance tensor $\mathcal{A} : \mathbb{S}^{n\times n} \to \mathbb{S}^{n\times n}$ is a bounded, symmetric positive definite operator that, for isotropic materials, takes the form

$$\mathcal{A}\underline{\boldsymbol{\sigma}} = \frac{1}{2\mu}\left(\underline{\boldsymbol{\sigma}} - \frac{\lambda}{2\mu + m\lambda}\mathrm{tr}(\underline{\boldsymbol{\sigma}}))I\right). \tag{3.3}$$

In order to describe the weak formulation we introduce the following function spaces.

$$\begin{aligned}
&L^2(\breve{\Omega}) = \{v : \int_{\breve{\Omega}} v^2 \, d\breve{\Omega} < \infty\}, \\
&\mathbf{L}^2(\breve{\Omega}) = \{\mathbf{v} : v_i \in L^2(\breve{\Omega}) \text{ for } i = 1, \cdots, n\}, \\
&\mathbf{H}^1(\breve{\Omega}) = \{\mathbf{v} : v_i \in L^2(\breve{\Omega}),\ \nabla v_i \in \mathbf{L}^2(\breve{\Omega})\,,\ \text{for } i = 1, \cdots, n\}, \\
&\underline{\mathbf{L}}^2(\breve{\Omega}; \mathbb{M}^n) = \{\underline{\boldsymbol{\sigma}} \in \mathbb{M}^n : \sigma_{ij} \in L^2(\breve{\Omega})\,,\ \text{for } i,j = 1, \cdots, n\}, \\
&\underline{\mathbf{L}}^2(\breve{\Omega}; \mathbb{K}^n) = \{\underline{\boldsymbol{\sigma}} \in \mathbb{K}^n : \sigma_{ij} \in L^2(\breve{\Omega})\,,\ \text{for } i,j = 1, \cdots, n\}, \\
&\underline{\mathbf{H}}^1(\breve{\Omega}, \mathbb{M}^n) = \{\underline{\boldsymbol{\sigma}} \in \underline{\mathbf{L}}^2(\breve{\Omega}, \mathbb{M}^n) : \nabla\sigma_{ij} \in \mathbf{L}^2(\breve{\Omega})\,,\ \text{for } i,j = 1, \cdots, n\}, \quad \text{and} \\
&\underline{\mathbf{H}}(\mathrm{div}, \breve{\Omega}, \mathbb{M}^n) = \{\underline{\boldsymbol{\sigma}} \in \underline{\mathbf{L}}^2(\breve{\Omega}, \mathbb{M}^n) : \nabla \cdot \underline{\boldsymbol{\sigma}} \in \mathbf{L}^2(\breve{\Omega})\}\,.
\end{aligned}$$

Letting $X = \underline{\mathbf{H}}(\mathrm{div}, \breve{\Omega}, \mathbb{M}^n)$, $Q = \mathbf{L}^2(\breve{\Omega})$, and $W = \underline{\mathbf{L}}^2(\breve{\Omega}; \mathbb{K}^n)$. Then, the weak formulation is given by [2, 20, 25, 27, 28]: *Given,* $\mathbf{f} \in Q$*, determine* $(\underline{\boldsymbol{\sigma}}, \mathbf{u}, \underline{\boldsymbol{\rho}}) \in X \times Q \times W$ *such that, for all*

$(\underline{\boldsymbol{\tau}}, \mathbf{v}, \underline{\boldsymbol{\xi}}) \in X \times Q \times W$

$$\int_{\breve{\Omega}} (\mathcal{A}\underline{\boldsymbol{\sigma}} : \underline{\boldsymbol{\tau}} + \nabla \cdot \underline{\boldsymbol{\tau}} \cdot \mathbf{u} + \underline{\boldsymbol{\tau}} : \underline{\boldsymbol{\rho}}) \, d\breve{\Omega} = 0 \tag{3.4}$$

$$\int_{\breve{\Omega}} \nabla \cdot \underline{\boldsymbol{\sigma}} \cdot \mathbf{v} \, d\breve{\Omega} = \int_{\breve{\Omega}} \mathbf{f} \cdot \mathbf{v} \, d\breve{\Omega} \tag{3.5}$$

$$\int_{\breve{\Omega}} \underline{\boldsymbol{\sigma}} : \underline{\boldsymbol{\xi}} \, d\breve{\Omega} = 0. \tag{3.6}$$

With the inner products,

$$\begin{aligned} a_c(\cdot,\cdot) &: X \times X \to \mathbb{R}, \quad a_c(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\tau}}) := \int_{\breve{\Omega}} \mathcal{A}\underline{\boldsymbol{\sigma}} : \underline{\boldsymbol{\tau}} \, d\breve{\Omega}, \\ b_c(\cdot,\cdot) &: Q \times X \to \mathbb{R}, \quad b_c(\mathbf{u}, \underline{\boldsymbol{\tau}}) := \int_{\breve{\Omega}} (\nabla \cdot \underline{\boldsymbol{\tau}}) \cdot \mathbf{u} \, d\breve{\Omega}, \\ c_c(\cdot,\cdot) &: W \times X \to \mathbb{R}, \quad c_c(\underline{\boldsymbol{\rho}}, \underline{\boldsymbol{\tau}}) := \int_{\breve{\Omega}} \underline{\boldsymbol{\rho}} : \underline{\boldsymbol{\tau}} \, d\breve{\Omega} \end{aligned} \tag{3.7}$$

and taking $A_c(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\tau}}) = a_c(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\tau}})$ and $B_c(\underline{\boldsymbol{\tau}}, (\mathbf{u}, \underline{\boldsymbol{\rho}})) = b_c(\mathbf{u}, \underline{\boldsymbol{\tau}}) + c_c(\underline{\boldsymbol{\rho}}, \underline{\boldsymbol{\tau}})$, (3.4) – (3.6) can be rewritten in the familiar saddle-point formulation: *Given,* $\mathbf{f} \in Q$*, determine* $(\underline{\boldsymbol{\sigma}}, (\mathbf{u}, \underline{\boldsymbol{\rho}})) \in X \times (Q \times W)$ *such that, for all* $(\underline{\boldsymbol{\tau}}, \mathbf{v}, \underline{\boldsymbol{\xi}}) \in X \times (Q \times W)$

$$\begin{aligned} A(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\tau}}) + B(\underline{\boldsymbol{\tau}}, (\mathbf{u}, \underline{\boldsymbol{\rho}})) &= 0 \\ B(\underline{\boldsymbol{\sigma}}, (\mathbf{v}, \underline{\boldsymbol{\xi}})) &= (\mathbf{f}, \mathbf{v}) \, . \end{aligned} \tag{3.8}$$

For a detailed analysis of (3.4) – (3.6), see [12].

4 Axisymmetric Function Spaces

When the three dimensional axisymmetric linear elasticity problem is expressed in cylindrical coordinates, it can be expressed as a decoupled meridian and azimuthal problem. Changing the coordinate system from Cartesian to cylindrical, however, alters the algebraic form of differential operators and requires a new set of function spaces and notation. In this section, we introduce the key changes needed to present and discuss the meridian axisymmetric linear elasticity problem. The Appendix in [9] provides additional details on cylindrical coordinates and the procedure for decoupling the axisymmetric problem.

For axisymmetric vectors $\mathbf{u} = u_r \mathbf{e}_r + u_z \mathbf{e}_z = (u_r, u_z)^t$, we define the gradient operators ∇ and ∇_{axi} as

$$\nabla \, \mathbf{u} = \begin{pmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{\partial u_r}{\partial z} \\ \dfrac{\partial u_z}{\partial r} & \dfrac{\partial u_z}{\partial z} \end{pmatrix}, \text{ and } \nabla_{\text{axi}} \, \mathbf{u} = \begin{pmatrix} \dfrac{\partial u_r}{\partial r} & 0 & \dfrac{\partial u_r}{\partial z} \\ 0 & \frac{1}{r} u_r & 0 \\ \dfrac{\partial u_z}{\partial r} & 0 & \dfrac{\partial u_z}{\partial z} \end{pmatrix}. \tag{4.1}$$

Note that it is necessary to represent the gradient and axisymmetric gradient as tensors with different sizes because the non-constant nature of the cylindrical coordinate unit vectors creates additional terms in axisymmetric derivatives. However, in order to express the meridian problem using a two-dimensional formulation, we represent the tensor $\nabla_{\text{axi}}\mathbf{u}$ as an ordered pair made up of a tensor and a scalar function. That is

$$\nabla_{\text{axi}}\, \mathbf{u} = (\nabla\, \mathbf{u}, \frac{1}{r}u_r). \tag{4.2}$$

Next, for the axisymmetric vector $\mathbf{u} = (u_r, u_z)^t$, the divergence operators $\nabla\cdot$ and $\nabla_{\text{axi}}\cdot$ are defined as

$$\nabla \cdot \mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z}, \text{ and } \nabla_{\text{axi}} \cdot \mathbf{u} = \frac{1}{r}\frac{\partial (r\, u_r)}{\partial r} + \frac{\partial u_z}{\partial z} = \nabla \cdot \mathbf{u} + \frac{1}{r}u_r\ . \tag{4.3}$$

As alluded to in (4.2), the stress tensor that appears in the meridian problem can be represented as $(\underline{\boldsymbol{\sigma}}, \sigma)$, where $\underline{\boldsymbol{\sigma}}$ denotes an $\mathbb{M}^2$ tensor function and σ represents a scalar function. The divergence of the meridian stress tensor is

$$\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\sigma}}, \sigma) = \begin{pmatrix} \nabla_{\text{axi}} \cdot \underline{\boldsymbol{\sigma}}_1 - \frac{1}{r}\sigma \\ \nabla_{\text{axi}} \cdot \underline{\boldsymbol{\sigma}}_2 \end{pmatrix}.$$

At times, the axisymmetric divergence operator will also be applied to an $\mathbb{M}^2$ tensor function $\underline{\boldsymbol{\sigma}}$, in which case

$$\nabla_{\text{axi}} \cdot \underline{\boldsymbol{\sigma}} = \nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\sigma}}, 0) = \begin{pmatrix} \nabla_{\text{axi}} \cdot \underline{\boldsymbol{\sigma}}_1 \\ \nabla_{\text{axi}} \cdot \underline{\boldsymbol{\sigma}}_2 \end{pmatrix}.$$

Note that for the skew symmetric component of $(\underline{\boldsymbol{\sigma}}, \sigma)$ we have

$$as((\underline{\boldsymbol{\sigma}}, \sigma)) = as(\underline{\boldsymbol{\sigma}}) = \mathcal{S}^2(q), \quad \text{where} \quad q = \frac{1}{2}(\sigma_{12} - \sigma_{21}).$$

The curl of an axisymmetric scalar function p is denoted by ∇_{ac} and is defined as

$$\nabla_{\text{ac}}\, p = \left(\frac{\partial p}{\partial z}, -\frac{1}{r}\frac{\partial (r\, p)}{\partial r} \right). \tag{4.4}$$

Note that ∇_{ac} returns a row-vector. For a vector function $\mathbf{p} = (p_r, p_z)^t$ we have

$$\nabla_{\text{ac}}\, \mathbf{p} = \begin{pmatrix} \nabla_{\text{ac}}\, p_r \\ \nabla_{\text{ac}}\, p_z \end{pmatrix}. \tag{4.5}$$

In addition to the divergence and curl, the cylindrical coordinate inner product also takes a different form from the Cartesian inner product.

As illustrates in Figure 1.1, Ω denotes the half cross section of the axisymmetric domain $\breve{\Omega}$.

Consider the change of variables for a Cartesian function $\breve{p} \in L^2(\breve{\Omega})$ into cylindrical coordinates

$$\int_{\breve{\Omega}} \breve{p}^2\, d\breve{\Omega} = \iint_{\Omega} \int_{\theta=0}^{2\pi} p^2 r\, d\theta\, dr\, dz. \tag{4.6}$$

Notice the $r = r(\mathbf{x})$ scaling in the measure. In the axisymmetric setting, $p \equiv p(r,z)$ and the θ integral can be computed to give a factor of 2π. As this term is a constant factor in all such integrals arising, we omit it. To distinguish the cylindrical coordinate inner product from the Cartesian inner product, we use the following notation

$$(p,q) = \int_\Omega p\; q\; r\; dr\; dz.$$

To account for this scaling in the inner product, we introduce the following function spaces

$$\begin{aligned}
{}_\alpha L^2(\Omega) &= \{v : \int_\Omega v^2 r^\alpha\; dr\; dz < \infty\},\\
{}_\alpha \mathbf{L}^2(\Omega) &= \{\mathbf{v} \in \mathbb{R}^n : v_i \in {}_\alpha L^2(\Omega) \text{ for } i = 1,...,n\},\\
{}_\alpha \underline{\mathbf{L}}^2(\Omega, \mathbb{M}^n) &= \{\underline{\boldsymbol{\sigma}} \in \mathbb{M}^n : \sigma_{ij} \in {}_\alpha L^2(\Omega) \text{ for } i = 1,\cdots n \text{ and } j = 1,\cdots n\}, \quad \text{and}\\
{}_\alpha \underline{\mathbf{L}}^2(\Omega, \mathbb{K}^n) &= \{\underline{\boldsymbol{\sigma}} \in \mathbb{K}^n : \sigma_{ij} \in {}_\alpha L^2(\Omega) \text{ for } i = 1,\cdots n \text{ and } j = 1,\cdots n\}.
\end{aligned}$$

The norms associated with these ${}_\alpha L^2$ spaces are

$$\begin{aligned}
&\|v\|^2_{{}_\alpha L^2(\Omega)} = \int_\Omega v^2 r^\alpha\; dr\; dz, \quad \|\mathbf{v}\|^2_{{}_\alpha \mathbf{L}^2(\Omega)} = \sum_{i=1}^n \|v_i\|^2_{{}_\alpha L^2(\Omega)},\\
&\|\underline{\boldsymbol{\sigma}}\|^2_{{}_\alpha \underline{\mathbf{L}}^2(\Omega,\mathbb{M}^n)} = \sum_{i=1}^n\sum_{j=1}^n \|\sigma_{ij}\|^2_{{}_\alpha L^2(\Omega)} \quad \text{and} \quad \|\underline{\boldsymbol{\sigma}}\|^2_{{}_\alpha \underline{\mathbf{L}}^2(\Omega,\mathbb{K}^n)} = \sum_{i=1}^n\sum_{j=1}^n \|\sigma_{ij}\|^2_{{}_\alpha L^2(\Omega)}.
\end{aligned}$$

In addition to the ${}_\alpha L^2$ spaces, the elasticity problem requires divergence spaces for the stress tensors. These spaces are

$$\begin{aligned}
{}_\alpha \mathbf{H}(\text{div}_{\text{axi}}, \Omega) &= \{\mathbf{v} \in {}_\alpha \mathbf{L}^2(\Omega) : \nabla_{\text{axi}} \cdot \mathbf{v} \in {}_\alpha L^2(\Omega)\},\\
{}_\alpha \underline{\mathbf{H}}(\text{div}_{\text{axi}}, \Omega; \mathbb{M}^n) &= \{\underline{\boldsymbol{\sigma}} \in {}_\alpha \underline{\mathbf{L}}^2(\Omega; \mathbb{M}^n) : \nabla_{\text{axi}} \cdot \underline{\boldsymbol{\sigma}} \in {}_\alpha \mathbf{L}^2(\Omega)\}, \quad \text{and}\\
{}_\alpha \underline{\mathbf{H}}(\text{div}_{\text{axi}}, \Omega; \mathbb{K}^n) &= \{\underline{\boldsymbol{\sigma}} \in {}_\alpha \underline{\mathbf{L}}^2(\Omega; \mathbb{K}^n) : \nabla_{\text{axi}} \cdot \underline{\boldsymbol{\sigma}} \in {}_\alpha \mathbf{L}^2(\Omega)\}.
\end{aligned}$$

with norms

$$\begin{aligned}
\|\mathbf{v}\|^2_{{}_\alpha \mathbf{H}(\text{div}_{\text{axi}},\Omega)} &= \|\nabla_{\text{axi}} \cdot \mathbf{v}\|^2_{{}_\alpha L^2(\Omega)} + \|\mathbf{v}\|^2_{{}_\alpha \mathbf{L}^2(\Omega)},\\
\|\underline{\boldsymbol{\sigma}}\|^2_{{}_\alpha \underline{\mathbf{H}}(\text{div}_{\text{axi}},\Omega;\, \mathbb{M}^n)} &= \|\nabla_{\text{axi}} \cdot \underline{\boldsymbol{\sigma}}\|^2_{{}_\alpha \mathbf{L}^2(\Omega)} + \|\underline{\boldsymbol{\sigma}}\|^2_{{}_\alpha \underline{\mathbf{L}}^2(\Omega;\mathbb{M}^n)},\\
\|\underline{\boldsymbol{\sigma}}\|^2_{{}_\alpha \underline{\mathbf{H}}(\text{div}_{\text{axi}},\Omega;\, \mathbb{K}^n)} &= \|\nabla_{\text{axi}} \cdot \underline{\boldsymbol{\sigma}}\|^2_{{}_\alpha \mathbf{L}^2(\Omega)} + \|\underline{\boldsymbol{\sigma}}\|^2_{{}_\alpha \underline{\mathbf{L}}^2(\Omega;\mathbb{K}^n)}.
\end{aligned}$$

For ζ a nonnegative integer and v a ζ times differentiable function, let

$$\nabla^\zeta v = \left[\frac{\partial^\zeta v}{(\partial r)^\zeta}, \frac{\partial^\zeta v}{(\partial r)^{\zeta-1}\partial z}, \cdots, \frac{\partial^\zeta v}{(\partial r)(\partial z)^{\zeta-1}}, \frac{\partial^\zeta v}{(\partial z)^\zeta}\right].$$

Then,

$$\begin{aligned}
{}_\alpha H^k(\Omega) &= \{v \in {}_\alpha L^2(\Omega) : \nabla^\zeta v \in {}_\alpha \mathbf{L}^2(\Omega) \text{ for all } \zeta \leq k\},\\
{}_\alpha \mathbf{H}^k(\Omega) &= \{\mathbf{v} \in {}_\alpha \mathbf{L}^2(\Omega) : \nabla^\zeta v_i \in {}_\alpha \mathbf{L}^2(\Omega) \text{ for all } \zeta \leq k \text{ and } i = 1, 2, \cdots n\}, \quad \text{and}\\
{}_\alpha \underline{\mathbf{H}}^k(\Omega; \mathbb{M}^n) &= \{\underline{\boldsymbol{\sigma}} \in {}_\alpha \mathbf{L}^2(\Omega; \mathbb{M}^n) : \nabla^\zeta \sigma_{i,j} \in {}_\alpha \mathbf{L}^2(\Omega) \text{ for all } \zeta \leq k \text{ and } i, j = 1, 2, \cdots n\},
\end{aligned}$$

with norms

$$\|v\|^2_{{}_\alpha H^k(\Omega)} = \|v\|^2_{{}_\alpha L^2(\Omega)} + \sum_{\zeta=1}^{k} \|\nabla^\zeta v\|^2_{{}_\alpha \mathbf{L}^2(\Omega)},$$

$$\|\mathbf{v}\|^2_{{}_\alpha \mathbf{H}^k(\Omega)} = \|\mathbf{v}\|^2_{{}_\alpha \mathbf{L}^2(\Omega)} + \sum_{i=1}^{n}\sum_{\zeta=1}^{k} \|\nabla^\zeta v_i\|^2_{{}_\alpha \mathbf{L}^2(\Omega)},$$

$$\|\underline{\boldsymbol{\sigma}}\|^2_{{}_\alpha \underline{\mathbf{H}}^k(\Omega;\mathbb{M}^n)} = \|\underline{\boldsymbol{\sigma}}\|^2_{{}_\alpha \underline{\mathbf{L}}^2(\Omega;\mathbb{M}^n)} + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\zeta=1}^{k} \|\nabla^\zeta \sigma_{ij}\|^2_{{}_\alpha \mathbf{L}^2(\Omega)}.$$

Next we consider some subtle details related to function spaces containing axisymmetric derivative terms. To begin, using (4.1),

$$\|\nabla_{\text{axi}}\mathbf{v}\|^2_{{}_1 L^2(\Omega)} = \int_\Omega \nabla_{\text{axi}}\mathbf{v} : \nabla_{\text{axi}}\mathbf{v}\; r\, d\Omega = \int_\Omega \nabla\mathbf{v} : \nabla\mathbf{v}\; r\, d\Omega + \int_\Omega \frac{1}{r} v_r^2\, d\Omega.$$

Therefore, in order that $\|\nabla_{\text{axi}}\mathbf{v}\|_{{}_1 L^2(\Omega)} < \infty$, it is necessary for $v_r \in {}_1H^1(\Omega)$ and $v_r \in {}_{-1}L^2(\Omega)$. To denote this important subspace of ${}_1H^1(\Omega)$, we define

$${}_1V^1(\Omega) = \{v \in {}_1H^1(\Omega) : v \in {}_{-1}L^2(\Omega)\} \text{ with associated norm } \|v\|_{{}_1V^1(\Omega)} = \left(\|v\|^2_{{}_{-1}L^2(\Omega)} + \|v\|^2_{{}_1H^1(\Omega)}\right)^{1/2}.$$

Also, we introduce

$${}_1\mathbf{VH}^1(\Omega) = \{\mathbf{v} = (v_r, v_z)^t : v_r \in {}_1V^1(\Omega),\ v_z \in {}_1H^1(\Omega)\}$$

$$\text{with associated norm } \|\mathbf{v}\|_{{}_1\mathbf{VH}^1(\Omega)} = \left(\|v_r\|^2_{{}_1V^1(\Omega)} + \|v_z\|^2_{{}_1H^1(\Omega)}\right)^{1/2}.$$

It is also important to observe, that unlike in the Cartesian setting, ${}_1\mathbf{H}^1(\Omega) \not\subset {}_1\mathbf{H}(\text{div}_{\text{axi}}, \Omega)$.

When referencing a function space whose functions have a vanishing trace along the boundary segment Γ, we use a zero subscript, i.e.,

$${}_1H^1_0(\Omega) = \{v \in {}_1H^1(\Omega) : v = 0 \text{ on } \Gamma\}.$$

Note that Γ here does not include the rotation axis portion of the boundary of Ω as illustrated in Figure 1.1.

As eluded to above, in transforming from $\breve{\Omega}$ to Ω, we have the following relationships.

Lemma 4.1. *[8, Proposition 1] The space of axisymmetric vector fields in $H^1(\breve{\Omega})^3$ with zero angular component is isomorphic to ${}_1\mathbf{VH}^1(\Omega)$.*

Lemma 4.2. *The space of axisymmetric tensors in $\underline{\mathbf{H}}^1(\breve{\Omega}, \mathbb{M}^3)$ with zero azimuthal components is isomorphic to*

$$\left\{ \underline{\boldsymbol{\tau}} = \begin{pmatrix} \tau_{rr} & 0 & \tau_{rz} \\ 0 & \tau_{\theta\theta} & 0 \\ \tau_{zr} & 0 & \tau_{zz} \end{pmatrix} : \tau_{rr},\, \tau_{\theta\theta},\, \tau_{zz} \in {}_1H^1(\Omega),\ \tau_{rz},\, \tau_{zr} \in {}_1V^1(\Omega),\ (\tau_{rr} - \tau_{\theta\theta}) \in {}_{-1}L^2(\Omega) \right\}. \tag{4.7}$$

Proof. The representation of an axisymmetric tensor $\underline{\boldsymbol{\tau}}$, in cylindrical coordinates with zero azimuthal components is given by the tensor in (4.7). In terms of the unit coordinate vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$, $\nabla\underline{\boldsymbol{\tau}}$ may be written as

$$\begin{aligned}
\nabla\underline{\boldsymbol{\tau}} = \;& \partial_r\tau_{rr}\,\mathbf{e}_r\otimes\mathbf{e}_r\otimes\mathbf{e}_r \;+\; 0\,\mathbf{e}_r\otimes\mathbf{e}_r\otimes\mathbf{e}_\theta \;+\; \partial_z\tau_{rr}\,\mathbf{e}_r\otimes\mathbf{e}_r\otimes\mathbf{e}_z \\
&+\; 0\,\mathbf{e}_r\otimes\mathbf{e}_\theta\otimes\mathbf{e}_r \;+\; \frac{1}{r}(\tau_{rr}\,-\,\tau_{\theta\theta})\,\mathbf{e}_r\otimes\mathbf{e}_\theta\otimes\mathbf{e}_\theta \;+\; 0\,\mathbf{e}_r\otimes\mathbf{e}_\theta\otimes\mathbf{e}_z \\
&+\; \partial_r\tau_{rz}\,\mathbf{e}_r\otimes\mathbf{e}_z\otimes\mathbf{e}_r \;+\; 0\,\mathbf{e}_r\otimes\mathbf{e}_z\otimes\mathbf{e}_\theta \;+\; \partial_z\tau_{rz}\,\mathbf{e}_r\otimes\mathbf{e}_z\otimes\mathbf{e}_z \\
&+\; 0\,\mathbf{e}_\theta\otimes\mathbf{e}_r\otimes\mathbf{e}_r \;+\; \frac{1}{r}(\tau_{rr}\,-\,\tau_{\theta\theta})\,\mathbf{e}_\theta\otimes\mathbf{e}_r\otimes\mathbf{e}_\theta \;+\; 0\,\mathbf{e}_\theta\otimes\mathbf{e}_r\otimes\mathbf{e}_z \\
&+\; \partial_r\tau_{\theta\theta}\,\mathbf{e}_\theta\otimes\mathbf{e}_\theta\otimes\mathbf{e}_r \;+\; 0\,\mathbf{e}_\theta\otimes\mathbf{e}_\theta\otimes\mathbf{e}_\theta \;+\; \partial_z\tau_{\theta\theta}\,\mathbf{e}_\theta\otimes\mathbf{e}_\theta\otimes\mathbf{e}_z \\
&+\; 0\,\mathbf{e}_\theta\otimes\mathbf{e}_z\otimes\mathbf{e}_r \;+\; \frac{1}{r}\,\tau_{rz}\,\mathbf{e}_\theta\otimes\mathbf{e}_z\otimes\mathbf{e}_\theta \;+\; 0\,\mathbf{e}_\theta\otimes\mathbf{e}_z\otimes\mathbf{e}_z \\
&+\; \partial_r\tau_{zr}\,\mathbf{e}_z\otimes\mathbf{e}_r\otimes\mathbf{e}_r \;+\; 0\,\mathbf{e}_z\otimes\mathbf{e}_r\otimes\mathbf{e}_\theta \;+\; \partial_z\tau_{zr}\,\mathbf{e}_z\otimes\mathbf{e}_r\otimes\mathbf{e}_z \\
&+\; 0\,\mathbf{e}_z\otimes\mathbf{e}_\theta\otimes\mathbf{e}_r \;+\; \frac{1}{r}\,\tau_{zr}\,\mathbf{e}_z\otimes\mathbf{e}_\theta\otimes\mathbf{e}_\theta \;+\; 0\,\mathbf{e}_z\otimes\mathbf{e}_\theta\otimes\mathbf{e}_z \\
&+\; \partial_r\tau_{zz}\,\mathbf{e}_z\otimes\mathbf{e}_z\otimes\mathbf{e}_r \;+\; 0\,\mathbf{e}_z\otimes\mathbf{e}_z\otimes\mathbf{e}_\theta \;+\; \partial_z\tau_{zz}\,\mathbf{e}_z\otimes\mathbf{e}_z\otimes\mathbf{e}_z\,.
\end{aligned}$$

Let,

$$D\underline{\boldsymbol{\tau}}_r \;=\; \begin{pmatrix} \partial_r\tau_{rr} & 0 & \partial_z\tau_{rr} \\ 0 & \frac{1}{r}(\tau_{rr}\,-\,\tau_{\theta\theta}) & 0 \\ \partial_r\tau_{rz} & 0 & \partial_z\tau_{rz} \end{pmatrix}, \quad D\underline{\boldsymbol{\tau}}_\theta \;=\; \begin{pmatrix} 0 & \frac{1}{r}(\tau_{rr}\,-\,\tau_{\theta\theta}) & 0 \\ \partial_r\tau_{\theta\theta} & 0 & \partial_z\tau_{\theta\theta} \\ 0 & \frac{1}{r}\,\tau_{rz} & 0 \end{pmatrix},$$

$$\text{and} \qquad D\underline{\boldsymbol{\tau}}_z \;=\; \begin{pmatrix} \partial_r\tau_{zr} & 0 & \partial_z\tau_{zr} \\ 0 & \frac{1}{r}\,\tau_{zr} & 0 \\ \partial_r\tau_{zz} & 0 & \partial_z\tau_{zz} \end{pmatrix}.$$

Then,

$$\begin{aligned}
\|\underline{\boldsymbol{\tau}}\|^2_{H^1(\breve{\Omega})} \;=&\; \int_{\breve{\Omega}} \|\underline{\boldsymbol{\tau}}(x,y,z)\|^2\,d\breve{\Omega} \;+\; \int_{\breve{\Omega}} \|D\underline{\boldsymbol{\tau}}(x,y,z)\|^2\,d\breve{\Omega} \\
=&\; 2\pi\int_{\Omega}\left(\tau_{rr}^2 \,+\, \tau_{rz}^2 \,+\, \tau_{\theta\theta}^2 \,+\, \tau_{zr}^2 \,+\, \tau_{zz}^2\right) r\,dr\,dz \\
&+\; 2\pi\int_{\Omega}\left(D\underline{\boldsymbol{\tau}}_r : D\underline{\boldsymbol{\tau}}_r \,+\, D\underline{\boldsymbol{\tau}}_\theta : D\underline{\boldsymbol{\tau}}_\theta \,+\, D\underline{\boldsymbol{\tau}}_z : D\underline{\boldsymbol{\tau}}_z\right) r\,dr\,dz\,.
\end{aligned}$$

Hence $\|\underline{\boldsymbol{\tau}}\|^2_{H^1(\breve{\Omega})} < \infty$ implies, $\tau_{rr},\,\tau_{\theta\theta},\,\tau_{zz} \in {}_1H^1(\Omega)$, $\tau_{rz},\,\tau_{zr} \in {}_1V^1(\Omega)$, $(\tau_{rr}\,-\,\tau_{\theta\theta}) \in {}_{-1}L^2(\Omega)$.

Reversing the argument establishes the isomorphism between the spaces. ■

In the discussions that follow, we take $U = {}_1\mathbf{L}^2(\Omega)$, $Q = {}_1L^2(\Omega)$. As the merdian stress tensor is made up of a tensor and scalar component, we introduce the space $\boldsymbol{\Sigma}(\Omega)$ defined by

$$\boldsymbol{\Sigma}(\Omega) = \{(\underline{\boldsymbol{\sigma}},\sigma) \in {}_1\underline{\mathbf{L}}^2(\Omega,\mathbb{M}^2)\times{}_1L^2(\Omega) \;:\; \nabla_{\text{axi}}\cdot(\underline{\boldsymbol{\sigma}},\sigma) \in {}_1L^2(\Omega)\}.$$

Associated with $\boldsymbol{\Sigma}(\Omega)$ we have the norm

$$\|(\underline{\boldsymbol{\sigma}},\sigma)\|_{\boldsymbol{\Sigma}(\Omega)} = \left(\|\nabla_{\text{axi}}\cdot(\underline{\boldsymbol{\sigma}},\sigma)\|^2_{{}_1\mathbf{L}^2(\Omega)} + \|\underline{\boldsymbol{\sigma}}\|^2_{{}_1\underline{\mathbf{L}}^2(\Omega;\,\mathbb{M}^2)} + \|\sigma\|^2_{{}_1L^2(\Omega)}\right)^{\frac{1}{2}}.$$

Additionally, we define $\boldsymbol{S}(\Omega) \subset \boldsymbol{\Sigma}(\Omega)$ by

$$\boldsymbol{S}(\Omega) = \{(\underline{\boldsymbol{\sigma}}, 0) \in \boldsymbol{\Sigma}(\Omega) : \underline{\boldsymbol{\sigma}} = \begin{pmatrix} \mathbf{w}^t \\ \mathbf{z}^t \end{pmatrix} \; ; \; \mathbf{w}, \mathbf{z} \in {}_1\mathbf{VH}^1(\Omega)\},$$

with norm

$$\|(\underline{\boldsymbol{\sigma}}, 0)\|_{\boldsymbol{S}(\Omega)} = \left(\|\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\sigma}}, 0)\|^2_{{}_1\mathbf{L}^2(\Omega)} + \|\mathbf{w}\|^2_{{}_1\mathbf{VH}^1(\Omega)} + \|\mathbf{z}\|^2_{{}_1\mathbf{VH}^1(\Omega)} \right)^{\frac{1}{2}} . \tag{4.8}$$

For convenience, when the context is clear, $\boldsymbol{\Sigma}(\Omega)$ and $\boldsymbol{S}(\Omega)$ will be denoted as $\boldsymbol{\Sigma}$ and $\boldsymbol{S}$.

5 Axisymmetric Variational Formulation

In this section we present the variational form of the axisymmetric meridian problem. This problem has many similarities with the elasticity problem in the Cartesian setting, however, new terms are introduced into the bilinear forms as a consequence of the change of variable from Cartesian to cylindrical coordinates. Details of the derivation can be found in [9].

Analogous to (3.7), define the bilinear forms

$$\tilde{a}(.,.) : \boldsymbol{\Sigma} \times \boldsymbol{\Sigma} \to \mathbb{R}, \quad \tilde{a}((\underline{\boldsymbol{\sigma}}, \sigma), (\underline{\boldsymbol{\tau}}, \tau)) = (\mathcal{A}\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\tau}}) + (\mathcal{A}\sigma, \tau) - \frac{1}{2\mu} \frac{\lambda}{2\mu + 3\lambda} ((\sigma, \text{tr}(\underline{\boldsymbol{\tau}})) + (\text{tr}(\underline{\boldsymbol{\sigma}}), \tau)), \tag{5.1}$$

$$\tilde{b}(.,.) : \boldsymbol{\Sigma} \times U \to \mathbb{R}, \quad \tilde{b}((\underline{\boldsymbol{\tau}}, \tau), \mathbf{u}) = (\nabla_{\text{axi}} \cdot \underline{\boldsymbol{\tau}}, \mathbf{u}) - (\frac{\tau}{r}, u_r) , \tag{5.2}$$

$$\tilde{c}(.,.) : \boldsymbol{\Sigma} \times Q \to \mathbb{R}, \quad \tilde{c}((\underline{\boldsymbol{\sigma}}, \sigma), p) = (\underline{\boldsymbol{\sigma}}, \mathcal{S}^2(p)) , \tag{5.3}$$

where the operator $\mathcal{A}$ applied to the scalar function $\sigma_{\theta\theta}$ is given by (3.3) for $m = 1$.

The axisymmetric meridian problem with weak symmetry is then: *Given* $\mathbf{f} \in {}_1\mathbf{L}^2(\Omega)$, *find* $((\underline{\boldsymbol{\sigma}}, \sigma), \mathbf{u}, p) \in \boldsymbol{\Sigma} \times U \times Q$ *such that for all* $((\underline{\boldsymbol{\tau}}, \tau), \mathbf{v}, q) \in \boldsymbol{\Sigma} \times U \times Q$

$$\tilde{a}((\underline{\boldsymbol{\sigma}}, \sigma), (\underline{\boldsymbol{\tau}}, \tau)) + \tilde{b}((\underline{\boldsymbol{\tau}}, \tau), \mathbf{u}) + \tilde{c}((\underline{\boldsymbol{\tau}}, \tau), p) = 0 \tag{5.4}$$

$$\tilde{b}((\underline{\boldsymbol{\sigma}}, \sigma), \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \tag{5.5}$$

$$\tilde{c}((\underline{\boldsymbol{\sigma}}, \sigma), q) = 0 \, . \tag{5.6}$$

Of interest is to develop discrete inf-sup stable elements for the approximation of (5.4)-(5.6). In cylindrical coordinates, the divergence operator does not map polynomial spaces into polynomial spaces, so some of the standard techniques for verifying inf-sup stability cannot be used. Thus, to help establish a variational formulation for which stable triples of finite elements may be verified to satisfy the discrete inf-sup condition, we make two modifications to (5.4)-(5.6).

First, we add a grad-div stabilization term to $\tilde{a}(\cdot, \cdot)$ and define a new bilinear form $a(\cdot, \cdot) : \boldsymbol{\Sigma} \times \boldsymbol{\Sigma} :\to \mathbb{R}$

$$a((\underline{\boldsymbol{\sigma}}, \sigma), (\underline{\boldsymbol{\tau}}, \tau)) = \tilde{a}((\underline{\boldsymbol{\sigma}}, \sigma), (\underline{\boldsymbol{\tau}}, \tau)) + \gamma(\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\sigma}}, \sigma), \nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}}, \tau)) \tag{5.7}$$

where γ is the grad-div stabilization term. This stabilization term ensures that $a((\cdot, \cdot), (\cdot, \cdot))$ is coercive in the $\| \cdot \|_{\boldsymbol{\Sigma}}$ norm. Unless specified otherwise, we take $\gamma = 1$.

Recall from (3.1) that in cylindrical coordinates, $\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\sigma}}, \sigma) = \mathbf{f}$. Therefore, to account for the grad-div stabilization term in the constituent equation, $(\mathbf{f}, \nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}}, \tau))$ must also be added to the right hand side of (5.4).

For the second modification, recall that $\tilde{c}((\underline{\boldsymbol{\sigma}}, \sigma), q) = (\underline{\boldsymbol{\sigma}}, \mathcal{S}^2(q))$, and let $\mathbf{x} = (r, z)^t$. As described in Lemma 6.1 below,

$$\begin{aligned} \int_\Omega \underline{\boldsymbol{\sigma}} : \mathcal{S}^2(q)\; r\; d\,\Omega = &- \int_\Omega (\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\sigma}}, \sigma) \wedge \mathbf{x})\; q\; r\; d\Omega \\ &+ \int_{\partial\Omega} (\underline{\boldsymbol{\sigma}} \cdot \mathbf{n}) \cdot \mathbf{x}^\perp q\; r\; ds - \int_\Omega \underline{\boldsymbol{\sigma}} : (\mathbf{x}^\perp \otimes \nabla q)\; r\; d\Omega - \int_\Omega \frac{1}{r} \sigma\; z\; q\; r\; d\Omega, \end{aligned} \tag{5.8}$$

or equivalently

$$\begin{aligned} \int_\Omega \underline{\boldsymbol{\sigma}} : \mathcal{S}^2(q)\; r\; d\Omega &+ \int_\Omega (\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\sigma}}, \sigma) \wedge \mathbf{x})\; q\; r\; d\Omega \\ &= \int_{\partial\Omega} (\underline{\boldsymbol{\sigma}} \cdot \mathbf{n}) \cdot \mathbf{x}^\perp q\; r\; ds - \int_\Omega \underline{\boldsymbol{\sigma}} : (\mathbf{x}^\perp \otimes \nabla q)\; r\; d\Omega - \int_\Omega \sigma\; z\; q\; d\Omega. \end{aligned} \tag{5.9}$$

In terms of establishing stable approximation elements via the construction of a suitable projection (see Theorem 6.1) it is more convenient to use equation (5.9) than (5.8). To introduce (5.9) into the weak form, we add $\int_\Omega (\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\sigma}}, \sigma) \wedge \mathbf{x})\; q\; r\; d\Omega$ to both sides of (5.6) giving

$$\tilde{c}((\underline{\boldsymbol{\sigma}}, \sigma), q) + \int_\Omega (\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\sigma}}, \sigma) \wedge \mathbf{x})\; q\; r\; d\Omega = \int_\Omega (\mathbf{f} \wedge \mathbf{x})\; q\; r\; d\Omega, \tag{5.10}$$

where we have used the relationship $\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\sigma}}, \sigma) = \mathbf{f}$ on the right hand side. To represent the left hand side of (5.10), we define a new bilinear form $c(\cdot, \cdot) : \boldsymbol{\Sigma} \times Q \to \mathbb{R}$ as

$$\begin{aligned} c((\underline{\boldsymbol{\sigma}}, \sigma), q) &:= \tilde{c}((\underline{\boldsymbol{\sigma}}, \sigma), q) + (\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\sigma}}, \sigma) \wedge \mathbf{x}, q) \\ &= (\underline{\boldsymbol{\sigma}}, \mathcal{S}^2(q)) + (\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\sigma}}, \sigma) \wedge \mathbf{x}, q). \end{aligned} \tag{5.11}$$

Therefore, (5.6) becomes

$$c((\underline{\boldsymbol{\sigma}}, \sigma), q) = (\mathbf{f} \wedge \mathbf{x}, q).$$

For notational consistency in the new formulation, we let $b(\cdot, \cdot) = \tilde{b}(\cdot, \cdot)$.

To maintain the saddle point structure of the variational formulation with the bilinear form $c(\cdot, \cdot)$, we need to add and subtract $(\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}}, \tau) \wedge \mathbf{x}, p)$ to the left hand side of (5.4). To understand the affect of this modification on the formulation, first observe that

$$\begin{aligned} \nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}}, \tau) \wedge \mathbf{x} &= \begin{pmatrix} \dfrac{\partial \tau_{11}}{\partial r} + \dfrac{\partial \tau_{12}}{\partial z} + \dfrac{1}{r}(\tau_{11} - \tau) \\ \dfrac{\partial \tau_{21}}{\partial r} + \dfrac{\partial \tau_{22}}{\partial z} + \dfrac{1}{r}\tau_{21} \end{pmatrix} \wedge \begin{pmatrix} r \\ z \end{pmatrix} \\ &= z\Big(\frac{\partial \tau_{11}}{\partial r} + \frac{\partial \tau_{12}}{\partial z} + \frac{1}{r}(\tau_{11} - \tau)\Big) - r\Big(\frac{\partial \tau_{21}}{\partial r} + \frac{\partial \tau_{22}}{\partial z} + \frac{1}{r}\tau_{21}\Big) \;=\; (\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}}, \tau)) \cdot \mathbf{x}^\perp. \end{aligned}$$

Therefore,

$$
\begin{aligned}
((\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}}, \tau)) \wedge \mathbf{x}, p) = \int_\Omega \nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}}, \tau) \wedge \mathbf{x} \, p \, r \, d\Omega \; &= \; \int_\Omega (\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}}, \tau)) \cdot \mathbf{x}^\perp \, p \, r \, d\Omega \\
&= b((\underline{\boldsymbol{\tau}}, \tau), \mathbf{x}^\perp \, p).
\end{aligned} \tag{5.12}
$$

This shows that $((\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}}, \tau) \wedge \mathbf{x}, p)$ can be expressed as $b((\underline{\boldsymbol{\tau}}, \tau), \mathbf{x}^\perp p)$. As a result, the negative part of $((\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}}, \tau) \wedge \mathbf{x}, p)$ that is used to balance the constituent equation enters into the expression as part of the bilinear form $b(\cdot, \cdot)$. That is,

$$
b((\underline{\boldsymbol{\tau}}, \tau), \mathbf{u}) - b((\underline{\boldsymbol{\tau}}, \tau), \mathbf{x}^\perp p) = b((\underline{\boldsymbol{\tau}}, \tau), \mathbf{u} - \mathbf{x}^\perp p). \tag{5.13}
$$

To reflect the fact that the expression within the bilinear form $b(\cdot, \cdot)$ no longer depends only on the displacement $\mathbf{u}$, we define a new variable $\mathbf{w} = \mathbf{u} - \mathbf{x}^\perp p$. As we discuss further in Sections 8 and 9, once the solution has be found, the displacement $\mathbf{u} = \mathbf{w} + \mathbf{x}^\perp p$ can be accurately recovered during a post-processing step.

Therefore, an equivalent but modified version of the axisymmetric linear elasticity problem (5.4)-(5.6) can be expressed as: *Given* $\mathbf{f} \in {}_1\mathbf{L}^2(\Omega)$ *find* $((\underline{\boldsymbol{\sigma}}, \sigma), \mathbf{w}, p) \in \boldsymbol{\Sigma} \times U \times Q$ *such that for all* $((\underline{\boldsymbol{\tau}}, \tau), \mathbf{v}, q) \in \boldsymbol{\Sigma} \times U \times Q$

$$
a((\underline{\boldsymbol{\sigma}}, \sigma), (\underline{\boldsymbol{\tau}}, \tau)) + b((\underline{\boldsymbol{\tau}}, \tau), \mathbf{w}) + c((\underline{\boldsymbol{\tau}}, \tau), p) = (\mathbf{f}, \nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}}, \tau)) \tag{5.14}
$$

$$
b((\underline{\boldsymbol{\sigma}}, \sigma), \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \tag{5.15}
$$

$$
c((\underline{\boldsymbol{\sigma}}, \sigma), q) = (\mathbf{f} \wedge \mathbf{x}, q) \, . \tag{5.16}
$$

5.1 Well posedness of the variational formulation (5.14)-(5.16)

To establish the well posedness of the saddle point formulation, (5.14)-(5.16), we show that $a(\cdot, \cdot)$ is bounded and coercive on $\boldsymbol{\Sigma} \times \boldsymbol{\Sigma}$, and that $((\underline{\boldsymbol{\tau}}, \tau), \mathbf{v}, p)$ satisfy the inf-sup condition

$$
\inf_{\mathbf{v} \in U, p \in Q} \sup_{(\underline{\boldsymbol{\tau}}, \tau) \in \boldsymbol{\Sigma}} \frac{b((\underline{\boldsymbol{\tau}}, \tau), \mathbf{v}) + c((\underline{\boldsymbol{\tau}}, \tau), p)}{\|(\underline{\boldsymbol{\tau}}, \tau)\|_{\boldsymbol{\Sigma}} (\|\mathbf{v}\|_U + \|p\|_Q)} \geq C. \tag{5.17}
$$

Lemma 5.1. *[9] The operator $a(.,.)$ defined in (5.7) is bounded. That is,*

$$
a((\underline{\boldsymbol{\sigma}}, \sigma), (\underline{\boldsymbol{\tau}}, \tau)) \leq C \|(\underline{\boldsymbol{\sigma}}, \sigma)\|_{\boldsymbol{\Sigma}} \|(\underline{\boldsymbol{\tau}}, \tau)\|_{\boldsymbol{\Sigma}} \tag{5.18}
$$

for some $C > 0$ and all $(\underline{\boldsymbol{\sigma}}, \sigma), (\underline{\boldsymbol{\tau}}, \tau) \in \boldsymbol{\Sigma}$.

Lemma 5.2. *[9] The operator $a(.,.)$ defined in (5.7) is coercive. That is,*

$$
a((\underline{\boldsymbol{\sigma}}, \sigma), (\underline{\boldsymbol{\sigma}}, \sigma)) \geq c \|(\underline{\boldsymbol{\sigma}}, \sigma)\|^2_{\boldsymbol{\Sigma}} \text{ where } c = \min\{\frac{1}{2\mu} \frac{1}{2\mu + 3\lambda}, 1\}. \tag{5.19}
$$

5.1.1 Satisfying the continuous inf-sup condition (5.17)

To establish the inf-sup condition (5.17), we follow a similar two step argument as used in [12] for the planar elasticity problem. In Step 1 a $(\underline{\boldsymbol{\tau}}_1, \tau_1)$ is found such that, for $\mathbf{v}, p$ given, $b((\underline{\boldsymbol{\tau}}_1, \tau_1), \mathbf{v}) = \|\mathbf{v}\|^2_U$. Then, in Step 2 $(\underline{\boldsymbol{\tau}}_2, \tau_2)$ is constructed to handle the $c(\cdot, \cdot)$ term, while satisfying $b((\underline{\boldsymbol{\tau}}_2, \tau_2), \mathbf{v}) = 0$. The following lemma is useful in the construction of $(\underline{\boldsymbol{\tau}}_2, \tau_2)$.

Lemma 5.3. *Given* $\beta \in {}_1L^2(\Omega)$ *there exist* $(\underline{\boldsymbol{\tau}}, \tau) \in \boldsymbol{\Sigma}$ *such that*

$$\nabla_{axi} \cdot (\underline{\boldsymbol{\tau}}, \tau) = \mathbf{0}, \qquad as(\underline{\boldsymbol{\tau}}) = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}, \qquad and \quad \|(\underline{\boldsymbol{\tau}}, \tau)\|_{\boldsymbol{\Sigma}} \ \le\ C\,\|\beta\|_{{}_1L^2(\Omega)}\,. \tag{5.20}$$

Proof. From (5.20),

$$\frac{1}{2}\big(\tau_{12} - \tau_{21}\big) = \beta\,, \quad \Rightarrow \quad \tau_{21} = \tau_{12} - 2\beta\,, \tag{5.21}$$

$$\partial_r \tau_{11} + \frac{1}{r}\tau_{11} + \partial_z \tau_{12} - \frac{1}{r}\tau \ = \ 0\,, \tag{5.22}$$

$$\partial_r \tau_{21} + \frac{1}{r}\tau_{21} + \partial_z \tau_{22} \ = \ 0\,. \tag{5.23}$$

Substituting (5.21) into (5.23), then multiplying (5.22) and (5.23) through by r and simplifying we obtain

$$\partial_r(r\,\tau_{11}) + \partial_z(r\,\tau_{12}) - \tau = 0\,, \tag{5.24}$$

$$\partial_r(r\,\tau_{12}) + \partial_z(r\,\tau_{22}) = \partial_r(2r\,\beta)\,. \tag{5.25}$$

Integrating (5.24) with respect to z, and (5.25) with respect to r, yields for arbitrary f_1 and f_2,

$$\int^z \partial_r(r\,\tau_{11})\,dz \ + \ r\,\tau_{12} \ - \int^z \tau\,dz = f_1(r)\,, \tag{5.26}$$

$$r\,\tau_{12} \ + \int^r \partial_z(r\,\tau_{22})\,dr = 2r\,\beta \ + \ f_2(z)\,. \tag{5.27}$$

Interchanging the order of integration and differentiation, and then subtracting (5.26) from (5.27), yields

$$\partial_r\left(-\int^z (r\,\tau_{11})\,dz\right) \ + \ \partial_z\left(\int^r (r\,\tau_{22})\,dr\right) \ + \ \int^z \tau\,dz \ = \ 2r\,\beta \ + \ f_2(z) \ - \ f_1(r)\,.$$

Dividing through by r, choosing $f_1 = f_2 = 0$, and rearranging we have

$$\frac{1}{r}\partial_r\left(r(-\int^z \tau_{11}\,dz)\right) \ + \ \partial_z\left(\frac{1}{r}\int^r (r\,\tau_{22})\,dr\right) \ - \ \frac{1}{r}\left(-\int^z \tau\,dz\right) \ = \ 2\beta\,. \tag{5.28}$$

$$\text{Let} \quad \sigma_{rr} = (-\int^z \tau_{11}\,dz)\,, \quad \sigma_{rz} = \frac{1}{r}\int^r (r\,\tau_{22})\,dr\,, \quad \text{and } \sigma_{\theta\theta} = -\int^z \tau\,dz\,. \tag{5.29}$$

Then, (5.28) can be embedded in the meridian problem

$$\nabla_{\text{axi}} \cdot \begin{bmatrix} \sigma_{rr} & 0 & \sigma_{rz} \\ 0 & \sigma_{\theta\theta} & 0 \\ \sigma_{zr} & 0 & \sigma_{zz} \end{bmatrix} \ = \ \begin{bmatrix} 2\,\beta \\ 0 \end{bmatrix} \quad \text{in } \ \Omega\,. \tag{5.30}$$

Lifting (5.30) from Ω to $\breve{\Omega}$ we obtain an axisymmetric elasticity problem in $\breve{\Omega}$, $\nabla \cdot \underline{\boldsymbol{\sigma}} \ = \ \boldsymbol{f}$ (see (3.1)), with $\boldsymbol{f} \in \mathbf{L}^2(\breve{\Omega})$. From [21], we have that, for $\breve{\Omega}$ a bounded polyhedral domain, $\underline{\boldsymbol{\sigma}} \in \underline{\mathbf{H}}^1(\breve{\Omega}, \mathbb{M}^3)$. Additionally, $\|\underline{\boldsymbol{\sigma}}\|_{\underline{\mathbf{H}}^1(\breve{\Omega},\mathbb{M}^3)} \ \le\ C\,\|\boldsymbol{f}\|_{\mathbf{L}^2(\breve{\Omega})} \ \le\ C\|\beta\|_{{}_1L^2(\Omega)}$.

Then, using Lemma 4.2 we have that

$$\sigma_{rr},\, \sigma_{\theta\theta} \in {}_1H^1(\Omega),\ \ \sigma_{rz} \in {}_1V^1(\Omega), \ \ \text{and} \ \ (\sigma_{rr} - \sigma_{\theta\theta}) \in {}_{-1}L^2(\Omega)\,. \tag{5.31}$$

From (5.29) and (5.31),

$$\begin{aligned}
\tau =&\ -\partial_z \sigma_{\theta\theta}. \ \text{ As } \ \sigma_{\theta\theta} \in {}_1H^1(\Omega), \ \text{ then } \ \tau \in {}_1L^2(\Omega)\,. \\
\tau_{11} =&\ -\partial_z \sigma_{rr}. \ \text{ As } \ \sigma_{rr} \in {}_1H^1(\Omega), \ \text{ then } \ \tau_{11} \in {}_1L^2(\Omega)\,. \\
\tau_{22} =&\ \frac{1}{r}\sigma_{rz} + \partial_r \sigma_{rz}\ . \ \text{ As } \ \sigma_{rz} \in {}_1V^1(\Omega), \ \text{ then } \ \tau_{22} \in {}_1L^2(\Omega)\,.
\end{aligned}$$

Also, it follows that

$$\|\tau\|_{{}_1L^2(\Omega)} \ + \ \|\tau_{11}\|_{{}_1L^2(\Omega)} \ + \ \|\tau_{22}\|_{{}_1L^2(\Omega)} \ \le \ C\,\|\underline{\boldsymbol{\sigma}}\|_{\underline{\mathbf{H}}^1(\breve{\Omega},\mathbb{M}^3)} \ \le \ C\|\beta\|_{{}_1L^2(\Omega)}\,. \tag{5.32}$$

Next, from (5.27) and (5.31),

$$\begin{aligned}
\tau_{12} =&\ 2\beta \ - \ \frac{1}{r}\int^r \partial_z(r\,\tau_{22})\,dr \ = \ 2\beta \ - \ \partial_z\left(\frac{1}{r}\int^r r\,\tau_{22}\,dr\right) \\
=&\ 2\beta \ - \ \partial_z\sigma_{rz}\,. \\
\Rightarrow&\ \tau_{12} \in {}_1L^2(\Omega)\,, \quad \text{with } \|\tau_{12}\|_{{}_1L^2(\Omega)} \ \le \ C\|\beta\|_{{}_1L^2(\Omega)}\,.
\end{aligned} \tag{5.33}$$

Also, from (5.21) and (5.33),

$$\tau_{21} \ = \ \tau_{12} - 2\beta \ \in {}_1L^2(\Omega)\,, \quad \text{with } \|\tau_{21}\|_{{}_1L^2(\Omega)} \ \le \ C\|\beta\|_{{}_1L^2(\Omega)}\,. \tag{5.34}$$

Finally, we confirm that (5.22) and (5.23) are satisfied.

$$\begin{aligned}
\partial_r\tau_{11} + \frac{1}{r}\tau_{11} + \partial_z\tau_{12} - \frac{1}{r}\tau =&\ \frac{1}{r}\partial_r\big(r\,\tau_{11}\big) \ + \ \partial_z\tau_{12} \ - \ \frac{1}{r}\tau \\
=&\ \frac{1}{r}\partial_r\big(r\,\partial_z(-\sigma_{rr})\big) \ + \ \partial_z\big(2\beta \ - \ \partial_z\sigma_{rz}\big) \ - \ \frac{1}{r}\big(- \partial_z\sigma_{\theta\theta}\big) \\
=&\ -\partial_z\big(\frac{1}{r}\partial_r(r\,\sigma_{rr}) \ + \ \partial_z\sigma_{rz} \ - \ \frac{1}{r}\sigma_{\theta\theta} \ - \ 2\beta\big) \\
=&\ -\partial_z\big(0\big) \ = \ 0 \quad (\text{from } (5.30))\ .
\end{aligned}$$

Also,

$$\begin{aligned}
\partial_r\tau_{21} + \frac{1}{r}\tau_{21} + \partial_z\tau_{22} =&\ \frac{1}{r}\partial_r\big(r\,\tau_{21}\big) \ + \ \partial_z\tau_{12} \\
=&\ \frac{1}{r}\partial_r\big(r\,(-\partial_z\sigma_{rz})\big) \ + \ \partial_z\big(\frac{1}{r}\partial_r(r\sigma_{rz})\big) \\
=&\ -\partial_z\big(\frac{1}{r}\partial_r(r\,\sigma_{rz})\big) \ + \ \partial_z\big(\frac{1}{r}\partial_r(r\sigma_{rz})\big) \ = \ 0\,.
\end{aligned}$$

This completes the proof. ■

The following lemma established the inf-sup condition (5.17).

Lemma 5.4. *For any* $\mathbf{v} \in U$ *and* $p \in Q$*, there exists a* $C > 0$ *and a* $(\underline{\boldsymbol{\tau}}, \tau) \in \mathbf{\Sigma}$ *such that*

$$b((\underline{\boldsymbol{\tau}}, \tau), \mathbf{v}) + c((\underline{\boldsymbol{\tau}}, \tau), p) = \|\mathbf{v}\|_U^2 + \|p\|_Q^2 \,, \tag{5.35}$$

$$\text{with} \quad \|(\underline{\boldsymbol{\tau}}, \tau)\|_{\mathbf{\Sigma}} \le C(\|\mathbf{v}\|_U + \|p\|_Q) \,. \tag{5.36}$$

Proof. Let $\mathbf{v} = (v_1, v_2)^t \in U$ and $p \in Q$ be given. Then, there exist vectors $\mathbf{w}, \mathbf{z} \in {}_1\mathbf{VH}^1(\Omega)$ such that

$$\nabla_{\text{axi}} \cdot \mathbf{w} = v_1 \text{ and } \nabla_{\text{axi}} \cdot \mathbf{z} = v_2 \tag{5.37}$$

where $\|\nabla_{\text{axi}} \cdot \mathbf{w}\|_{{}_1L^2(\Omega)} + \|\mathbf{w}\|_{{}_1\mathbf{VH}^1(\Omega)} \le C\, \|v_1\|_{{}_1L^2(\Omega)}$ and $\|\nabla_{\text{axi}} \cdot \mathbf{z}\|_{{}_1L^2(\Omega)} + \|\mathbf{z}\|_{{}_1\mathbf{VH}^1(\Omega)} \le C\, \|v_2\|_{{}_1L^2(\Omega)}$. To compute the vectors $\mathbf{w}$ and $\mathbf{z}$, one can map the axisymmetric scalar functions v_1 and v_2 into 3D Cartesian space and solve scalar Laplace equations to obtain functions t_1 and t_2. The gradient functions $\breve{\mathbf{w}} = \nabla_{(x,y,z)} t_1$ and $\breve{\mathbf{z}} = \nabla_{(x,y,z)} t_2$ are then computed. Finally, using Lemma 4.1, mapping $\breve{\mathbf{w}}$ and $\breve{\mathbf{z}}$ from $\breve{\Omega}$ to Ω, we obtain $\mathbf{w}$ and $\mathbf{z}$.

Using $\mathbf{w}$ and $\mathbf{z}$, we then construct a matrix $\underline{\boldsymbol{\tau}}^1$, where

$$\underline{\boldsymbol{\tau}}^1 = \begin{pmatrix} \mathbf{w}^t \\ \mathbf{z}^t \end{pmatrix}. \tag{5.38}$$

Taking $\tau^1 = 0$, and using (5.37) and (5.38),

$$\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}}^1, \tau^1) = \begin{pmatrix} \nabla_{\text{axi}} \cdot \mathbf{w} \\ \nabla_{\text{axi}} \cdot \mathbf{z} \end{pmatrix} = \mathbf{v}, \quad \text{hence} \quad b((\underline{\boldsymbol{\tau}}^1, \tau^1), \mathbf{v}) = \|\mathbf{v}\|_U^2, \tag{5.39}$$

$$\text{and } \|(\underline{\boldsymbol{\tau}}^1, \tau^1)\|_{\mathbf{\Sigma}} \le \|(\underline{\boldsymbol{\tau}}^1, 0)\|_{\boldsymbol{S}} \le C\|\mathbf{v}\|_U \le C(\|\mathbf{v}\|_U + \|p\|_Q). \tag{5.40}$$

To build $(\underline{\boldsymbol{\tau}}^2, \tau^2) \in \mathbf{\Sigma}$, we first choose $\theta, \gamma \in {}_1L^2(\Omega)$ such that

$$\mathcal{S}^2(\theta) = as(\underline{\boldsymbol{\tau}}^1), \quad \text{and } \gamma = \frac{1}{2}(v_1 z - v_2 r) = \frac{1}{2}(\nabla_{\text{axi}} \cdot \underline{\boldsymbol{\tau}}^1 \wedge \mathbf{x}). \tag{5.41}$$

Next, set $\beta = (\gamma + \theta - \frac{1}{2}p) \in {}_1L^2(\Omega)$. Note that

$$\begin{aligned} \|\theta\|_{{}_1L^2(\Omega)}^2 &= \int_\Omega (\tau_{12} - \tau_{21})^2 \, r \, d\Omega \le 2 \int_\Omega (\tau_{11}^2 + \tau_{12}^2 + \tau_{21}^2 + \tau_{22}^2) \, r \, d\Omega \\ &\le 2\|(\underline{\boldsymbol{\tau}}^1, \tau^1)\|_{\mathbf{\Sigma}} \le C(\|\mathbf{v}\|_U^2 + \|p\|_Q^2) \end{aligned}$$

Also,

$$\begin{aligned} \|\gamma\|_{{}_1L^2(\Omega)}^2 &= \int_\Omega \frac{1}{4}(v_1 z - v_2 r)^2 \, r \, d\Omega \le \frac{1}{2} \int_\Omega (v_1^2 z^2 + v_2^2 r^2) \, r \, d\Omega \\ &\le C \int_\Omega \mathbf{v} \cdot \mathbf{v} \, r \, d\Omega \;=\; C\|\mathbf{v}\|_U^2 \,. \end{aligned}$$

Therefore,

$$\|\beta\|_{{}_1L^2(\Omega)} \le C(\|p\|_Q + \|\theta\|_{{}_1L^2(\Omega)} + \|\gamma\|_{{}_1L^2(\Omega)}) \ \le \ C(\|\mathbf{v}\|_U + \|p\|_Q). \tag{5.42}$$

Next, $(\underline{\boldsymbol{\tau}}^2, \tau^2) \in \boldsymbol{\Sigma}$ is constructed using Lemma 5.3, with $\beta \to -\beta$. For such a $(\underline{\boldsymbol{\tau}}^2, \tau^2)$, it follows that

$$\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}}^2, \tau^2) = \mathbf{0}, \quad \text{and hence} \quad b((\underline{\boldsymbol{\tau}}^2, \tau^2), \mathbf{v}) = 0\,. \tag{5.43}$$

Also, as

$$\text{as}(\underline{\boldsymbol{\tau}}^2) \ = \ \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}, \quad \text{then} \quad \left(\text{as}(\underline{\boldsymbol{\tau}}^2)\,,\, \mathcal{S}^2(p)\right) \ = \ \left(\mathcal{S}^2(-\beta)\,,\, \mathcal{S}^2(p)\right)\,, \tag{5.44}$$

$$\text{and,} \quad \|(\underline{\boldsymbol{\tau}}^2, \tau^2)\|_{\boldsymbol{\Sigma}} \le C\|\beta\|_{{}_1L^2(\Omega)} \le C(\|\mathbf{v}\|_U + \|p\|_Q)\,. \tag{5.45}$$

Thus, for $(\underline{\boldsymbol{\tau}}, \tau) = (\underline{\boldsymbol{\tau}}^1, \tau^1) + (\underline{\boldsymbol{\tau}}^2, \tau^2)$ (using (5.39) and (5.43))

$$b((\underline{\boldsymbol{\tau}}, \tau), \mathbf{v}) = b((\underline{\boldsymbol{\tau}}^1, \tau^1), \mathbf{v}) + b((\underline{\boldsymbol{\tau}}^2, \tau^2), \mathbf{v}) = \|\mathbf{v}\|_U^2, \tag{5.46}$$

and

$$\begin{aligned}
c((\underline{\boldsymbol{\tau}}, \tau), p) &= c((\underline{\boldsymbol{\tau}}^1, \tau^1), p) + c((\underline{\boldsymbol{\tau}}^2, \tau^2), p) \\
&= (\underline{\boldsymbol{\tau}}^1, \mathcal{S}^2(p)) + (\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}}^1, \tau^1) \wedge \mathbf{x}, p) + (\underline{\boldsymbol{\tau}}^2, \mathcal{S}^2(p)) + (\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}}^2, \tau^2) \wedge \mathbf{x}, p) \\
&= (\underline{\boldsymbol{\tau}}^1, \mathcal{S}^2(p)) + (\nabla_{\text{axi}} \cdot \underline{\boldsymbol{\tau}}^1 \wedge \mathbf{x}, p) + (\underline{\boldsymbol{\tau}}^2, \mathcal{S}^2(p)), \quad (\text{using } \nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}}^2, \tau^2) = \mathbf{0}) \\
&= (as(\underline{\boldsymbol{\tau}}^1), \mathcal{S}^2(p)) + 2(\gamma, p) + (as(\underline{\boldsymbol{\tau}}^2), \mathcal{S}^2(p)), \quad (\text{using } (5.41)) \\
&= (\mathcal{S}^2(\theta), \mathcal{S}^2(p)) + (\mathcal{S}^2(\gamma), \mathcal{S}^2(p)) + (\mathcal{S}^2(-\beta), \mathcal{S}^2(p)), \quad (\text{using } (5.44)) \\
&= (\mathcal{S}^2(\tfrac{1}{2}p), \mathcal{S}^2(p)) \\
&= \|p\|_Q^2.
\end{aligned} \tag{5.47}$$

Thus, from (5.46) and (5.47), $(\underline{\boldsymbol{\tau}}, \tau)$ satisfies (5.35), and using (5.40) and (5.45),

$$\|(\underline{\boldsymbol{\tau}}, \tau)\|_{\boldsymbol{\Sigma}} \le \|(\underline{\boldsymbol{\tau}}^1, \tau^1)\|_{\boldsymbol{\Sigma}} + \|(\underline{\boldsymbol{\tau}}^2, \tau^2)\|_{\boldsymbol{\Sigma}} \ \le \ C(\|\mathbf{v}\|_Q + \|p\|_Q). \tag{5.48}$$

■

6 Discrete Axisymmetric Variational Formulation

In this section we present the setting for the approximation of (5.14)-(5.16). We begin by introducing the approximation spaces used:

$$\boldsymbol{\Sigma}_h := \Sigma_{h,\underline{\boldsymbol{\sigma}}} \times \Sigma_{h,\sigma} = \{(\underline{\boldsymbol{\sigma}}_h, \sigma_h) : \underline{\boldsymbol{\sigma}}_h \in \Sigma_{h,\underline{\boldsymbol{\sigma}}}, \sigma_h \in \Sigma_{h,\sigma}\} \subset \boldsymbol{\Sigma}\,, \quad U_h \subset U, \quad \text{and} \quad Q_h \subset Q\,. \tag{6.1}$$

We assume that there exists a piecewise polynomial space $(\Theta_h)^2$ such that $((\Theta_h)^2, Q_h)$ is a stable axisymmetric Stokes pair. Additionally we assume that the solution, $\mathbf{w}_h = (w_{h1}\,,\, w_{h2})^t \in (\Theta_h)^2$

to the modified discrete axisymmetric Stokes problem: *Given* $\beta \in {}_1L^2(\Omega)$, *determine* $(\mathbf{w}_h, p_h) \in ((\Theta_h)^2, Q_h)$, *such that for all* $(\mathbf{v}_h, q_h) \in ((\Theta_h)^2, Q_h)$

$$(\nabla \mathbf{w}_h \,:\, \nabla \mathbf{v}_h) \;+\; (\frac{1}{r} w_{h1}\,,\, \frac{1}{r} v_{h1}) \;+\; (p_h, \nabla_{\mathrm{axi}} \cdot \mathbf{v}_h) + \sum_{T \in \mathcal{T}_h} \left(r \frac{\partial^2 w_{h2}}{\partial z^2}, r \frac{\partial^2 v_{h2}}{\partial z^2} \right)_T = 0, \tag{6.2}$$

$$(\nabla_{\mathrm{axi}} \cdot \mathbf{w}_h, q_h) = (\beta, q_h), \tag{6.3}$$

satisfies

$$\|\mathbf{w}_h\|_{{}_1\mathbf{VH}^1(\Omega)} \;+\; \left(\sum_{T \in \mathcal{T}_h} \left\| r \frac{\partial^2 w_{h2}}{\partial z^2} \right\|^2_{{}_1L^2(T)} \right)^{\frac{1}{2}} \leq C \, \|\beta\|_{{}_1L^2(\Omega)} \,. \tag{6.4}$$

Remark: The discrete space $(\Theta_h)^2$ is a subspace of ${}_1\mathbf{VH}^1(\Omega)$.

The discrete axisymmetric meridan problem with weak symmetry is then: *Given* $\mathbf{f} \in {}_1\mathbf{L}^2(\Omega)$ *find* $((\underline{\boldsymbol{\sigma}}_h, \sigma_h), \mathbf{w}_h, p_h) \in \mathbf{\Sigma}_h \times U_h \times Q_h$ *such that for all* $((\underline{\boldsymbol{\tau}}_h, \tau_h), \mathbf{v}_h, q_h) \in \mathbf{\Sigma}_h \times U_h \times Q_h$

$$a((\underline{\boldsymbol{\sigma}}_h, \sigma_h), (\underline{\boldsymbol{\tau}}_h, \tau_h)) + b((\underline{\boldsymbol{\tau}}_h, \tau_h), \mathbf{w}_h) + c((\underline{\boldsymbol{\tau}}_h, \tau_h), p_h) = (\mathbf{f}, \nabla_{\mathrm{axi}} \cdot (\underline{\boldsymbol{\tau}}_h, \tau_h)) \tag{6.5}$$

$$b((\underline{\boldsymbol{\sigma}}_h, \sigma_h), \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \tag{6.6}$$

$$c((\underline{\boldsymbol{\sigma}}_h, \sigma_h), q_h) = (\mathbf{f} \wedge \mathbf{x}, q_h)\,. \tag{6.7}$$

6.1 Well posedness of the discrete variational formulation (6.5)-(6.7)

Analogous to the continuous formulation, the well posedness of (6.5)-(6.7) relies on the boundedness and coercivity of $a(\cdot,\cdot)$ on $\mathbf{\Sigma}_h \times \mathbf{\Sigma}_h$ and that $\mathbf{\Sigma}_h \times (U_h \times Q_h)$ satisfy the inf-sup condition

$$\inf_{\mathbf{v}_h \in U_h, p_h \in Q_h} \; \sup_{(\underline{\boldsymbol{\tau}}_h, \tau_h) \in \mathbf{\Sigma}_h} \frac{b((\underline{\boldsymbol{\tau}}_h, \tau_h), \mathbf{v}_h) + c((\underline{\boldsymbol{\tau}}_h, \tau_h), p_h)}{\|(\underline{\boldsymbol{\tau}}, \tau)\|_{\mathbf{\Sigma}} (\|\mathbf{v}_h\|_U + \|p_h\|_Q)} \geq C. \tag{6.8}$$

To establish (6.8) we use Fortin's Lemma [13]. Given $\mathbf{u}_h \in U_h \subset U$, $p_h \in Q_h \subset Q$ we determine, as in the proof of Lemma 5.4, a $(\underline{\boldsymbol{\tau}}, \tau) = (\underline{\boldsymbol{\tau}}^1, \tau^1) + (\underline{\boldsymbol{\tau}}^2, \tau^2)$ such that the continuous inf-sup condition is satisfied. Then, using a suitably defined projection (see (6.14)-(6.16)), we obtain $(\underline{\boldsymbol{\tau}}_h, \tau_h) \in \mathbf{\Sigma}_h$ satisfying (6.8).

Helpful in this discussion is to define the restriction of the operators $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$ to $T \in \mathcal{T}_h$:

$$b((\underline{\boldsymbol{\tau}}, \tau), \mathbf{u})_T = (\nabla_{\mathrm{axi}} \cdot \underline{\boldsymbol{\tau}}, \mathbf{u})_T - (\frac{\tau}{r}, u_r)_T \tag{6.9}$$

$$c((\underline{\boldsymbol{\tau}}, \tau), p)_T = (as(\underline{\boldsymbol{\tau}}), \mathcal{S}^2(p))_T + ((\nabla_{\mathrm{axi}} \cdot (\underline{\boldsymbol{\tau}}, \tau)) \wedge \mathbf{x}, p)_T. \tag{6.10}$$

Next, we present the following identity for the operator $c(\cdot,\cdot)_T$.

Lemma 6.1. *For* $T \in \mathcal{T}_h$,

$$c((\underline{\boldsymbol{\tau}}, \tau), p)_T = \int_{\partial T} (\underline{\boldsymbol{\tau}} \cdot \mathbf{n}) \cdot \mathbf{x}^{\perp} \; p \; r \; ds - \int_T \underline{\boldsymbol{\tau}} : (\mathbf{x}^{\perp} \otimes \nabla p) \; r \; dT - \int_T \tau \; z \; p \; dT. \tag{6.11}$$

Proof. See Section A in the appendix. ■

Theorem 6.1. *Assume* $\mathbf{\Sigma}_h, U_h, Q_h$ *satisfy* (6.1). *Let*

$$\boldsymbol{S}_{\Theta_h} = \left\{ (\underline{\boldsymbol{\tau}}, \tau) \,:\, \underline{\boldsymbol{\tau}} \,=\, \begin{pmatrix} \dfrac{\partial w_{h1}}{\partial z} & -\dfrac{1}{r}\dfrac{\partial}{\partial r}(r\, w_{h1}) - \dfrac{\partial w_{2h}}{\partial z} \\ 0 & 0 \end{pmatrix} \,,\ \tau \,=\, r\,\frac{\partial^2\, w_{h2}}{\partial z^2};\ \mathbf{w}_h = (w_{h1}, w_{h2})^t \in (\Theta_h)^2 \right\}.$$

If there exists a mapping $\mathbf{\Pi}_h = \Pi_h \times \pi_h : (\boldsymbol{S} + \boldsymbol{S}_{\Theta_h}) \to \mathbf{\Sigma}_h$ *such that :*

$$\|\Pi_h \times \pi_h(\underline{\boldsymbol{\tau}}, \tau)\|_{\mathbf{\Sigma}} \leq\ C\,\|(\underline{\boldsymbol{\tau}}, \tau)\|_{\boldsymbol{S}}, \quad \forall (\underline{\boldsymbol{\tau}}, \tau) \in \boldsymbol{S}, \tag{6.12}$$

$$\|\Pi_h \times \pi_h(\underline{\boldsymbol{\tau}}, \tau)\|_{\mathbf{\Sigma}} \leq\ C\,\|\mathbf{w}_h\|_{{}_1\mathbf{VH}^1(\Omega)}\ +\ \left(\sum_{T\in\mathcal{T}_h} \left\| r\frac{\partial^2 w_{h2}}{\partial z^2} \right\|^2_{{}_1L^2(T)}\right)^{\frac{1}{2}}, \quad \forall (\underline{\boldsymbol{\tau}}, \tau) \in \boldsymbol{S}_{\Theta_h}, \tag{6.13}$$

and for all $T \in \mathcal{T}_h$

$$\int_T (\underline{\boldsymbol{\tau}} - \Pi_h\underline{\boldsymbol{\tau}}) : (\nabla\mathbf{u}_h + \mathbf{x}^\perp \otimes \nabla q_h)\, r\, dT \ =\ 0, \quad \forall \mathbf{u}_h \in U_h,\ \forall\, q_h \in Q_h, \tag{6.14}$$

$$\int_\ell ((\underline{\boldsymbol{\tau}} - \Pi_h\underline{\boldsymbol{\tau}}) \cdot \mathbf{n}_K) \cdot (\mathbf{u}_h + \mathbf{x}^\perp q_h)\, r\, ds \ =\ 0, \quad \forall \text{ edges } \ell,\ \forall\, \mathbf{u}_h \in U_h,\ \forall\, q_h \in Q_h, \tag{6.15}$$

$$\int_T \frac{1}{r}(\tau - \pi_h\tau)\, \sigma\, r\, dT \ =\ 0, \quad \forall\, \sigma \in \{z\, q_h \,:\, q_h \in Q_h\} \cup \{v_{h1} \,:\, (v_{h1}\,,\, 0) \in U_h\}\,, \tag{6.16}$$

then $\mathbf{\Sigma}_h \times (U_h \times Q_h)$ *are inf-sup stable.*

Proof. The approach to this proof is similar to that used in [12]. Let $\mathbf{v}_h = (v_{h1}, v_{h2})^t \in U_h \subset {}_1\mathbf{L}^2(\Omega, \mathbb{R}^2)$ and $p_h \in Q_h \subset {}_1L^2(\Omega)$.

Recall from Lemma 5.4 that for $\mathbf{v}_h$ given, there exists $(\underline{\boldsymbol{\tau}}^1, 0) \in \boldsymbol{S}(\Omega) \subset \mathbf{\Sigma}(\Omega)$ such that

$$\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}}^1, 0) = \mathbf{v}_h \quad \text{ and } \quad \|(\underline{\boldsymbol{\tau}}^1, 0)\|_{\boldsymbol{S}} \leq C\,(\|\mathbf{v}_h\|_U + \|p_h\|_Q)\,. \tag{6.17}$$

Also, from (5.39), $b((\underline{\boldsymbol{\tau}}^1, 0), \mathbf{v}_h) = (\mathbf{v}_h, \mathbf{v}_h)$, hence

$$\begin{aligned} b((\underline{\boldsymbol{\tau}}^1, 0) - \mathbf{\Pi}_h(\underline{\boldsymbol{\tau}}^1, 0), \mathbf{v}_h) &= b((\underline{\boldsymbol{\tau}}^1 - \Pi_h\underline{\boldsymbol{\tau}}^1, 0), \mathbf{v}_h) \\ &= \sum_{T\in\mathcal{T}_h} (\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}}^1 - \Pi_h\underline{\boldsymbol{\tau}}^1), \mathbf{v}_h)_T + (\frac{0}{r}, v_{h1})_T \\ &= 0 \ \ (\text{using (6.14)-(6.15) with } q_h = 0). \end{aligned} \tag{6.18}$$

Furthermore, from (6.12) and (6.17),

$$\|\mathbf{\Pi}_{\boldsymbol{h}}(\underline{\boldsymbol{\tau}}^1, 0)\|_{\mathbf{\Sigma}} \leq C\|(\underline{\boldsymbol{\tau}}^1, 0)\|_{\boldsymbol{S}} \leq C(\|\mathbf{v}_h\|_U + \|p_h\|_Q). \tag{6.19}$$

Next, combining (6.11) with (6.14)-(6.15) (with $\mathbf{u}_h = 0$),

$$\begin{aligned} c((\underline{\boldsymbol{\tau}}^1, 0) - \mathbf{\Pi}_{\boldsymbol{h}}(\underline{\boldsymbol{\tau}}^1, 0), p_h) \ &=\ \sum_{T\in\mathcal{T}_h} c((\underline{\boldsymbol{\tau}}^1 - \Pi_h\underline{\boldsymbol{\tau}}^1, 0), p_h)_T \\ = \sum_{T\in\mathcal{T}_h} \Bigg(\int_{\partial T} ((\underline{\boldsymbol{\tau}}^1 - \Pi_h\underline{\boldsymbol{\tau}}^1) \cdot \mathbf{n}_T) \cdot \mathbf{x}^\perp\, p_h\, r\, ds \ &-\ \int_T (\underline{\boldsymbol{\tau}}^1 - \Pi_h\underline{\boldsymbol{\tau}}^1) : (\mathbf{x}^\perp \otimes \nabla\, p_h)\, r\, dT \\ - \int_T \frac{1}{r}\, 0\, z\, p_h\, r\, dT \Bigg) \ &=\ 0. \end{aligned} \tag{6.20}$$

The $(\underline{\boldsymbol{\tau}}^2, \tau^2)$ used in establishing the continuous inf-sup condition is not sufficiently regular in order to construct a suitable projection. To circumvent this problem we use (6.2),(6.3) to determine a suitable replacement for $(\underline{\boldsymbol{\tau}}^2, \tau^2)$, namely $(\underline{\boldsymbol{\tau}}_h^2, \tau_h^2)$, and then use a projection of $(\underline{\boldsymbol{\tau}}_h^2, \tau_h^2)$ to help satisfy (6.8).

Let $\mathbf{w}_h \in (\Theta_h)^2$ be determined by (6.2),(6.3), and define

$$\underline{\boldsymbol{\tau}}_h^2 \;=\; 2 \begin{pmatrix} \dfrac{\partial w_{h1}}{\partial z} & -\dfrac{1}{r}\dfrac{\partial}{\partial r}(r\; w_{h1}) - \dfrac{\partial w_{h2}}{\partial z} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tau_h^2 = -2\;\pi_h\left(r\;\frac{\partial^2\; w_{h2}}{\partial z^2}\right).$$

Then, (cf. (5.43)-(5.45))

$$\begin{aligned}
b((\underline{\boldsymbol{\tau}}_h^2, \tau_h^2), \mathbf{v}_h) &= \sum_{T\in\mathcal{T}_h} \Big(\nabla_{\mathrm{axi}} \cdot \underline{\boldsymbol{\tau}}_{h\,1}^2 \;-\; \frac{1}{r}\tau_h^2), v_{h1})_T \\
&= \sum_{T\in\mathcal{T}_h} \left(\nabla_{\mathrm{axi}} \cdot (\frac{\partial w_{h1}}{\partial z}\,,\, -\frac{1}{r}\frac{\partial}{\partial r}(r\; w_{h1}) - \frac{\partial w_{h2}}{\partial z}) \;-\; \frac{1}{r}(-2\;\pi_h(r\;\frac{\partial^2\; w_{h2}}{\partial z^2})), v_{h1}\right)_T \\
&= \sum_{T\in\mathcal{T}_h} 2\,(\frac{\partial^2 w_{h1}}{\partial r \partial z} + \frac{1}{r}\frac{\partial w_{h1}}{\partial z} - \frac{1}{r}\frac{\partial w_{h1}}{\partial z} - \frac{\partial^2 w_{h1}}{\partial r \partial z} - \frac{\partial^2 w_{h2}}{\partial^2 z} + \frac{\partial^2 w_{h2}}{\partial^2 z}, v_{h1})_T \qquad \text{(using (6.16))} \\
&= 0 \;=\; b((\underline{\boldsymbol{\tau}}^2, \tau^2), \mathbf{v}_h)\,. \qquad (6.21)
\end{aligned}$$

From (6.3)),

$$\big(\mathrm{as}(\underline{\boldsymbol{\tau}}_h^2)\,,\,\mathcal{S}^2(p_h)\big) = \;\big(\mathcal{S}^2(-\beta)\,,\,\mathcal{S}^2(p_h)\big)\;. \qquad (6.22)$$

For $c((\underline{\boldsymbol{\tau}}_h^2, \tau_h^2), p_h)$ we have

$$\begin{aligned}
c((\underline{\boldsymbol{\tau}}_h^2, \tau_h^2), p_h) &= (\underline{\boldsymbol{\tau}}_h^2, \mathcal{S}^2(p_h)) + (\nabla_{\mathrm{axi}} \cdot (\underline{\boldsymbol{\tau}}_h^2, \tau_h^2) \wedge \mathbf{x}, p_h) \\
&= \;(as(\underline{\boldsymbol{\tau}}_h^2, \mathcal{S}^2(p_h)) \;+\; \sum_{T\in\mathcal{T}_h} \Big(\nabla_{\mathrm{axi}} \cdot \underline{\boldsymbol{\tau}}_{h\,1}^2 \;-\; \frac{1}{r}\tau_h^2), z\,p_h)_T \\
&= \;(\mathcal{S}^2(-\beta), \mathcal{S}^2(p_h)) \;+\; 0 \quad \text{(as in (6.21))} \\
&= \;c((\underline{\boldsymbol{\tau}}^2, \tau^2), p_h)\,.
\end{aligned}$$

Using (6.13) and (6.4)

$$\|\tau_h^2\|_{{}_1L^2(\Omega)} = \;\|2\;\pi_h\left(r\;\frac{\partial^2\; w_{h2}}{\partial z^2}\right)\|_{{}_1L^2(\Omega)} \;\le\; C\left(\sum_{T\in\mathcal{T}_h}\left\|r\frac{\partial^2 w_{h2}}{\partial z^2}\right\|_{{}_1L^2(T)}^2\right)^{\frac{1}{2}} \;\le\; C\,\beta\,.$$

Then, from (6.4) and $\|\beta\|_{{}_1L^2(\Omega)} \le C\,(\|\mathbf{v}_h\|_U + \|p_h\|_Q)$, it follows that

$$\|(\underline{\boldsymbol{\tau}}_h^2, \tau_h^2)\|_{\boldsymbol{\Sigma}} \le C\,(\|\mathbf{v}_h\|_U + \|p_h\|_Q)\,. \qquad (6.23)$$

Now, for $(\Pi_h \underline{\boldsymbol{\tau}}_h^2, \tau_h^2) \in \boldsymbol{\Sigma}_h$, proceeding as in (6.18); using (6.14)-(6.15) (with $q_h = 0$), and (6.16)

$$\begin{aligned} b((\underline{\boldsymbol{\tau}}_h^2, \tau_h^2) - (\Pi_h \underline{\boldsymbol{\tau}}_h^2, \tau_h^2) \ , \ \mathbf{v}_h) \ &= \ b((\underline{\boldsymbol{\tau}}_h^2 - \Pi_h \underline{\boldsymbol{\tau}}_h^2, 0) \ , \ \mathbf{v}_h) \\ &= \sum_{T \in \mathcal{T}_h} \left(\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}}_h^2 - \Pi_h \underline{\boldsymbol{\tau}}_h^2), \mathbf{v}_h \right)_T \ = \ 0. \end{aligned} \tag{6.24}$$

Also, as in (6.20), and using (6.16),

$$\begin{aligned} c((\underline{\boldsymbol{\tau}}_h^2, \tau_h^2) - (\Pi_h \underline{\boldsymbol{\tau}}_h^2, \tau_h^2), p_h) \ &= \ \sum_{T \in \mathcal{T}_h} c((\underline{\boldsymbol{\tau}}_h^2 - \Pi_h \underline{\boldsymbol{\tau}}_h^2), 0), p_h)_T \\ = \sum_{T \in \mathcal{T}_h} \Big(\int_{\partial T} ((\underline{\boldsymbol{\tau}}_h^2 - \Pi_h \underline{\boldsymbol{\tau}}_h^2) \cdot \mathbf{n}_T) \cdot \mathbf{x}^\perp \ p_h \ r \ ds \ &- \ \int_T (\underline{\boldsymbol{\tau}}_h^2 - \Pi_h \underline{\boldsymbol{\tau}}_h^2) : (\mathbf{x}^\perp \otimes \nabla p_h) \ r \ dT \ = 0. \end{aligned} \tag{6.25}$$

Finally, with $(\underline{\boldsymbol{\tau}}_h, \tau_h) = (\Pi_h \underline{\boldsymbol{\tau}}^1, \pi_h \tau^1) + (\Pi_h \underline{\boldsymbol{\tau}}_h^2, \tau_h^2) \in \boldsymbol{\Sigma}_h$,

$$\begin{aligned} &\sup_{(\underline{\boldsymbol{\sigma}}_h, \sigma_h) \in \boldsymbol{\Sigma}_h} \frac{b((\underline{\boldsymbol{\sigma}}_h, \sigma_h), \mathbf{v}_h) + c((\underline{\boldsymbol{\sigma}}_h, \sigma_h), p_h)}{\|(\underline{\boldsymbol{\sigma}}_h, \sigma_h)\|_{\boldsymbol{\Sigma}} \, (\|\mathbf{v}_h\|_U + \|p_h\|_Q)} \geq \frac{b((\underline{\boldsymbol{\tau}}_h, \tau_h), \mathbf{v}_h) + c((\underline{\boldsymbol{\tau}}_h, \tau_h), p_h)}{\|(\underline{\boldsymbol{\tau}}_h, \tau_h)\|_{\boldsymbol{\Sigma}} \, (\|\mathbf{v}_h\|_U + \|p_h\|_Q)} \\ &\geq \frac{b((\Pi_h \underline{\boldsymbol{\tau}}^1, \pi_h \tau^1), \mathbf{v}_h) + c((\Pi_h \underline{\boldsymbol{\tau}}^1, \pi_h \tau^1), p_h) + b((\Pi_h \underline{\boldsymbol{\tau}}_h^2, \tau_h^2), \mathbf{v}_h) + c((\Pi_h \underline{\boldsymbol{\tau}}_h^2, \tau_h^2), p_h)}{\left(\|(\Pi_h \underline{\boldsymbol{\tau}}^1, \pi_h \tau^1)\|_{\boldsymbol{\Sigma}} + \|(\Pi_h \underline{\boldsymbol{\tau}}_h^2, \tau_h^2)\|_{\boldsymbol{\Sigma}}\right) (\|\mathbf{v}_h\|_U + \|p_h\|_Q)} \\ &\geq C \, \frac{b((\underline{\boldsymbol{\tau}}^1, \tau^1), \mathbf{v}_h) + c((\underline{\boldsymbol{\tau}}^1, \tau^1), p_h) + b((\underline{\boldsymbol{\tau}}_h^2, \tau_h^2), \mathbf{v}_h) + c((\underline{\boldsymbol{\tau}}_h^2, \tau_h^2), p_h)}{\left(\|(\underline{\boldsymbol{\tau}}^1, \tau^1)\|_{\boldsymbol{S}} + \|(\underline{\boldsymbol{\tau}}_h^2, \tau_h^2)\|_{\boldsymbol{\Sigma}}\right) (\|\mathbf{v}_h\|_U + \|p_h\|_Q)} \\ &\geq C \, \frac{b((\underline{\boldsymbol{\tau}}^1, \tau^1), \mathbf{v}_h) + c((\underline{\boldsymbol{\tau}}^1, \tau^1), p_h) + b((\underline{\boldsymbol{\tau}}^2, \tau^2), \mathbf{v}_h) + c((\underline{\boldsymbol{\tau}}^2, \tau^2), p_h)}{\left(\|(\underline{\boldsymbol{\tau}}^1, \tau^1)\|_{\boldsymbol{S}} + \|(\underline{\boldsymbol{\tau}}_h^2, \tau_h^2)\|_{\boldsymbol{\Sigma}}\right) (\|\mathbf{v}_h\|_U + \|p_h\|_Q)} \\ &\geq C \, \frac{\|\mathbf{v}_h\|_U^2 + \|p_h\|_Q^2}{(\|\mathbf{v}_h\|_U + \|p_h\|_Q + \|\mathbf{v}_h\|_U + \|p_h\|_Q)(\|\mathbf{v}_h\|_U + \|p_h\|_Q)} \\ &\geq C. \end{aligned}$$

■

Throughout the remainder of this document, we will denote the space $(\boldsymbol{S} + \boldsymbol{S}_{\Theta_h})$ as $\boldsymbol{\Sigma}^S$. Additionally, we denote the tensor and scalar components of $\boldsymbol{\Sigma}^S$ as $\Sigma^S_{\underline{\boldsymbol{\sigma}}}$ and Σ^S_σ, i.e., $\boldsymbol{\Sigma}^S = \Sigma^S_{\underline{\boldsymbol{\sigma}}} \times \Sigma^S_\sigma$.

7 Approximation spaces satisfying (6.14),(6.15)

In this section we investigate approximation spaces for $\boldsymbol{\Sigma}^S$, U_h, and Q_h such that there exists a projection operator $\boldsymbol{\Pi_h}(\underline{\boldsymbol{\tau}}, \tau)$ satisfying (6.14)-(6.16).

7.1 $\boldsymbol{\Sigma}_h = (\mathbf{BDM}_1(\mathcal{T}_h))^2 \times P_1(\mathcal{T}_h)$, $U_h = (P_0(\mathcal{T}_h))^2$, and $Q_h = P_0(\mathcal{T}_h)$

In this section we show that for the choice of spaces $\Sigma_{h,\underline{\boldsymbol{\sigma}}} = (\mathbf{BDM}_1(\mathcal{T}_h))^2$, $\Sigma_{h,\sigma} = P_1(\mathcal{T}_h)$, $U_h = (P_0(\mathcal{T}_h))^2$, and $Q_h = P_0(\mathcal{T}_h)$ there exists a projection operator, $\boldsymbol{\Pi_h}(\underline{\boldsymbol{\tau}}, \tau)$, satisfying (6.14)-(6.16).

Lemma 7.1. *Let $T \in \mathcal{T}_h$. The mappings $\Pi_h : \Sigma^S_{\underline{\boldsymbol{\sigma}}}(T) \to (P_1(T))^4$ and $\pi_h : \Sigma^S_{\sigma}(T) \to P_1(T)$ given by*

$$\int_\ell (\underline{\boldsymbol{\tau}} - \Pi_h \underline{\boldsymbol{\tau}}) \cdot \mathbf{n}_k \cdot \mathbf{p}_1 \; r \; ds = 0 \text{ for all edges } \ell \in \partial T \text{ and } \mathbf{p}_1 \in (P_1(\ell))^2 \tag{7.1}$$

$$\int_T \frac{1}{r} (\tau - \pi_h \tau) \; p_1 \, r \; dT = 0 \text{ for all } p_1 \in P_1(T) \tag{7.2}$$

are well defined. Hence the spaces

$$\Sigma_{h,\underline{\boldsymbol{\sigma}}} = (\mathbf{BDM}_1(\mathcal{T}_h))^2 \qquad \Sigma_{h,\sigma} = P_1(\mathcal{T}_h) \qquad U_h = (P_0(\mathcal{T}_h))^2 \qquad Q_h = P_0(\mathcal{T}_h) \tag{7.3}$$

satisfy (6.14)-(6.16).

Proof. Observe that π_h is the well defined L^2 projection.

Next we show that Π_h is well defined. Note that $\Pi_h \underline{\boldsymbol{\tau}} \in (P_1(T))^4 = (\mathbf{BDM}_1(T))^2$ has 12 degrees of freedom, and $(P_1(\ell))^2$ has 4 degrees of freedom per edge. Thus the number of unknowns in $\Pi_h \underline{\boldsymbol{\tau}}$ is equal to the number of constraints in (7.1). It follows that if $\underline{\boldsymbol{\tau}} = \mathbf{0}$ implies that $\Pi_h \underline{\boldsymbol{\tau}} = \mathbf{0}$, then the projection Π_h is well defined.

Consider a single row of the tensor projection (7.1). In this case, for $\underline{\boldsymbol{\tau}} = (\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)^t$ the projection (7.1) takes the form

$$\int_\ell (\underline{\boldsymbol{\tau}}_s - \Pi_h \underline{\boldsymbol{\tau}}_s) \cdot \mathbf{n}_k \; p_1 \; r \; ds = 0 \text{ for } p_1 \in P_1(\ell), \;\; s = 1, 2. \tag{7.4}$$

Next, observe that the function $\Pi_h \underline{\boldsymbol{\tau}}_s \cdot \mathbf{n}_k \; p_1 \; r$ is a cubic polynomial. Recalling that a degree n Gauss quadrature rule integrates polynomials of degree $2n-1$ exactly, we select two Gauss quadrature points $\{q_i^{\ell_k}\}_{i=1}^2$ on each edge ℓ_k for $k = 1, 2, 3$.

For $\ell_k \in \partial K$, define a basis for $P_1(\ell_k)$ so that

$$p_1^{\ell_k}(x) = \begin{cases} 1 \text{ if } x = q_1^{\ell_k} \\ 0 \text{ if } x = q_2^{\ell_k} \end{cases} \quad \text{and} \quad p_2^{\ell_k}(x) = \begin{cases} 0 \text{ if } x = q_1^{\ell_k} \\ 1 \text{ if } x = q_2^{\ell_k} \end{cases} . \tag{7.5}$$

Let $\{\boldsymbol{\phi}_i^{\ell_k}\}$ be a basis for $\mathbf{BDM}_1(T)$ [17] such that

$$(\boldsymbol{\phi}_i^{\ell_m} \cdot \mathbf{n})(q_j^{\ell_n}) = \delta_{(i,j),(\ell_m,\ell_n)} \text{ for } i, j = 1, 2 \text{ and } m, n = 1, 2, 3.$$

Note that the normal component of the basis functions satisfy a Lagrangian property at the boundary quadrature points. Since $\Pi_h \underline{\boldsymbol{\tau}}_s \in \mathbf{BDM}_1(T)$, it can be written as

$$\Pi_h \underline{\boldsymbol{\tau}}_s = \sum_{k=1}^{3} \sum_{i=1}^{2} \alpha_i^{\ell_k} \boldsymbol{\phi}_i^{\ell_k}.$$

With $\underline{\boldsymbol{\tau}}_s = \mathbf{0}$, taking the basis function $p_1^{\ell_k}$ for $\ell_k \in \partial K$ and using (7.1) and Gaussian quadrature gives

$$\begin{aligned} 0 = \int_{\ell_k} \Pi_h \underline{\boldsymbol{\tau}}_s \cdot \mathbf{n}\, p_1\, r\, ds &= \sum_{j=1}^{2} (\Pi_h \underline{\boldsymbol{\tau}}_s \cdot \mathbf{n})(q_j^{\ell_k}) \cdot p_1^{\ell_k}(q_j^{\ell_k})\, r(q_j^{\ell_k})\, w(q_j^{\ell_k}) \\ &= \alpha_1^{\ell_k}\, p_1^{\ell_k}(q_1^{\ell_k})\, r(q_1^{\ell_k}) w(q_1^{\ell_k}) + \alpha_2^{\ell_k}\, p_1^{\ell_k}(q_2^{\ell_k})\, r(q_2^{\ell_k}) w(q_2^{\ell_k}) \\ &= \alpha_1^{\ell_k} r(q_1^{\ell_k}) w(q_1^{\ell_k}). \end{aligned}$$

In the case where $r(q_1^{\ell_k}) \neq 0$, this implies $\alpha_1^{\ell_k} = 0$. If, however, $r(q_1^{\ell_k}) = 0$, then $\alpha_1^{\ell_k}$ and $\beta_1^{\ell_k}$ must be zero, otherwise, the normal stress along the axis of symmetry will be non-zero implying that the solution is not axisymmetric. A similar argument can be used to show that the other α terms are also zero. Hence the vector projection from (7.4) is well defined.

To extend the vector projection from (7.4) to the tensor projection given in (7.1), we extend the basis for $P_1(\ell_k)$ from (7.5) to $(P_1(\ell_k))^2$ by using

$$(P_1(\ell_k))^2 \;=\; \text{span}\left\{ \begin{pmatrix} p_1^{\ell_k} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} p_2^{\ell_k} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ p_1^{\ell_k} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ p_2^{\ell_k} \end{pmatrix} \right\}.$$

With this basis, the arguments presented above for the vector case can be applied to each row of (7.1) to show that Π_h is well defined.

Lastly, we verify that the spaces given in (7.3) satisfy the conditions outlined in (6.14)-(6.16). Since gradients of the piecewise constant spaces U_h and Q_h are zero on each element T, (6.14) is trivially satisfied. Next, observe that the test space of (7.1) includes all $\mathbf{p}_1 \in (P_1(\ell_k))^2$ for $k = 1, 2, 3$, while (6.15) only requires that the projection is satisfied on a subspace of $(P_1(\ell_k))^2$. Finally, since $P_0(T) \subset P_1(T)$, (7.2) ensures that (6.16) is satisfied. ■

7.2 $\Sigma_h = (\mathbf{BDM}_2(\mathcal{T}_h))^2 \times P_2(\mathcal{T}_h)$, $U_h = (P_1(\mathcal{T}_h))^2$, and $Q_h = P_1(\mathcal{T}_h)$

In this section we show that for the choice of spaces $\Sigma_{h,\underline{\boldsymbol{\sigma}}} = (\mathbf{BDM}_2(\mathcal{T}_h))^2$, $\Sigma_{h,\sigma} = P_2(\mathcal{T}_h)$, $U_h = (P_1(\mathcal{T}_h))^2$, and $Q_h = P_1(\mathcal{T}_h)$ there exists a projection operator, $\boldsymbol{\Pi_h}(\underline{\boldsymbol{\tau}}, \tau)$, satisfying (6.14)-(6.16).

Lemma 7.2. *Let $T \in \mathcal{T}_h$. The projection operators $\Pi_h : \Sigma^S_{\underline{\boldsymbol{\sigma}}}(T) \to (P_2(T))^4$ and $\pi_h : \Sigma^S_{\sigma}(T) \to P_2(T)$ given by*

$$\int_T (\underline{\boldsymbol{\tau}} - \Pi_h \underline{\boldsymbol{\tau}}) : (\underline{\mathbf{p_0}} + \mathbf{x}^{\perp} \otimes \mathbf{p_0})\, r\, dT = 0 \quad \forall\, \underline{\mathbf{p_0}} \in (P_0(T))^{2\times 2} \quad \forall\, \mathbf{p_0} \in (P_0(T))^2 \tag{7.6}$$

$$\int_\ell (\underline{\boldsymbol{\tau}} - \Pi_h \underline{\boldsymbol{\tau}}) \cdot \mathbf{n}_k \cdot \mathbf{p}_2\, r\, ds = 0 \quad \forall \text{ edges } \ell \quad \forall\, \mathbf{p}_2 \in (P_2(\ell))^2 \tag{7.7}$$

$$\int_T \frac{1}{r} (\tau - \pi_h \tau)\, p_2\, r\, dT = 0 \text{ for all } p_2 \in P_2(T) \tag{7.8}$$

are well defined. Hence the spaces

$$\Sigma_{h,\underline{\boldsymbol{\sigma}}} = (\mathbf{BDM}_2(\mathcal{T}_h))^2 \qquad \Sigma_{h,\sigma} = P_2(\mathcal{T}_h) \qquad U_h = (P_1)^2(\mathcal{T}_h) \qquad Q_h = P_1(\mathcal{T}_h) \tag{7.9}$$

satisfy (6.14)-(6.16).

Proof. Observe that π_h is the well defined L^2 projection.

Next we show that Π_h is well defined. First observe that the number of constraints defined by Π_h, 24, is the same as number of degrees of freedom in $(P_2(T))^4 = (\mathbf{BDM}_2(T))^2$. We verify that the projection is injective by showing that

$$\int_T \Pi_h \underline{\boldsymbol{\tau}} : (\underline{\mathbf{p_0}} + \mathbf{x}^{\perp} \otimes \mathbf{p_0})\, r\, dT = 0 \quad \forall\, \underline{\mathbf{p_0}} \in (P_0(T))^{2\times 2} \quad \forall\, \mathbf{p_0} \in (P_0(T))^2 \tag{7.10}$$

$$\int_\ell \Pi_h \underline{\boldsymbol{\tau}} \cdot \mathbf{n}_k \cdot \mathbf{p}_2\, r\, ds = 0 \quad \forall \text{ edges } \ell \quad \forall\, \mathbf{p_2} \in (P_2(\ell))^2 \tag{7.11}$$

has the unique solution $\Pi_h \underline{\boldsymbol{\tau}} = \mathbf{0}$.

We can represent $\Pi_h \underline{\boldsymbol{\tau}}$ in terms of the basis for $(\mathbf{BDM}_2(\widehat{T}))^2$, where $\mathbf{BDM}_2(\widehat{T})$ is the reference element representation presented in [17, Section 4.2]. This $\mathbf{BDM}_2(\widehat{T})$ basis is expressed in terms of edge and interior element functions. Using equation (7.11) with three Gauss quadrature points and an argument analogous to that used in the proof of Lemma 7.1, it follows that all 18 of the $\mathbf{BDM}_2(\widehat{T})$ edge basis functions must equal zero.

Therefore, the only possible non-zero basis functions on $\widehat{T}$ are the interior element functions

$$\underline{\phi}_1 = \frac{\sqrt{2}}{(g_2 - g_1)}(1 - \xi - \eta)\begin{pmatrix} g_2\xi \\ (g_2-1)\eta \end{pmatrix} \quad \underline{\phi}_2 = \frac{1}{(g_2 - g_1)}\xi \begin{pmatrix} g_2\xi + \eta - g_2 \\ (g_2-1)\eta \end{pmatrix}$$
$$\underline{\phi}_3 = \frac{1}{(g_2 - g_1)}\eta \begin{pmatrix} (g_2-1)\xi \\ \xi + g_2\eta - g_2 \end{pmatrix} \tag{7.12}$$

where $g_1 = 1/2 - \sqrt{3}/6$ and $g_2 = 1/2 + \sqrt{3}/6$ are the Gaussian quadrature points on $[0, 1]$. Thus, $\widehat{\Pi_h \underline{\boldsymbol{\tau}}}$, the representation of $\Pi_h \underline{\boldsymbol{\tau}}$ on $\widehat{T}$, must have the form $\widehat{\Pi_h \underline{\boldsymbol{\tau}}} = \begin{pmatrix} \underline{\phi_\alpha}^t \\ \underline{\phi_\beta}^t \end{pmatrix}$ where

$$\underline{\phi_\alpha}^t = \alpha_1 \underline{\phi_1}^t + \alpha_2 \underline{\phi_2}^t + \alpha_3 \underline{\phi_3}^t \quad \text{and} \quad \underline{\phi_\beta}^t = \beta_1 \underline{\phi_1}^t + \beta_2 \underline{\phi_2}^t + \beta_3 \underline{\phi_3}^t.$$

It remains to show that $\alpha_i = \beta_i = 0$ for $i = 1, 2, 3$. To do so, we consider the matrix representation of equation (7.6). The functions in (7.12) can be used as the six trial basis functions of (7.6), while the test space of (7.6) has dimension 6, and is spanned by the functions

$$\underline{\psi}_i = \begin{pmatrix} \delta_{i1} & \delta_{i2} \\ \delta_{i3} & \delta_{i4} \end{pmatrix} + \begin{pmatrix} \eta\,\delta_{i5} & \eta\,\delta_{i6} \\ -\xi\delta_{i5} & -\xi\delta_{i6} \end{pmatrix} \text{ for } i = 1, \cdots, 6 \text{ and } \delta_{ij} \in \mathbb{R} \text{ for } i, j = 1, 2, \cdots 6. \tag{7.13}$$

Taking ψ_i as the test function for row i, we obtain a 6×6 matrix, M_T, whose entries can be explicitly calculated (see [9]). For (r_0, z_0), (r_1, z_1), (r_2, z_2) denoting the vertices of T, labeled such that $r_0 \le r_1$, r_2, let $r_1^* = (r_1 - r_0)/r_0$, $r_2^* = (r_2 - r_0)/r_0$ if $r_0 > 0$, and $r_1^* = r_1$, $r_2^* = r_2$ if $r_0 = 0$, the determinant of M_T is

$$|M_T| = \frac{1}{36}(r_1^* + r_2^* + 3)(2r_1^* + r_2^* + 5)(r_1^* + 2r_2^* + 5)$$
$$(2r_1^* + 2r_2^* + 5)((r_1^*)^2 + 4r_1^* r_2^* + (r_2^*)^2 + 10r_1^* + 10r_2^* + 15)\,.$$

Since $r_1^*, r_2^* \geq 0$, it follows that $|M_T| > 0$ implying that the matrix representation of the projection operator is full rank. Therefore $\Pi_h \underline{\boldsymbol{\tau}} = \{\mathbf{0}\}$ is the unique solution. Hence Π_h is well defined.

Finally, we verify that the spaces given in (7.9) satisfy (6.14)-(6.16). Observe that for $U_h = (P_1)^2$ and $Q_h = P_1$ the test space of (6.14) is the set

$$\left\{ \begin{pmatrix} \delta_1 + z\delta_5 & \delta_2 + z\delta_6 \\ \delta_3 - r\delta_5 & \delta_4 - r\delta_6 \end{pmatrix} \mid \quad \forall\, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6 \in \mathbb{R} \right\},$$

which is the same as the test space described in (7.6). Furthermore, Theorem 6.1 requires that (6.15) is satisfied on a subset of

$$\left\{ \begin{pmatrix} s_1 + s_2 s + s_3 s^2 \\ s_4 + s_5 s + s_6 s^2 \end{pmatrix} \mid \quad \forall\, s_1, s_2, s_3, s_4, s_5, s_6 \in \mathbb{R} \right\}$$

for all ℓ. Since the boundary integral (7.7) is satisfied for all quadratic polynomials on all ℓ, this condition is also satisfied. Lastly, for $U_h(T) = (P_1(T))^2$, $Q_h(T) = P_1(T)$, the test functions in (6.16) are a subset of the test functions in (7.8). ■

8 Error Analysis

In this section we present an error analysis for the approximation $((\underline{\boldsymbol{\sigma}}_h, \sigma_h), \mathbf{w}_h, p_h)$ determined by (6.5)-(6.7). From the general theory of mixed finite element methods [13], (see [9] for details) we have the following.

Theorem 8.1. *Let* $((\underline{\boldsymbol{\sigma}}, \sigma), \mathbf{w}, p) \in \boldsymbol{\Sigma} \times U \times Q$ *be the solution of* (5.14)-(5.16)*. Assuming* (6.8) *is satisfied, let* $((\underline{\boldsymbol{\sigma}}_h, \sigma_h), \mathbf{w}_h, p_h) \in \boldsymbol{\Sigma}_h \times U_h \times Q_h$ *denote the solution of* (6.5)-(6.7)*. Then,*

$$\begin{aligned} &\|(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h, \sigma - \sigma_h)\|_{\boldsymbol{\Sigma}} + \|\mathbf{w} - \mathbf{w}_h\|_U + \|p - p_h\|_Q \\ &\quad \leq C\Big(\inf_{(\underline{\boldsymbol{\tau}}_h, \tau_h) \in \boldsymbol{\Sigma}_h} \|(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\tau}}_h, \sigma - \tau_h)\|_{\boldsymbol{\Sigma}} + \inf_{\mathbf{v}_h \in U_h} \|\mathbf{w} - \mathbf{v}_h\|_U + \inf_{q_h \in Q_h} \|p - q_h\|_Q \Big). \end{aligned} \tag{8.1}$$

■

With some additional smoothness assumptions, we can form an error bound in terms of the mesh parameter h. For the axisymmetric BDM_k interpolation operator $\tilde{\rho}_h : {}_1\mathbf{H}^1(\Omega) \to \mathrm{BDM}_k(\mathcal{T}_h)$ as defined in [18], if $\mathbf{u} \in {}_1\mathbf{H}^{k+1}(\Omega)$, then for some $C > 0$,

$$\|\mathbf{u} - \tilde{\rho}_h(\mathbf{u})\|_{{}_1L^2(\Omega)} \leq C\, h^{k+1} |\mathbf{u}|_{{}_1H^{k+1}(\Omega)}. \tag{8.2}$$

In addition, if $\nabla_{\text{axi}} \cdot \mathbf{u} \in {}_1H^k(\Omega)$ where $\left(\Sigma_{T \in \mathcal{T}_h} |\nabla_{\text{axi}} \cdot \tilde{\rho}_h(\mathbf{u})|^2_{{}_1H^{k+1}(T)}\right)^2 < C_1$, then for some $C > 0$,

$$\|\nabla_{\text{axi}} \cdot \mathbf{u} - \nabla_{\text{axi}} \cdot \tilde{\rho}_h(\mathbf{u})\|_{{}_1L^2(\Omega)} \leq C h^k. \tag{8.3}$$

Combining (8.2), (8.3), with well known polynomial interpolation results we obtain the following corollary.

Corollary 8.1. *Assume* $(\underline{\boldsymbol{\sigma}}, \sigma, \mathbf{w}, p) \in {}_1\underline{\mathbf{H}}^{k+1}(\Omega) \times {}_1H^{k+1}(\Omega) \times {}_1\mathbf{H}^k(\Omega) \times {}_1H^k(\Omega)$ *is the solution to* (5.14)-(5.16)*, and* $(\underline{\boldsymbol{\sigma}}_h, \sigma_h, \mathbf{w}_h, p_h) \in (BDM_k)^2(\mathcal{T}_h) \times P_k(\mathcal{T}_h) \times (P_{k-1}(\mathcal{T}_h))^2 \times P_{k-1}(\mathcal{T}_h)$ *solves* (6.5)-(6.7)*, for* $k = 1, 2$*. Assuming that the projection operators in Lemma 7.1 and Lemma 7.2 are bounded (independent of* h*), then*

$$\|(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h, \sigma - \sigma_h)\|_{\mathbf{\Sigma}} + \|\mathbf{w} - \mathbf{w}_h\|_U + \|p - p_h\|_Q \le C\, h^k. \tag{8.4}$$

■

To conclude this section, we establish an error bound for the true displacement $\mathbf{u}$. Recall from Section 5 that $\mathbf{w} = \mathbf{u} - \mathbf{x}^{\perp} p$.

Corollary 8.2. *Let* $((\underline{\boldsymbol{\sigma}}, \sigma), \mathbf{w}, p) \in \mathbf{\Sigma} \times U \times Q$ *be the solution of* (5.14)-(5.16) *and* $((\underline{\boldsymbol{\sigma}}_h, \sigma_h), \mathbf{w}_h, p_h) \in \mathbf{\Sigma}_h \times U_h \times Q_h$ *the solution of* (6.5)-(6.7)*. Furthermore, let* $\mathbf{u} = \mathbf{w} + \mathbf{x}^{\perp} p$ *denote the true displacement, and* $\mathbf{u}_h = \mathbf{w}_h + \mathbf{x}^{\perp} p_h$ *its discrete approximation. Then, there exists a* $C > 0$ *such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_U \;\le C \Big(\inf_{(\underline{\boldsymbol{\tau}}_h, \tau_h) \in \mathbf{\Sigma}_h} \|(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\tau}}_h, \sigma - \tau_h)\|_{\mathbf{\Sigma}} \;+\; \inf_{\mathbf{v}_h \in U_h} \|\mathbf{w} - \mathbf{v_h}\|_U + \inf_{q_h \in Q_h} \|p - q_h\|_Q \Big). \tag{8.5}$$

Proof. For a bounded domain Ω, there exists $C > 0$ such that $\|\mathbf{x}^{\perp}(p - p_h)\|_U \le C \|p - p_h\|_Q$. Therefore, using Theorem 8.1 we have that

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_U &= \|(\mathbf{w} - \mathbf{w}_h) + \mathbf{x}^{\perp}(p - p_h)\|_U \le \|\mathbf{w} - \mathbf{w}_h\|_U + \|\mathbf{x}^{\perp}(p - p_h)\|_U \\ &\le C \Big(\inf_{(\underline{\boldsymbol{\tau}}_h, \tau_h) \in \mathbf{\Sigma}_h} \|(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\tau}}_h, \sigma - \tau_h)\|_{\mathbf{\Sigma}} + \inf_{\mathbf{v}_h \in U_h} \|\mathbf{u} - \mathbf{v_h}\|_U \;+\; \inf_{q_h \in Q_h} \|p - q_h\|_Q \Big). \end{aligned} \tag{8.6}$$

■

9 Numerical Experiments

In this section we present two numerical experiments to investigate our theoretical results. For both experiments we consider $\Omega = (0,1) \times (0,1)$, and compute approximations using the approximation elements $((\mathbf{BDM}_1(\mathcal{T}_h))^2 \times P_1(\mathcal{T}_h)) \times (P_0(\mathcal{T}_h))^2 \times P_0(\mathcal{T}_h)$ (shown in Table 9.1 and Table 9.3), and $((\mathbf{BDM}_2(\mathcal{T}_h))^2 \times P_2(\mathcal{T}_h)) \times (P_1(\mathcal{T}_h))^2 \times P_1(\mathcal{T}_h)$ (shown in Table 9.2 and Table 9.4). For both experiments, the value for the grad-div parameter (see (5.7)) used was $\gamma = 1$, and the values for the Lamé constants were $\mu = 1/2$ and $\lambda = 1$.

Experiment 1

For Experiment 1 the displacement solution was taken to be

$$\mathbf{u}(r, z) = \begin{pmatrix} 4r^3(1-r)z(1-z) \\ -4r^3(1-r)z(1-z) \end{pmatrix}. \tag{9.1}$$

Correspondingly, the true symmetric stress tensor is

$$\underline{\boldsymbol{\sigma}} = \begin{pmatrix} 4r^2(-2r^2z + r^2 + 9rz^2 - 7rz - r - 7z^2 + 7z) & 2r^2(2r^2z - r^2 - 4rz^2 + 2rz + r + 3z^2 - 3z) \\ 2r^2(2r^2z - r^2 - 4rz^2 + 2rz + r + 3z^2 - 3z) & 4r^2(-4r^2z + 2r^2 + 5rz^2 - rz - 2r - 4z^2 + 4z) \end{pmatrix} \tag{9.2}$$

$$\sigma = 4r^2(-2r^2z + r^2 + 6rz^2 - 4rz - r - 5z^2 + 5z) \tag{9.3}$$

Table 9.1: Experiment 1: Convergence rates for $((\mathbf{BDM}_1(\mathcal{T}_h))^2 \times P_1(\mathcal{T}_h)) \times (P_0(\mathcal{T}_h))^2 \times P_0(\mathcal{T}_h)$ finite elements with grad-div stabilization parameter $\gamma = 1$.

h	$\|(\underline{\boldsymbol{\sigma}}, \sigma) - (\underline{\boldsymbol{\sigma}}_h, \sigma_h)\|_\Sigma$	Cvg. Rate	$\|\mathbf{u} - \mathbf{u}_h\|_U$	Cvg. Rate	$\|\text{as}(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h)\|_Q$	Cvg. Rate
$\frac{1}{4}$	1.273E+00	1.0	2.908E-02	1.0	1.912E-01	1.1
$\frac{1}{6}$	8.444E-01	1.0	1.911E-02	1.1	1.200E-01	1.1
$\frac{1}{8}$	6.308E-01	1.0	1.410E-02	1.0	8.636E-02	1.1
$\frac{1}{10}$	5.034E-01	1.0	1.115E-02	1.0	6.727E-02	1.1
$\frac{1}{12}$	4.189E-01	–	9.227E-03	–	5.508E-02	–
Pred.		1.0		1.0		1.0

Table 9.2: Experiment 1: Convergence rates for $((\mathbf{BDM}_2(\mathcal{T}_h))^2 \times P_2(\mathcal{T}_h)) \times (P_1(\mathcal{T}_h))^2 \times P_1(\mathcal{T}_h)$ finite elements with grad-div stabilization parameter $\gamma = 1$.

h	$\|(\underline{\boldsymbol{\sigma}}, \sigma) - (\underline{\boldsymbol{\sigma}}_h, \sigma_h)\|_\Sigma$	Cvg. Rate	$\|\mathbf{u} - \mathbf{u}_h\|_U$	Cvg. Rate	$\|\text{as}(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h)\|_Q$	Cvg. Rate
$\frac{1}{4}$	6.797E-02	2.0	8.381E-03	1.9	1.602E-02	2.1
$\frac{1}{6}$	3.061E-02	2.0	3.915E-03	1.9	6.753E-03	2.1
$\frac{1}{8}$	1.730E-02	2.0	2.238E-03	2.0	3.647E-03	2.1
$\frac{1}{10}$	1.109E-02	2.0	1.442E-03	2.0	2.264E-03	2.1
$\frac{1}{12}$	7.711E-03	–	1.005E-03	–	1.536E-03	–
Pred.		2.0		2.0		2.0

and the divergence of the stress tensor is

$$\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\sigma}}, \sigma) = \begin{pmatrix} 2r(2r^3 - 24r^2z + 10r^2 + 60rz^2 - 42rz - 9r - 32z^2 + 32z) \\ -2r(8r^3 + r^2(7 - 30z) + 4r(4z^2 + 2z - 3) - 9(z-1)z) \end{pmatrix}. \tag{9.4}$$

The solution was chosen to be consistent with homogenous Dirichlet conditions while having a sufficiently high order polynomial degree to investigate the orders of convergence.

Presented in Table 9.1-9.2 are the results of the simulation. We note that the convergence rate for the displacement reflects the true displacement, $\|\mathbf{u} - \mathbf{u}_h\|_U$.

Experiment 2

For this numerical experiment, we considered the displacement solution

$$\mathbf{u}(r, z) = \begin{pmatrix} r^3 \sin(r\pi)\cos((z - 0.5)\pi) \\ -r^3 \sin(r\pi)\cos((z - 0.5)\pi) \end{pmatrix}. \tag{9.5}$$

This solution was selected to be consistent with homogenous Dirichlet conditions while also providing a non-polynomial validation example. Based on $\mathbf{u}$, the true solution for $\underline{\boldsymbol{\sigma}}$ was determined from the

Table 9.3: Experiment 2: Convergence Rates for $((\mathbf{BDM}_1(\mathcal{T}_h))^2 \times P_1(\mathcal{T}_h) \times (P_0(\mathcal{T}_h))^2 \times P_0(\mathcal{T}_h)$ finite elements with grad-div stabilization parameter $\gamma = 1$.

h	$\|(\underline{\boldsymbol{\sigma}}, \sigma) - (\underline{\boldsymbol{\sigma}}_h, \sigma_h)\|_\Sigma$	Cvg. Rate	$\|\mathbf{u} - \mathbf{u}_h\|_U$	Cvg. Rate	$\|\mathrm{as}(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h)\|_Q$	Cvg. Rate
$\frac{1}{4}$	3.235E+00	1.0	8.675E-02	1.1	6.103E-01	1.1
$\frac{1}{6}$	2.136E+00	1.0	5.619E-02	1.1	3.862E-01	1.1
$\frac{1}{8}$	1.596E+00	1.0	4.111E-02	1.1	2.811E-01	1.1
$\frac{1}{10}$	1.275E+00	1.0	3.239E-02	1.1	2.209E-01	1.1
$\frac{1}{12}$	1.062E+00	–	2.674E-02	–	1.821E-01	–
Pred.		1.0		1.0		1.0

Table 9.4: Experiment 2: Convergence Rates for $((\mathbf{BDM}_2(\mathcal{T}_h))^2 \times P_2(\mathcal{T}_h) \times (P_1(\mathcal{T}_h))^2 \times P_1(\mathcal{T}_h)$ finite elements with grad-div stabilization parameter $\gamma = 1$.

h	$\|(\underline{\boldsymbol{\sigma}}, \sigma) - (\underline{\boldsymbol{\sigma}}_h, \sigma_h)\|_\Sigma$	Cvg. Rate	$\|\mathbf{u} - \mathbf{u}_h\|_U$	Cvg. Rate	$\|\mathrm{as}(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h)\|_Q$	Cvg. Rate
$\frac{1}{4}$	4.291E-01	1.9	2.720E-02	1.9	6.467E-02	2.1
$\frac{1}{6}$	1.966E-01	2.0	1.243E-02	2.0	2.816E-02	2.1
$\frac{1}{8}$	1.119E-01	2.0	7.036E-03	2.0	1.556E-02	2.1
$\frac{1}{10}$	7.208E-02	2.0	4.514E-03	2.0	9.825E-03	2.1
$\frac{1}{12}$	5.023E-02	–	3.138E-03	–	6.752E-03	–
Pred.		2.0		2.0		2.0

relationship

$$\mathcal{A}\underline{\boldsymbol{\sigma}} = \epsilon(\mathbf{u}) \quad \text{where} \quad \mathcal{A}\underline{\boldsymbol{\sigma}} = \frac{1}{2\mu}\left(\underline{\boldsymbol{\sigma}} - \frac{\lambda}{2\mu + 3\lambda}\mathrm{tr}(\underline{\boldsymbol{\sigma}})\right). \tag{9.6}$$

For brevity, the expressions for $(\underline{\boldsymbol{\sigma}}, \sigma)$ and $\nabla_{\mathrm{axi}} \cdot (\underline{\boldsymbol{\sigma}}, \sigma)$ are omitted here.

The results of the simulations are presented in Tables 9.3 and 9.4.

The computational results are consistent with the theoretically predicted results from Corollaries 8.1 and 8.2.

10 Conclusion

We have developed a computational framework for the axisymmetric linear elasticity problem with weak symmetry. Provided the projection bounds (6.12)-(6.13) are satisfied, Lemmas 7.1 and 7.2 establish that the finite element spaces $((\mathbf{BDM}_1(\mathcal{T}_h))^2 \times P_1(\mathcal{T}_h)) \times (P_0(\mathcal{T}_h))^2 \times P_0(\mathcal{T}_h)$ and $((\mathbf{BDM}_2(\mathcal{T}_h))^2 \times P_2(\mathcal{T}_h)) \times (P_1(\mathcal{T}_h))^2 \times P_1(\mathcal{T}_h)$ are inf-sup stable, resulting in approximations satisfying the error bounds stated in Corollary 8.1. Computational presented in Section 9 support these results.

Table 10.1: Experiment 1: Convergence rates for $((\mathbf{BDM}_3(\mathcal{T}_h))^2 \times P_3(\mathcal{T}_h)) \times (P_2(\mathcal{T}_h))^2 \times P_2(\mathcal{T}_h)$ finite elements with $\gamma = 1$.

h	$\|(\underline{\boldsymbol{\sigma}}, \sigma) - (\underline{\boldsymbol{\sigma}}_h, \sigma_h)\|_\Sigma$	Cvg. Rate	$\|\mathbf{u} - \mathbf{u}_h\|_U$	Cvg. Rate	$\|\mathrm{as}(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h)\|_Q$	Cvg. Rate
$\frac{1}{4}$	1.155E-02	3.0	1.359E-03	2.9	1.454E-03	3.1
$\frac{1}{6}$	3.459E-03	3.0	4.233E-04	2.9	4.192E-04	3.1
$\frac{1}{8}$	1.465E-03	3.0	1.816E-04	3.0	1.733E-04	3.1
$\frac{1}{10}$	7.517E-04	3.0	9.370E-05	3.0	8.742E-05	3.1
$\frac{1}{12}$	4.356E-04	–	5.445E-05	–	5.002E-05	–

Table 10.2: Example 2: Convergence Rates for $((\mathbf{BDM}_3(\mathcal{T}_h))^2 \times P_3(\mathcal{T}_h) \times (P_2(\mathcal{T}_h))^2 \times P_2(\mathcal{T}_h)$ finite elements with grad-div stabilization parameter $\gamma = 1$.

h	$\|(\underline{\boldsymbol{\sigma}}, \sigma) - (\underline{\boldsymbol{\sigma}}_h, \sigma_h)\|_\Sigma$	Cvg. Rate	$\|\mathbf{u} - \mathbf{u}_h\|_U$	Cvg. Rate	$\|\mathrm{as}(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h)\|_Q$	Cvg. Rate
$\frac{1}{4}$	5.470E-02	2.9	4.610E-03	2.9	7.849E-03	3.0
$\frac{1}{6}$	1.658E-02	3.0	1.422E-03	3.0	2.355E-03	3.0
$\frac{1}{8}$	7.046E-03	3.0	6.085E-04	3.0	9.937E-04	3.0
$\frac{1}{10}$	3.620E-03	3.0	3.136E-04	3.0	5.082E-04	3.0
$\frac{1}{12}$	2.098E-03	–	1.821E-04	–	2.937E-04	–

It is an open question if for $k \geq 3$, $((\mathbf{BDM}_k(\mathcal{T}_h))^2 \times P_k(\mathcal{T}_h)) \times (P_{k-1}(\mathcal{T}_h))^2 \times P_{k-1}(\mathcal{T}_h)$ form an inf-sup stable set of approximation spaces for this problem.

In the Cartesian setting, the spaces $(\mathbf{BDM}_k(\mathcal{T}_h))^2 \times (P_{k-1}(\mathcal{T}_h))^2 \times P_{k-1}(\mathcal{T}_h)$ form an inf-sup stable set of approximation spaces for the linear elasticity problem with weak symmetry [12]. Therefore, it is reasonable to conjecture that $((\mathbf{BDM}_k(\mathcal{T}_h))^2 \times P_k(\mathcal{T}_h)) \times (P_{k-1}(\mathcal{T}_h))^2 \times P_{k-1}(\mathcal{T}_h)$ are inf-sup stable for the axisymmetric problem. To test this conjecture, Tables 10.1 and 10.2 present convergence results for $((\mathbf{BDM}_3(\mathcal{T}_h))^2 \times P_3(\mathcal{T}_h)) \times (P_2(\mathcal{T}_h))^2 \times P_2(\mathcal{T}_h)$ for the numerical experiments described in Section 9. For these experiments the approximations converge with convergence rate $O(h^k) = O(h^3)$.

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A Proof of Lemma 6.1

Let $\mathbf{x} = (r, z)$. Then

$$\begin{aligned}
\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}} \wedge \mathbf{x}) &= \nabla_{\text{axi}} \cdot \begin{pmatrix} \tau_{11} z - \tau_{21} r \\ \tau_{12} z - \tau_{22} r \end{pmatrix} \\
&= \frac{\partial}{\partial r}(\tau_{11} z - \tau_{21} r) + \frac{\partial}{\partial z}(\tau_{12} z - \tau_{22} r) + \frac{1}{r}(\tau_{11} z - \tau_{21} r) \\
&= z\frac{\partial \tau_{11}}{\partial r} - \tau_{21} - r\frac{\partial \tau_{21}}{\partial r} + \tau_{12} + z\frac{\partial \tau_{12}}{\partial z} - r\frac{\partial \tau_{22}}{\partial z} + \frac{z}{r}\tau_{11} - \tau_{21} \\
&= z(\frac{\partial \tau_{11}}{\partial r} + \frac{\partial \tau_{12}}{\partial z} + \frac{1}{r}\tau_{11}) - r(\frac{\partial \tau_{21}}{\partial r} + \frac{\partial \tau_{22}}{\partial z} + \frac{1}{r}\tau_{21}) + \tau_{12} - \tau_{21} \\
&= (\nabla_{\text{axi}} \cdot \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix}) \wedge \begin{pmatrix} r \\ z \end{pmatrix} + \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} : \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= (\nabla_{\text{axi}} \cdot \underline{\boldsymbol{\tau}}) \wedge \mathbf{x} + \underline{\boldsymbol{\tau}} : \mathbb{P}, \qquad \text{(A.1)}
\end{aligned}$$

where $\mathbb{P} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Therefore,

$$\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}} \wedge \mathbf{x}) - \frac{z}{r}\tau = \nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}}, \tau) \wedge \mathbf{x} + \underline{\boldsymbol{\tau}} : \mathbb{P}.$$

Next we multiply the left and right hand sides of (A.1) by $p\, r$ and integrate over T to yield

$$\begin{aligned}
\int_T \nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}} \wedge \mathbf{x})\, p\, r\, dT - \int_T z\, \tau\, p\, dT &= \int_T (\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}}, \tau)) \wedge \mathbf{x}\, p\, r\, dT + \int_T \underline{\boldsymbol{\tau}} : \mathbb{P}\, p\, r\, dT \\
&= ((\nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}}, \tau)) \wedge \mathbf{x}, p)_T + (as(\underline{\boldsymbol{\tau}}), \mathcal{S}^2(p))_T \\
&= c((\underline{\boldsymbol{\tau}}, \tau), p)_T \quad \text{(see (6.10))}. \qquad \text{(A.2)}
\end{aligned}$$

Note that we have used the relationship $\underline{\boldsymbol{\tau}} : \mathbb{P}\; p = as(\underline{\boldsymbol{\tau}}) : \mathcal{S}^2(p)$. Next, applying integration by parts to the first term on the left-hand side of (A.2) gives

$$\begin{aligned}
\int_T \nabla_{\text{axi}} \cdot (\underline{\boldsymbol{\tau}} \wedge \mathbf{x})\, p\, r\, dT &= \int_T \nabla \cdot (r\, \underline{\boldsymbol{\tau}} \wedge \mathbf{x})\, p\, dT \\
&= \int_{\partial T} (\underline{\boldsymbol{\tau}} \wedge \mathbf{x}) \cdot \mathbf{n}\, p\, r\, \partial s - \int_T (\underline{\boldsymbol{\tau}} \wedge \mathbf{x}) \cdot \nabla p\;\, r\, dT. \qquad \text{(A.3)}
\end{aligned}$$

Then combining (A.2), and (A.3) yields

$$c(\underline{\boldsymbol{\tau}}, p) = \int_{\partial T} (\underline{\boldsymbol{\tau}} \wedge \mathbf{x}) \cdot \mathbf{n}\, p\, r\, ds - \int_T (\underline{\boldsymbol{\tau}} \wedge \mathbf{x}) \cdot \nabla p\;\, r\, dT - \int_T \tau\, z\, p\, dT.$$

Finally, since

$$(\underline{\boldsymbol{\tau}} \wedge \mathbf{x}) \cdot \nabla p = \underline{\boldsymbol{\tau}} : (\mathbf{x}^{\perp} \otimes \nabla p) \text{ and } (\underline{\boldsymbol{\tau}} \wedge \mathbf{x}) \cdot \mathbf{n} = (\underline{\boldsymbol{\tau}} \cdot \mathbf{n}) \cdot \mathbf{x}^{\perp},$$

we have

$$c((\underline{\boldsymbol{\tau}}, \tau), p)_T = \int_{\partial T} (\underline{\boldsymbol{\tau}} \cdot \mathbf{n}) \cdot \mathbf{x}^{\perp}\; p\, r\, ds - \int_T \underline{\boldsymbol{\tau}} : (\mathbf{x}^{\perp} \otimes \nabla p)\, r\, dT - \int_T \tau\, z\, p\, dT.$$

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