

# Approximation of Time-Dependent, Multi-Component, Viscoelastic Fluid Flow

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**Abstract.** In this article we analyse a fully discrete approximation to the time dependent viscoelasticity equations allowing for multicomponent fluid flow. The Oldroyd B constitutive equation is used to model the viscoelastic stress. For the discretization, time derivatives are replaced by backward difference quotients, and the non-linear terms are linearized by lagging appropriate factors. The modeling equations for the individual fluids are combined into a single system of equations using a continuum surface model. The numerical approximation is stabilized by using a SUPG approximation for the constitutive equation. Under a small data assumption on the true solution, existence of the approximate solution is proven. A priori error estimates for the approximation in terms of the mesh parameter  $h$ , the time discretization parameter  $\Delta t$ , and the SUPG coefficient  $\nu$  are also derived. A numerical simulation of viscoelastic fluid flow involving two immiscible fluids is also presented.

**Key words.** viscoelasticity, finite element method, fully discrete, SUPG, multicomponent, interfacial tension

**AMS Mathematics subject classifications.** 65N30

## 1 Introduction

Presented in this paper is an error analysis for a fully discrete approximation to a time dependent, multicomponent, viscoelastic fluid flow problem. In [3] and [10], results are given for a fully discrete approximation to the flow of a single viscoelastic fluid. This paper extends these results to include flows which involve more than one fluid. Our motivation for considering multicomponent viscoelastic fluid flow arises from its application in material science, where new materials with novel properties are being developed by mixing several viscoelastic fluids [18].

For the governing equations of multicomponent viscoelastic fluid flow, we have that within each fluid component the viscoelastic equations must hold. In addition, along the interfaces separating components, a free-surface boundary condition must be satisfied. This boundary condition accounts for the discontinuity in the stress tensor across the interface between components and, in part,

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determines the shape of the interface. Using a continuum surface force model (CSF) [6], we replace the interfacial surface with an interfacial region in which we use a continuous interpolate to describe the fluid characteristics. The CSF approach enables us to model and analyse the multicomponent fluid problem as a single fluid with varying material parameters.

In viscoelasticity, under a “slow flow” assumption, the non-linearity in the momentum equation is often neglected. For an Oldroyd B type fluid, the stress is defined by a differential constitutive equation. The difficulty in performing accurate numerical computations arises from the hyperbolic character of the constitutive equation. Care must be used in discretizing the constitutive equation to avoid the introduction of spurious oscillations into the approximation.

The first error analysis for the steady-state finite element approximation of viscoelastic fluid flow was presented by Baranger and Sandri [2]. In this paper a discontinuous finite element formulation was used for the discretization of the constitutive equation, with the approximation for the stress being discontinuous. Motivated by implementation consideration, Najib and Sandri in [14] modified the discretization in [2] to obtain a decoupled system of two equations, showed the algorithm was convergent, and gave error estimates. In [16], Sandri presented an analysis of a finite element approximation to this problem wherein the constitutive equation was discretized using a Streamline Upwind Petrov Galerkin (SUPG) method. For the constitutive equation discretized using the method of characteristics, Baranger and Machmoum in [1] analysed this approach and gave error estimates for the approximations.

In the analyses described above for steady-state viscoelastic flow there are three main steps: (i) the definition of a iteration operator, (ii) showing that the iteration operator is well defined, and (iii) applying Brouwer’s fixed point theorem.

For the fully discrete approximation to the time dependent, multicomponent problem case presented herein, the analysis is completely different from the aforementioned method. Instead it follows closely the method of [10]. Time derivatives are replaced by backward difference quotients, and the non-linear terms are linearized by lagging appropriate factors. A key part in the error analysis is an induction argument on properties of the approximation. The approach follows that of Liu [13] for compressible Navier-Stokes equations. For completeness we present the analysis with the non-linear term in the momentum equation included.

This paper is organized as follows. In section 2, the general equations which govern the flow of multicomponent behavior are discussed, and the continuum surface force model is presented. In section 2 we describe the equations for viscoelastic fluid flow and present the numerical approximation scheme. The main approximation result is then given in Theorem 4.1 in section 4, followed by its proof. In section 5 we present a numerical simulation of viscoelastic fluid flow involving two immiscible fluids. The experimental convergence rates for the error in the numerical simulation agree with the theoretical rates established in section 4.

## 2 The Modeling Equations of Multicomponent Fluid Flow

In this section, we briefly present the modeling equations describing multicomponent, viscoelastic fluid flow. We use  $\mathbf{u}$ ,  $\rho$ , and  $\mathbf{T}$  to denote the velocity, density, and total stress (tensor) of the fluid.

Let  $\Omega$  denote a bounded domain in  $\mathbb{R}^d$  ( $d = 2, 3$ ), with boundary  $\partial\Omega$ . For ease of exposition, we will

present the formulation for two viscoelastic fluids in  $\Omega$ . Let  $\Omega_1, \Omega_2$  denote the region in  $\Omega$  occupied by fluids 1 and 2, respectively, and  $\mathcal{I}$ , the interface between the two fluids. Note that  $\Omega_1, \Omega_2$  and  $\mathcal{I}$  are functions of time, and  $\Omega = \Omega_1 \cup \Omega_2 \cup \mathcal{I}$ .

Within each  $\Omega_i$ :

For  $V$  a fixed region in  $\Omega_i$ , with boundary  $\partial V$ , the conservation of momentum and mass equations imply

$$\frac{d}{dt} \int_V \rho \mathbf{u} \, d\mathbf{x} = \int_V \mathbf{b} \, d\mathbf{x} + \int_{\partial V} \mathbf{T} \cdot \mathbf{n} \, dS - \int_{\partial V} \rho \mathbf{u} (\mathbf{u} \cdot \mathbf{n}) \, dS, \quad (2.1)$$

$$\frac{d}{dt} \int_V \rho \, d\mathbf{x} = - \int_{\partial V} \rho \mathbf{u} \cdot \mathbf{n} \, dS, \quad (2.2)$$

where  $\mathbf{n}$  denotes the unit outward normal on  $\partial V$ , and  $\mathbf{b}$  the body forces acting on  $V$ .

Along the Interface  $\mathcal{I}$ :

The boundary condition which holds along, and determines the interface  $\mathcal{I}$  is [4]

$$[\mathbf{T} \cdot \mathbf{n}] = -\sigma \kappa \mathbf{n} - \nabla_s \sigma, \quad (2.3)$$

where  $\kappa$  denotes the mean curvature of  $\mathcal{I}$ ,  $\sigma$  the coefficient of interfacial tension,  $\nabla_s \sigma$  the surface gradient of  $\sigma$ ,  $\mathbf{n}$  the unit normal on  $\mathcal{I}$  pointing into fluid 2, and  $[\mathbf{T} \cdot \mathbf{n}]$  the jump of the normal component of stress across  $\mathcal{I}$  defined by

$$[\mathbf{T} \cdot \mathbf{n}]|_{\mathbf{x}} = \lim_{\epsilon \rightarrow 0^+} \left( \mathbf{T}|_{\mathbf{x}+\epsilon \mathbf{n}} - \mathbf{T}|_{\mathbf{x}-\epsilon \mathbf{n}} \right).$$

Using the continuum surface force model of Brackbill et. al. [6], the force along the interface is rewritten as a volume force using a delta distribution, i.e.

$$\begin{aligned} \int_{V \cap \mathcal{I}} [\mathbf{T} \cdot \mathbf{n}] \, dS &= \int_V [\mathbf{T} \cdot \mathbf{n}] \delta(\mathbf{x} - \mathbf{x}_s) \\ &= \int_V (-\sigma \kappa \mathbf{n} - \nabla_s \sigma) \delta(\mathbf{x} - \mathbf{x}_s) \, d\mathbf{x} \end{aligned}$$

where  $\mathbf{x}_s$  denotes a nearest point to  $\mathbf{x}$  on  $\mathcal{I}$ .

Using the divergence theorem to replace the surface integrals in (2.1), (2.2) with volume integrals, the fact that  $V$  is an arbitrary volume, and the incompressibility of the fluid, we obtain the following pointwise equations for the conservation of momentum and mass:

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{b} + \nabla \cdot \mathbf{T} - (\sigma \kappa \mathbf{n} + \nabla_s \sigma) \delta_{\mathcal{I}}, \quad \text{in } \Omega, \quad (2.4)$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega. \quad (2.5)$$

Modeling Equation for the Stress Tensor  $\mathbf{T}$ :

The stress tensor  $\mathbf{T}$  is written in the form

$$\mathbf{T} = -p\mathbf{I} + \tau \quad (2.6)$$

where  $p$  denotes the internal fluid pressure,  $I$  the identity tensor, and  $\tau$  the extra stress tensor. For a Newtonian fluid  $\tau$  is modeled as

$$\tau = 2\eta D(\mathbf{u}) \quad (2.7)$$

where  $D(\mathbf{u}) := \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$  is the deformation tensor and  $\eta$  is the fluid viscosity. For viscoelastic fluids, because of the internal elasticity of the fluid, the modeling equation for the extra stress is in general considerably more complicated than (2.7), (see [5] for a description of various models).

In this paper we assume that the extra stress is governed by an Oldroyd B model. For this model  $\tau$  is expressed as

$$\tau = \tau_n + \tau_v \quad (2.8)$$

where the Newtonian contribution to the extra stress,  $\tau_n$  satisfies

$$\tau_n = 2(1 - \alpha)D(\mathbf{u}), \quad (2.9)$$

and the viscoelastic contribution  $\tau_v$  is given by

$$\tau_v + \lambda \frac{\hat{\partial}\tau_v}{\partial t} - 2\alpha D(\mathbf{u}) = 0, \quad (2.10)$$

where

$$\frac{\hat{\partial}\tau_v}{\partial t} := \frac{\partial\tau_v}{\partial t} + \mathbf{u} \cdot \nabla\tau_v + g_a(\tau_v, \nabla\mathbf{u}), \quad a \in [-1, 1] \quad (2.11)$$

and

$$g_a(\sigma, \nabla\mathbf{u}) := \frac{1-a}{2} \left( \tau_v \nabla\mathbf{u} + (\nabla\mathbf{u})^T \tau_v \right) - \frac{1+a}{2} \left( (\nabla\mathbf{u})\tau_v + \tau_v(\nabla\mathbf{u})^T \right). \quad (2.12)$$

In (2.9),  $\alpha \in (0, 1)$  may be interpreted as the proportion of the viscosity which is considered to be viscoelastic in nature. The *Weissenberg number*,  $\lambda$ , is a dimensionless constant which is defined as the product of the relaxation time and a characteristic strain rate [5]. In (2.11) the choices  $a = 1, -1, 0$  correspond to the upper, lower, and corrotational convected derivatives of  $\tau_v$ , respectively.

In what follows, for ease of notation, we use  $\tau$  to denote  $\tau_v$ . Using (2.6), (2.8)-(2.11) and (2.4), (2.5) we obtain, on nondimensionalization of the problem, the modeling system of equations:

$$Re \left( \frac{\partial\mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla\mathbf{u} \right) + \nabla p - 2(1 - \alpha)\nabla \cdot D(\mathbf{u}) - \nabla \cdot \tau = \mathbf{f} \quad \text{in } \Omega, \quad (2.13)$$

$$\tau + \lambda \left( \frac{\partial\tau}{\partial t} + \mathbf{u} \cdot \nabla\tau + g_a(\tau, \nabla\mathbf{u}) \right) - 2\alpha D(\mathbf{u}) = 0 \quad \text{in } \Omega, \quad (2.14)$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \quad (2.15)$$

where

$$\mathbf{f} := \mathbf{b} - (\sigma\kappa\mathbf{n} + \nabla_s\sigma)\delta_{\mathcal{I}}, \quad (2.16)$$

$$Re := \frac{L V \bar{\rho}}{\bar{\eta}}. \quad (2.17)$$

In (2.17),  $L, V, \bar{\rho}, \bar{\eta}$  denote a characteristic length scale, velocity scale, density, and viscosity.

To fully specify the problem, together with (2.13)-(2.15), we require initial conditions for the velocity and stress, boundary conditions for the velocity, and the stress specified on the inflow boundary of  $\Omega$ ,  $\partial\Omega_{\text{in}}$ ,

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \quad \text{in } \Omega, \quad (2.18)$$

$$\tau(\mathbf{x}, 0) = \tau_0(\mathbf{x}) \quad \text{in } \Omega. \quad (2.19)$$

$$\mathbf{u} = \mathbf{u}_{\text{bdy}} \quad \text{on } \partial\Omega, \quad (2.20)$$

$$\tau = \tau_{\text{bdy}} \quad \text{on } \partial\Omega_{\text{in}}. \quad (2.21)$$

Note: Equations (2.13)-(2.15), (2.18)-(2.21) only specify the pressure  $p$  up to an arbitrary constant.

The existence and uniqueness of  $(\mathbf{u}, \tau, p)$  satisfying (2.13)-(2.15), (2.18)-(2.21) is still largely an open research question. The local existence (in time), and under a ‘‘small data’’ assumption on  $\mathbf{f}, \mathbf{f}', \mathbf{u}_0, \tau_0$ , global existence (in time) of solutions to (2.13)-(2.15), (2.18)-(2.21) have been established [11]. For a more complete discussion of existence and uniqueness issues, see [15].

In order to simplify the numerical analysis of the approximation scheme to (2.13)-(2.15), (2.18)-(2.21), we will assume homogeneous boundary conditions for the velocity (i.e.  $\mathbf{u}_{\text{bdy}} = 0$ ). Consequently, as there is no inflow boundary, below we study the specific system of equations (2.13)-(2.15), (2.18)-(2.20) with  $\mathbf{u}_{\text{bdy}} = 0$ .

### 3 The Variational Formulation

In this section, we develop the variational formulation of (2.13)-(2.15), (2.18)-(2.20). The following notation will be used. The  $L^2(\Omega)$  norm and inner product will be denoted by  $\|\cdot\|$  and  $(\cdot, \cdot)$ . Likewise, the  $L^p(\Omega)$  norms and the Sobolev  $W^{k,p}(\Omega)$  norms are denoted by  $\|\cdot\|_{L^p}$  and  $\|\cdot\|_{k,p}$ , respectively.  $H^k$  is used to represent the Sobolev space  $W^{k,2}$ , and  $\|\cdot\|_k$  denotes the norm in  $H^k$ . The following function spaces are used in the analysis:

$$\text{Velocity Space} : X := H_0^1(\Omega) := \left\{ \mathbf{u} \in H^1(\Omega) : \mathbf{u} = 0 \text{ on } \partial\Omega \right\},$$

$$\begin{aligned} \text{Stress Space} : S := & \left\{ \tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji}; \tau_{ij} \in L^2(\Omega); 1 \leq i, j \leq 3 \right\} \\ & \cap \left\{ \tau = (\tau_{ij}) : \mathbf{u} \cdot \nabla \tau \in L^2(\Omega), \forall \mathbf{u} \in X \right\}, \end{aligned}$$

$$\text{Pressure Space} : Q := L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_{\Omega} q \, dx = 0 \right\},$$

$$\text{Divergence – free Space} : Z := \left\{ v \in X : \int_{\Omega} q(\nabla \cdot v) \, dx = 0, \forall q \in Q \right\}.$$

The variational formulation of (2.13)-(2.15), (2.18)-(2.20) proceeds in the usual manner. Taking the inner product of (2.13), (2.14), and (2.15) with a velocity test function, a stress test function, and a pressure test function respectively, we obtain

$$\left( Re \frac{\partial \mathbf{u}}{\partial t} + Re \mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v} \right) - (p, \nabla \cdot \mathbf{v}) + (2(1 - \alpha)D(\mathbf{u}) + \tau, D(\mathbf{v})) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in X, \quad (3.1)$$

$$\left( \tau + \left( \lambda \frac{\partial \tau}{\partial t} + \lambda \mathbf{u} \cdot \nabla \tau + \lambda g_a(\tau_V, \nabla \mathbf{u}) \right) - 2\alpha D(\mathbf{u}), \psi \right) = 0, \quad \forall \psi \in S, \quad (3.2)$$

$$(\nabla \cdot \mathbf{u}, q) = 0, \quad \forall q \in Q. \quad (3.3)$$

Note that  $Re$  and  $\lambda$  are functions of time and space, determined by which fluid is occupying the point  $\mathbf{x}$  at time  $t$ . We use

$$\begin{aligned} 0 < Re_m &:= \min_{\mathbf{x} \in \Omega} Re, \\ 0 < Re_M &:= \max_{\mathbf{x} \in \Omega} Re, \\ 0 < \lambda_m &:= \min_{\mathbf{x} \in \Omega} \lambda, \\ 0 < \lambda_M &:= \max_{\mathbf{x} \in \Omega} \lambda. \end{aligned}$$

The space  $Z$  is the space of weakly divergence free functions. The condition

$$(\nabla \cdot \mathbf{u}, q) = 0, \quad \forall q \in Q, \quad \mathbf{u} \in X,$$

is equivalent in a “distributional” sense to

$$(\mathbf{u}, \nabla q) = 0, \quad \forall q \in Q, \quad \mathbf{u} \in X, \quad (3.4)$$

where in (3.4),  $(\cdot, \cdot)$  denotes the duality pairing between  $H^{-1}$  and  $H_0^1$  functions. In addition, note that the velocity and pressure spaces,  $X$  and  $Q$ , satisfy the *inf-sup* condition

$$\inf_{q \in Q} \sup_{v \in X} \frac{(q, \nabla \cdot v)}{\|q\| \|v\|_1} \geq \beta > 0. \quad (3.5)$$

Since the inf-sup condition (3.5) holds, an equivalent variational formulation to (3.1)-(3.3) is: *find*  $\mathbf{u} \in Z, \tau \in S$  *satisfying*

$$\left( Re \frac{\partial \mathbf{u}}{\partial t} + Re \mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v} \right) + (2(1 - \alpha)D(\mathbf{u}) + \tau, \mathbf{D}(\mathbf{v})) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in Z, \quad (3.6)$$

$$\left( \tau + \lambda \left( \frac{\partial \tau}{\partial t} + \mathbf{u} \cdot \nabla \tau + g_a(\tau, \nabla \mathbf{u}) \right) - 2\alpha D(\mathbf{u}), \psi \right) = 0, \quad \forall \psi \in S. \quad (3.7)$$

We assume that the fluid flow satisfies the following properties:

$$\|\mathbf{u}\|_\infty, \|\tau\|_\infty, \|\nabla \mathbf{u}\|_\infty, \|\nabla \tau\|_\infty \leq M, \quad (3.8)$$

for all  $t \in [0, T]$ .

The following definitions are used in the analysis below:

$$b(\mathbf{u}, \tau, \psi) := (\mathbf{u} \cdot \nabla \tau, \psi), \quad (3.9)$$

$$c(\mathbf{w}, \mathbf{u}, \mathbf{v}) := (\mathbf{w} \cdot \nabla \mathbf{u}, \mathbf{v}). \quad (3.10)$$

### 3.1 Finite Element Approximation

In this section we formulate a fully discrete finite element method for solving the viscoelastic fluid flow equations, and prove the solvability of the approximation at each step (for sufficiently small

$\Delta t$  and  $h$ ). We begin by describing the finite element approximation framework and listing the approximating properties and inverse estimates used in the analysis.

Let  $T_h$  be a triangulation of  $\Omega$  made of triangles (in  $\mathbb{R}^2$ ) or tetrahedrals (in  $\mathbb{R}^3$ ). Thus, the computational domain is defined by

$$\Omega = \bigcup K; \quad K \in T_h.$$

We assume that there exist constants  $c_1, c_2$  such that

$$c_1 h \leq h_K \leq c_2 \rho_K$$

where  $h_K$  is the diameter of triangle (tetrahedral)  $K$ ,  $\rho_K$  is the diameter of the greatest ball (sphere) included in  $K$ , and  $h = \max_{K \in T_h} h_K$ . Let  $P_k(K)$  denote the space of polynomials on  $A$  of degree no greater than  $k$ . Then we define the finite element spaces as follows.

$$\begin{aligned} X_h &:= \left\{ \mathbf{v} \in X \cap C(\bar{\Omega})^2 : \mathbf{v}|_K \in P_k(K), \forall K \in T_h \right\}, \\ S_h &:= \left\{ \sigma \in S \cap C(\bar{\Omega})^4 : \sigma|_K \in P_m(K), \forall K \in T_h \right\}, \\ Q_h &:= \left\{ q \in Q \cap C(\bar{\Omega}) : q|_K \in P_q(K), \forall K \in T_h \right\}, \\ Z_h &:= \left\{ \mathbf{v} \in X_h : (q, \nabla \cdot \mathbf{v}) = 0, \forall q \in Q_h \right\}. \end{aligned}$$

We assume that the velocity and pressure spaces are chosen so as to satisfy the discrete *inf-sup* condition:

$$\inf_{q \in Q_h} \sup_{v \in X_h} \frac{(q, \nabla \cdot v)}{\|q\| \|v\|_1} \geq \beta > 0. \quad (3.11)$$

Let  $\Delta t$  denote the step size for  $t$ ,  $t_n := n\Delta t, n = 0, 1, 2, \dots, N$ , and let

$$d_t f^n := \frac{f(t_n) - f(t_{n-1})}{\Delta t}.$$

We also define the following additional norms:

$$\begin{aligned} \|v\|_{\infty, k} &:= \max_{1 \leq n \leq N} \|v^n\|_k, \\ \|v\|_{0, k} &:= \left( \sum_{n=1}^N \|v^n\|_k^2 \Delta t \right)^{\frac{1}{2}}. \end{aligned}$$

When  $v(\mathbf{x}, t)$  is defined on the entire time interval  $(0, T)$ , we use

$$\begin{aligned} \|v\|_{\infty, k} &:= \sup_{0 < t < T} \|v(\cdot, t)\|_k, \\ \|v\|_{0, k} &:= \left( \int_0^T \|v(\cdot, t)\|_k^2 dt \right)^{1/2}. \end{aligned}$$

In addition, we make use of the following approximation properties,[8]:

$$\inf_{\mathbf{v} \in X_h} \|\mathbf{u} - \mathbf{v}\| \leq Ch^{k+1} \|\mathbf{u}\|_{k+1}, \quad \mathbf{u} \in H^{k+1}(\Omega)^d,$$

$$\begin{aligned}
\inf_{\mathbf{v} \in X_h} \|\mathbf{u} - \mathbf{v}\|_1 &\leq Ch^k \|\mathbf{u}\|_{k+1}, \quad \mathbf{u} \in H^{k+1}(\Omega)^{\dot{d}}, \\
\inf_{\sigma \in S_h} \|\tau - \sigma\| &\leq Ch^{m+1} \|\tau\|_{m+1}, \quad \tau \in H^{m+1}(\Omega)^{\dot{d} \times \dot{d}}, \\
\inf_{\sigma \in S_h} \|\tau - \sigma\|_1 &\leq Ch^m \|\tau\|_{m+1}, \quad \tau \in H^{m+1}(\Omega)^{\dot{d} \times \dot{d}}, \\
\inf_{r \in Q_h} \|p - r\| &\leq Ch^{q+1} \|p\|_{q+1}, \quad p \in H^{q+1}(\Omega).
\end{aligned} \tag{3.12}$$

$$\tag{3.13}$$

The following inverse estimates, [8], are also used:

$$\begin{aligned}
\|\mathbf{u}_h\|_\infty &\leq ch^{-\frac{\dot{d}}{2}} \|\mathbf{u}_h\| \quad \forall \mathbf{u}_h \in X_h, \\
\|q_h\|_\infty &\leq ch^{-\frac{\dot{d}}{2}} \|q_h\| \quad \forall q_h \in Q_h.
\end{aligned} \tag{3.14}$$

To solve the time-dependent flow equations numerically, time derivatives are replaced by backward differences, and nonlinear terms are lagged. As we are assuming ‘‘slow flow’’, i.e.  $Re \equiv O(1)$ , we use a conforming finite element method to discretize the momentum equation. For the constitutive equation for stress, which is hyperbolic, we use a streamline upwind Petrov-Galerkin (SUPG) discretization to control the production of spurious oscillations in the approximation. The discrete approximating system of equations is then:

### Approximating System

For  $n = 1, 2, \dots, N$ , find  $\mathbf{u}_h^n \in Z_h, \tau_h^n \in S_h$  such that

$$\begin{aligned}
(Re \, d_t \mathbf{u}_h^n, \mathbf{v}) + c \left( Re \, \mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \mathbf{v} \right) + 2(1 - \alpha) (D(\mathbf{u}_h^n), D(\mathbf{v})) \\
+ (\tau_h^n, D(\mathbf{v})) = (\mathbf{f}^n, \mathbf{v}), \quad \forall \mathbf{v} \in Z_h,
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
(\tau_h^n, \tilde{\sigma}) + (\lambda \, d_t \tau_h^n, \sigma) + b \left( \lambda \, \mathbf{u}_h^{n-1}, \tau_h^n, \tilde{\sigma} \right) - 2\alpha (D(\mathbf{u}_h^n), \tilde{\sigma}) \\
= - \left( \lambda \, g_\alpha(\tau_h^{n-1}, \nabla \mathbf{u}_h^{n-1}), \tilde{\sigma} \right), \quad \forall \sigma \in S_h
\end{aligned} \tag{3.16}$$

where  $\tilde{\sigma} := \sigma + \nu \sigma_u^n$ ,  $\sigma_u^n := \mathbf{u}_h^{n-1} \cdot \nabla \sigma$ , and  $\nu$  is a small positive constant.

The parameter  $\nu > 0$  is used to suppress the production of spurious oscillations in the approximation. Note that for  $\nu = 0$  the discretization of the constitutive equation is the usual Galerkin method. The goal in choosing  $\nu$  is to keep it as small as possible, but large enough to control the generation of catastrophic spurious oscillations in the approximate stress.

To ensure computability of the algorithm, we begin by showing that (3.15)-(3.16) is uniquely solvable for  $\mathbf{u}_h$  and  $\tau_h$  at each time step  $n$ . We use the following induction hypothesis.

$$(IH1) \quad \left\| \mathbf{u}_h^{n-1} \right\|_\infty, \left\| \tau_h^{n-1} \right\|_\infty \leq K.$$

**Lemma 1** *Assume (IH1) is true. For sufficiently small step size  $\Delta t$ , there exists a unique solution  $(\mathbf{u}_h^n, \tau_h^n) \in Z_h \times S_h$  satisfying (3.15)-(3.16).*

**Proof:** For notational simplicity, in this proof we drop the subscript  $h$  from the variables. Choosing  $\mathbf{v} = \mathbf{u}_h^n, \sigma = \tau_h^n$ , multiplying (3.15) by  $2\alpha$  and adding to (3.16) we obtain

$$a(\mathbf{u}^n, \tau^n; \mathbf{u}^n, \tau^n) = 2\alpha (\mathbf{f}^n, \mathbf{u}^n) + 2\alpha \frac{1}{\Delta t} \left( Re \, \mathbf{u}^{n-1}, \mathbf{u}^n \right)$$

$$- \left( \lambda g_a \left( \tau^{n-1}, \nabla \mathbf{u}^{n-1} \right), \tilde{\tau}^n \right) + \frac{1}{\Delta t} \left( \lambda \tau^{n-1}, \tau^n \right), \quad (3.17)$$

where the bilinear form  $a(\mathbf{u}, \tau; \mathbf{v}, \sigma)$  is defined as:

$$\begin{aligned} a(\mathbf{u}, \tau; \mathbf{v}, \sigma) &:= \frac{2\alpha}{\Delta t} (Re \mathbf{u}, \mathbf{u}) + 2\alpha c(Re \mathbf{u}^{n-1}, \mathbf{u}, \mathbf{v}) + 4\alpha(1 - \alpha) (D(\mathbf{u}), D(\mathbf{v})) + 2\alpha (\tau, D(\mathbf{v})) \\ &+ (\tau, \tilde{\sigma}) + \frac{1}{\Delta t} (\lambda \tau, \sigma) + b \left( \lambda \mathbf{u}^{n-1}, \tau, \sigma \right) + b \left( \lambda \mathbf{u}^{n-1}, \tau, \nu \mathbf{u}^{n-1} \cdot \nabla \sigma \right) \\ &- 2\alpha (D(\mathbf{u}), \sigma) - 2\alpha \left( D(\mathbf{u}), \nu \mathbf{u}^{n-1} \cdot \nabla \sigma \right). \end{aligned}$$

Thus,

$$\begin{aligned} a(\mathbf{u}, \tau; \mathbf{u}, \tau) &= \frac{2\alpha}{\Delta t} (Re \mathbf{u}, \mathbf{u}) + 2\alpha c(Re \mathbf{u}^{n-1}, \mathbf{u}, \mathbf{u}) + 4\alpha(1 - \alpha) (D(\mathbf{u}), D(\mathbf{u})) + (\tau, \tau) + \nu (\tau, \tau_u) \\ &+ \frac{1}{\Delta t} (\lambda \tau, \tau) + b \left( \lambda \mathbf{u}^{n-1}, \tau, \tau \right) + b \left( \lambda \mathbf{u}^{n-1}, \tau, \nu \tau_u \right) - 2\alpha \nu (D(\mathbf{u}), \tau_u). \end{aligned}$$

We now estimate the terms in  $a(\mathbf{u}^n, \tau^n; \mathbf{u}^n, \tau^n)$ . We have

$$\begin{aligned} \frac{2\alpha}{\Delta t} (Re \mathbf{u}^n, \mathbf{u}^n) &\geq \frac{2\alpha Re_m}{\Delta t} \|\mathbf{u}^n\|^2 \\ \left| 2\alpha c(Re \mathbf{u}^{n-1}, \mathbf{u}^n, \mathbf{u}^n) \right| &= 2\alpha \left| (Re \mathbf{u}^{n-1} \cdot \nabla \mathbf{u}^n, \mathbf{u}^n) \right| \\ &\leq 2\alpha d^{\frac{1}{2}} \left\| Re \mathbf{u}^{n-1} \right\|_{\infty} \|\nabla \mathbf{u}^n\| \|\mathbf{u}^n\| \\ &\leq 2\alpha d^{\frac{1}{2}} C_K Re_M \left\| \mathbf{u}^{n-1} \right\|_{\infty} \|D(\mathbf{u}^n)\| \quad (\text{using Korn's lemma}) \\ &\leq \epsilon_1 \|D(\mathbf{u}^n)\|^2 + \frac{\alpha^2 d K^2 C_K^2 Re_M^2}{\epsilon_1} \|\mathbf{u}^n\|^2, \\ 4\alpha(1 - \alpha) (D(\mathbf{u}^n), D(\mathbf{u}^n)) &= 4\alpha(1 - \alpha) \|D(\mathbf{u}^n)\|^2, \\ (\tau^n, \tau^n) &= \|\tau^n\|^2, \\ |\nu (\tau^n, \tau_u^n)| &\leq \nu \|\tau^n\| \|\tau_u^n\| \\ &\leq \|\tau^n\|^2 + \frac{\nu^2}{4} \|\tau_u^n\|^2 \\ \frac{1}{\Delta t} (\lambda \tau^n, \tau^n) &\geq \frac{\lambda_m}{\Delta t} \|\tau^n\|^2, \\ \left| b(\lambda \mathbf{u}^{n-1}, \tau^n, \tau^n) \right| &= \left| (\lambda \mathbf{u}^{n-1} \cdot \nabla \tau^n, \tau^n) \right| \leq \lambda_M \|\tau_u^n\| \|\tau^n\| \\ &\leq \epsilon_2 \|\tau_u^n\|^2 + \frac{\lambda_M^2}{4\epsilon_2} \|\tau^n\|^2, \\ b(\lambda \mathbf{u}^{n-1}, \tau^n, \nu \tau_u^n) &= (\lambda \tau_u^n, \nu \tau_u^n) \\ &\geq \lambda_m \nu \|\tau_u^n\|^2, \\ |2\alpha \nu (D(\mathbf{u}^n), \tau_u^n)| &\leq 2\alpha \nu \|D(\mathbf{u}^n)\| \|\tau_u^n\| \\ &\leq \epsilon_3 \|D(\mathbf{u}^n)\|^2 + \frac{\alpha^2 \nu^2}{\epsilon_3} \|\tau_u^n\|^2. \end{aligned}$$

Applying these inequalities to the bilinear form  $a(\cdot, \cdot; \cdot, \cdot)$  yields

$$a(\mathbf{u}^n, \tau^n; \mathbf{u}^n, \tau^n) \geq \left( \frac{2\alpha Re_m}{\Delta t} - \frac{\alpha^2 d K^2 C_K^2 Re_M^2}{\epsilon_1} \right) \|\mathbf{u}^n\|^2 + (4\alpha(1 - \alpha) - \epsilon_1 - \epsilon_3) \|D(\mathbf{u}^n)\|^2$$

$$+ \left( \frac{\lambda_m}{\Delta t} - \frac{\lambda_M^2}{4\epsilon_2} \right) \|\tau^n\|^2 + \left( \lambda_m \nu - \epsilon_2 - \frac{\nu^2}{4} - \frac{\alpha^2 \nu^2}{\epsilon_3} \right) \|\tau_u^n\|^2.$$

For

$$\nu \leq \frac{2\lambda_m(1-\alpha)}{1+3\alpha} \quad (3.18)$$

and choosing  $\epsilon_1 = \epsilon_3 = \alpha(1-\alpha)$ ,  $\epsilon_2 = \frac{\lambda_m \nu}{2}$ , we have that for

$$\Delta t < \min \left\{ \frac{4\epsilon_2 \lambda_m}{\lambda_M^2}, \frac{2\epsilon_1 \alpha Re_m}{d K^2 C_K^2 \alpha^2 Re_M} \right\}, \quad (3.19)$$

the bilinear form  $a(\cdot, \cdot; \cdot, \cdot)$  is positive. Hence, (3.17) has at most one solution. Since (3.17) is a finite dimensional linear system, the uniqueness of the solution implies the existence of the solution.  $\blacksquare$

**Lemma 2 (Discrete Gronwall's Lemma)** [12] *Let  $\Delta t$ ,  $H$ , and  $a_n, b_n, c_n, \gamma_n$ , (for integers  $n \geq 0$ ), be nonnegative numbers such that*

$$a_l + \Delta t \sum_{n=0}^l b_n \leq \Delta t \sum_{n=0}^l \gamma_n a_n + \Delta t \sum_{n=0}^l c_n + H \quad \text{for } l \geq 0.$$

Suppose that  $\Delta t \gamma_n < 1$ , for all  $n$ , and set  $\sigma_n = (1 - \Delta t \gamma_n)^{-1}$ . Then,

$$a_l + \Delta t \sum_{n=0}^l b_n \leq \exp \left( \Delta t \sum_{n=0}^l \sigma_n \gamma_n \right) \left\{ \Delta t \sum_{n=0}^l c_n + H \right\} \quad \text{for } l \geq 0. \quad (3.20)$$

## 4 A Priori Error Estimate

In this section we analyze the error between the finite element approximation given by (3.15)-(3.16) and the true solution. A priori error estimates for the approximation are in Theorem 4.1.

**Theorem 4.1** *There exists constants  $c_1, c_2 > 0$  such that if  $\Delta t < c_1 h^{d/2}$ ,  $\nu < c_2 h^{d/2}$ , the finite element approximation (3.15)-(3.16) is convergent to the solution of (3.6)-(3.7) on the interval  $(0, T)$  as  $h \rightarrow 0$ . In addition, the approximation  $(\mathbf{u}_h, \tau_h)$  and the true solution  $(\mathbf{u}, \tau)$  satisfy the following error estimates:*

$$\|\mathbf{u}_h - \mathbf{u}\|_{\infty, 0} + \|\tau_h - \tau\|_{\infty, 0} \leq \mathbf{F}(\Delta t, h) \quad (4.1)$$

$$\|\mathbf{u}_h - \mathbf{u}\|_{0, 1} + \|\tau_h - \tau\|_{0, 0} \leq \mathbf{F}(\Delta t, h) \quad (4.2)$$

where

$$\begin{aligned} \mathbf{F}(\Delta t, h) &= Ch^k \|\mathbf{u}\|_{0, k+1} + Ch^{k+1} \|\mathbf{u}_t\|_{0, k+1} + Ch^m \|\tau\|_{0, m+1} + Ch^{m+1} \|\tau_t\|_{0, m+1} \\ &+ Ch^{q+1} \|p\|_{0, q+1} + C \left( h^{k+1} \|\mathbf{u}\|_{\infty, k+1} + h^{m+1} \|\tau\|_{\infty, m+1} \right) \\ &+ C |\Delta t| \left( \|\mathbf{u}_t\|_{0, 1} + \|\mathbf{u}_{tt}\|_{0, 0} + \|\tau_t\|_{0, 1} + \|\tau_{tt}\|_{0, 0} \right) \\ &+ C \nu \left( \|\tau_t\|_{0, 1} + \|\tau_t\|_{\infty, 0} \right). \end{aligned}$$

In order to establish the estimates (4.1)-(4.2), we begin by introducing the following notation. Let  $\mathbf{u}^n = \mathbf{u}(t_n), \tau^n = \tau(t_n)$  represent the solution of (3.6)-(3.7) at time  $t_n$ , and  $\mathbf{u}_h^n, \tau_h^n$  denote the solution of (3.15)-(3.16). Let  $(\mathcal{U}^n, \mathcal{P}^n)$  denote the Stokes projection of  $(\mathbf{u}^n, p^n)$  into  $(Z_h, Q_h)$ , and  $\mathcal{T}^n$  a Clément interpolant of  $\tau^n$ , [9]. We have the approximating properties:

$$\begin{aligned}
\|\mathbf{u}^n - \mathcal{U}^n\| &\leq Ch^{k+1} \|\mathbf{u}^n\|_{k+1}, \\
\|\tau^n - \mathcal{T}^n\| &\leq Ch^{m+1} \|\tau^n\|_{m+1}, \\
\|p^n - \mathcal{P}^n\| &\leq Ch^{q+1} \|p^n\|_{q+1}, \\
\|\nabla(\mathbf{u}^n - \mathcal{U}^n)\| &\leq Ch^k \|\mathbf{u}^n\|_{k+1}, \\
\|\nabla(\tau^n - \mathcal{T}^n)\| &\leq Ch^m \|\tau^n\|_{m+1}.
\end{aligned} \tag{4.3}$$

From [7], we have the following results.

**Lemma 3** : Let  $\{T_h\}$ ,  $0 < h \leq 1$ , denote a quasi-uniform family of subdivisions of a polyhedral domain  $\Omega \subset \mathbb{R}^d$ . Let  $(\hat{K}, P, N)$  be a reference finite element such that  $P \subset W_p^l(\hat{K}) \cap W_q^m(\hat{K})$  where  $1 \leq p \leq \infty$ ,  $1 \leq q \leq \infty$  and  $0 \leq m \leq l$ . For  $K \in T_h$ , let  $(K, P_K, N_K)$  be the affine equivalent element, and  $V_h = \{v : v \text{ is measurable and } v|_K \in P_K, \forall K \in T_h\}$ . Then there exists  $C = C(l, p, q)$  such that

$$\left( \sum_{K \in T_h} \|v\|_{W_p^l(K)}^p \right)^{1/p} \leq Ch^{m-l+\min(0, \frac{d}{p}-\frac{d}{q})} \left( \sum_{K \in T_h} \|v\|_{W_q^m(K)}^q \right)^{1/q}, \tag{4.4}$$

for all  $v \in V_h$ . ■

Note that it follows from (3.8) and inverse estimates, [7], that

$$\|\mathcal{U}^n\|_\infty, \|\nabla \mathcal{U}^n\|_\infty \leq \tilde{M} \approx M. \tag{4.5}$$

Below, for simplicity, we take  $\tilde{M} = M$ .

Define  $\mathbf{\Lambda}^n, \mathbf{E}^n, \mathbf{\Gamma}^n, \mathbf{F}^n, \epsilon_u, \epsilon_\tau$  as

$$\begin{aligned}
\mathbf{\Lambda}^n &= \mathbf{u}^n - \mathcal{U}^n, & \mathbf{E}^n &= \mathcal{U}^n - \mathbf{u}_h^n, \\
\mathbf{\Gamma}^n &= \tau^n - \mathcal{T}^n, & \mathbf{F}^n &= \mathcal{T}^n - \tau_h^n, \\
\epsilon_u &= \mathbf{u} - \mathbf{u}_h^n, & \epsilon_\tau &= \tau - \tau_h^n.
\end{aligned}$$

The proof of theorem 4.1 is established in three steps.

1. Prove a lemma, assuming two induction hypotheses.
2. Show that the induction hypotheses are true.
3. Prove the error estimates given in (4.1),(4.2).

**Step 1.** We prove the following lemma.

**Lemma 4** Under the induction hypothesis (IH1) and the additional assumption

$$(IH2) \quad \sum_{n=1}^{l-1} \Delta t \|\nabla E^n\|_\infty \leq 1,$$

we have that

$$\|\mathbf{E}^l\|^2 + \|\mathbf{F}^l\|^2 \leq G(\Delta t, h, \nu), \quad (4.6)$$

where

$$\begin{aligned} G(\Delta t, h, \nu) &= C \left( h^{2k} \|\mathbf{u}\|_{0,k+1}^2 + h^{2k+2} \|\mathbf{u}_t\|_{0,k+1}^2 \right) + C \left( h^{2m} \|\tau\|_{0,m+1}^2 + h^{2m+2} \|\tau_t\|_{0,m+1}^2 \right) \\ &+ C h^{2q+2} \|p\|_{0,q+1}^2 + C |\Delta t|^2 \left( \|\mathbf{u}_t\|_{0,1}^2 + \|\mathbf{u}_{tt}\|_{0,0}^2 + \|\tau_t\|_{0,1}^2 + \|\tau_{tt}\|_{0,0}^2 \right) \\ &+ C \nu^2 \left( \|\tau_t\|_{0,1}^2 + \|\tau_t\|_{\infty,0}^2 \right). \end{aligned}$$

**Proof of lemma 4:** From (3.6)-(3.7), we have that the true solution  $(\mathbf{u}, \tau)$  satisfies

$$\begin{aligned} (Re d_t \mathbf{u}^n, \mathbf{v}) &+ c \left( Re \mathbf{u}_h^{n-1}, \mathbf{u}^n, \mathbf{v} \right) + 2(1-\alpha) (D(\mathbf{u}^n), D(\mathbf{v})) + (\tau^n, D(\mathbf{v})) \\ &= (\mathbf{f}^n, \mathbf{v}) + (p^n, \nabla \cdot \mathbf{v}) + R_1(\mathbf{v}), \quad \forall \mathbf{v} \in Z_h, \end{aligned} \quad (4.7)$$

$$\begin{aligned} (\lambda d_t \tau^n, \sigma) &+ b \left( \lambda \mathbf{u}_h^{n-1}, \tau^n, \tilde{\sigma} \right) - 2\alpha (D(\mathbf{u}^n), \tilde{\sigma}) + (\tau^n, \tilde{\sigma}) \\ &= - \left( \lambda g_a \left( \tau_h^{n-1}, \nabla \mathbf{u}_h^{n-1} \right), \tilde{\sigma} \right) + R_2(\sigma), \quad \forall \sigma \in S_h, \end{aligned} \quad (4.8)$$

where

$$R_1(\mathbf{v}) := (Re d_t \mathbf{u}^n, \mathbf{v}) - (Re \mathbf{u}_t^n, \mathbf{v}) + c(Re \mathbf{u}_h^{n-1}, \mathbf{u}^n, \mathbf{v}) - c(Re \mathbf{u}^n, \mathbf{u}^n, \mathbf{v}),$$

and

$$\begin{aligned} R_2(\sigma) &:= (\lambda d_t \tau^n, \sigma) - (\lambda \tau_t^n, \sigma) - (\lambda \tau_t^n, \nu \sigma_u) + b(\lambda \mathbf{u}_h^{n-1}, \tau^n, \tilde{\sigma}) \\ &- b(\lambda \mathbf{u}^n, \tau^n, \tilde{\sigma}) + \left( \lambda g_a \left( \tau_h^{n-1}, \nabla \mathbf{u}_h^{n-1} \right), \tilde{\sigma} \right) - (\lambda g_a (\tau^n, \nabla \mathbf{u}^n), \tilde{\sigma}). \end{aligned}$$

Subtracting (3.15)-(3.16) from (4.7)-(4.8) we obtain the following equations for  $\epsilon_u$  and  $\epsilon_\tau$ :

$$\begin{aligned} (Re d_t \epsilon_u, \mathbf{v}) + c(Re \mathbf{u}_h^{n-1}, \epsilon_u, \mathbf{v}) &+ 2(1-\alpha) (D(\epsilon_u), D(\mathbf{v})) + (\epsilon_\tau, D(\mathbf{v})) \\ &= (p^n, \nabla \cdot \mathbf{v}) + R_1(\mathbf{v}), \quad \forall \mathbf{v} \in Z_h, \end{aligned} \quad (4.9)$$

$$(\lambda d_t \epsilon_\tau, \sigma) + b(\lambda \mathbf{u}_h^{n-1}, \epsilon_\tau, \tilde{\sigma}) - 2\alpha (D(\epsilon_u), \tilde{\sigma}) + (\epsilon_\tau, \tilde{\sigma}) = R_2(\sigma), \quad \forall \sigma \in S_h. \quad (4.10)$$

Substituting  $\epsilon_u = \mathbf{E}^n + \mathbf{\Lambda}^n$ ,  $\epsilon_\tau = \mathbf{F}^n + \mathbf{\Gamma}^n$ ,  $\mathbf{v} = \mathbf{E}^n$ ,  $\sigma = \mathbf{F}^n$  into (4.9)-(4.10), we obtain

$$(Re d_t \mathbf{E}^n, \mathbf{E}^n) + c(Re \mathbf{u}_h^{n-1}, \mathbf{E}^n, \mathbf{E}^n) + 2(1-\alpha) (D(\mathbf{E}^n), D(\mathbf{E}^n)) + (\mathbf{F}^n, D(\mathbf{E}^n)) = \mathcal{F}_1(\mathbf{E}^n), \quad (4.11)$$

$$(\lambda d_t \mathbf{F}^n, \mathbf{F}^n) + b(\lambda \mathbf{u}_h^{n-1}, \mathbf{F}^n, \tilde{\mathbf{F}}^n) - 2\alpha (D(\mathbf{E}^n), \tilde{\mathbf{F}}^n) + (\mathbf{F}^n, \tilde{\mathbf{F}}^n) = \mathcal{F}_2(\mathbf{F}^n), \quad (4.12)$$

where,

$$\begin{aligned} \mathcal{F}_1(\mathbf{E}^n) &= (p^n, \nabla \cdot \mathbf{E}^n) + R_1(\mathbf{E}^n) - (Re d_t \mathbf{\Lambda}^n, \mathbf{E}^n) - c(Re \mathbf{u}_h^{n-1}, \mathbf{\Lambda}^n, \mathbf{E}^n) \\ &- 2(1-\alpha) (D(\mathbf{\Lambda}^n), D(\mathbf{E}^n)) - (\mathbf{\Gamma}^n, D(\mathbf{E}^n)), \\ \mathcal{F}_2(\mathbf{F}^n) &= R_2(\mathbf{F}^n) - (\lambda d_t \mathbf{\Gamma}^n, \mathbf{F}^n) - b(\lambda \mathbf{u}_h^{n-1}, \mathbf{\Gamma}^n, \tilde{\mathbf{F}}^n) + 2\alpha (D(\mathbf{\Lambda}^n), \tilde{\mathbf{F}}^n) - (\mathbf{\Gamma}^n, \tilde{\mathbf{F}}^n). \end{aligned}$$

Note that

$$\begin{aligned}
(Re \, d_t \mathbf{E}^n, \mathbf{E}^n) &= \frac{1}{\Delta t} \left( (Re \, \mathbf{E}^n, \mathbf{E}^n) - (Re \, \mathbf{E}^{n-1}, \mathbf{E}^n) \right) \\
&\geq \frac{1}{\Delta t} \left( \|Re^{1/2} \mathbf{E}^n\|^2 - \|Re^{1/2} \mathbf{E}^n\| \|Re^{1/2} \mathbf{E}^{n-1}\| \right) \\
&\geq \frac{1}{2\Delta t} \left( \|Re^{1/2} \mathbf{E}^n\|^2 - \|Re^{1/2} \mathbf{E}^{n-1}\|^2 \right) \\
&\geq \frac{Re_n^*}{2\Delta t} \left( \|\mathbf{E}^n\|^2 - \|\mathbf{E}^{n-1}\|^2 \right),
\end{aligned}$$

where  $Re_n^* = Re_m$  or  $Re_M$  depending on the sign of  $(\|\mathbf{E}^n\|^2 - \|\mathbf{E}^{n-1}\|^2)$ . Similarly,  $(\lambda \, d_t \mathbf{F}^n, \mathbf{F}^n) \geq \frac{\lambda_n^*}{2\Delta t} (\|\mathbf{F}^n\|^2 - \|\mathbf{F}^{n-1}\|^2)$ . Then, from (4.11), we have that

$$\begin{aligned}
\frac{1}{2\Delta t} \left( \|\mathbf{E}^n\|^2 - \|\mathbf{E}^{n-1}\|^2 \right) + \frac{1}{Re_n^*} c(Re \, \mathbf{u}_h^{n-1}, \mathbf{E}^n, \mathbf{E}^n) + \frac{1}{Re_n^*} 2(1-\alpha) (D(\mathbf{E}^n), D(\mathbf{E}^n)) \\
+ \frac{1}{Re_n^*} (\mathbf{F}^n, D(\mathbf{E}^n)) \leq \frac{1}{Re_n^*} \mathcal{F}_1(\mathbf{E}^n). \quad (4.13)
\end{aligned}$$

Multiplying (4.13) by  $Re_m \Delta t$  and summing from  $n = 1$  to  $l$  gives

$$\begin{aligned}
\frac{Re_m}{2} \left( \|\mathbf{E}^l\|^2 - \|\mathbf{E}^0\|^2 \right) + \sum_{n=1}^l \left\{ \frac{Re_m \Delta t}{Re_n^*} c(Re \, \mathbf{u}_h^{n-1}, \mathbf{E}^n, \mathbf{E}^n) + \frac{Re_m \Delta t}{Re_n^*} 2(1-\alpha) \|D(\mathbf{E}^n)\|^2 \right. \\
\left. + \frac{Re_m \Delta t}{Re_n^*} (\mathbf{F}^n, D(\mathbf{E}^n)) \right\} \leq \sum_{n=1}^l \frac{Re_m \Delta t}{Re_n^*} \mathcal{F}_1(\mathbf{E}^n). \quad (4.14)
\end{aligned}$$

Similarly, from (4.12) we have that

$$\begin{aligned}
\frac{\lambda_m}{2} \left( \|\mathbf{F}^l\|^2 - \|\mathbf{F}^0\|^2 \right) + \sum_{n=1}^l \frac{\lambda_m \Delta t}{\lambda_n^*} \left\{ (\mathbf{F}^n, \tilde{\mathbf{F}}^n) + b(\lambda \, \mathbf{u}_h^{n-1}, \mathbf{F}^n, \tilde{\mathbf{F}}^n) - 2\alpha (D(\mathbf{E}^n), \tilde{\mathbf{F}}^n) \right\} \\
\leq \sum_{n=1}^l \frac{\lambda_m \Delta t}{\lambda_n^*} \mathcal{F}_2(\mathbf{F}^n). \quad (4.15)
\end{aligned}$$

Multiplying (4.14) by  $2\alpha$  and adding to (4.15) yields

$$\begin{aligned}
\alpha Re_m \left( \|\mathbf{E}^l\|^2 - \|\mathbf{E}^0\|^2 \right) + \frac{\lambda_m}{2} \left( \|\mathbf{F}^l\|^2 - \|\mathbf{F}^0\|^2 \right) + 4\alpha(1-\alpha) \sum_{n=1}^l \frac{Re_m \Delta t}{Re_n^*} \|D(\mathbf{E}^n)\|^2 \\
+ \nu \sum_{n=1}^l \frac{\lambda_m^2 \Delta t}{\lambda_n^*} \|\mathbf{F}_u^n\|^2 + \sum_{n=1}^l \frac{\lambda_m \Delta t}{\lambda_n^*} \|\mathbf{F}^n\|^2 \\
\leq \Delta t \sum_{n=1}^l \frac{2\alpha Re_m}{Re_n^*} \left( \mathcal{F}_1(\mathbf{E}^n) - c(Re \, \mathbf{u}_h^{n-1}, \mathbf{E}^n, \mathbf{E}^n) - (\mathbf{F}^n, D(\mathbf{E}^n)) \right) \\
+ \Delta t \sum_{n=1}^l \frac{\lambda_m}{\lambda_n^*} \left( \mathcal{F}_2(\mathbf{F}^n) - b(\lambda \, \mathbf{u}_h^{n-1}, \mathbf{F}^n, \mathbf{F}^n) + 2\alpha (D(\mathbf{E}^n), \tilde{\mathbf{F}}^n) - \nu (\mathbf{F}^n, \mathbf{F}_u^n) \right). \quad (4.16)
\end{aligned}$$

Noting that  $Re_m \leq Re_n^* \leq Re_M, \lambda_m \leq \lambda_n^* \leq \lambda_M$ , applying the triangle inequality to the right hand side of (4.16), we have that

$$\begin{aligned}
& \alpha Re_m \left( \|\mathbf{E}^l\|^2 - \|\mathbf{E}^0\|^2 \right) + \frac{\lambda_m}{2} \left( \|\mathbf{F}^l\|^2 - \|\mathbf{F}^0\|^2 \right) + 4\alpha(1-\alpha) \sum_{n=1}^l \frac{Re_m \Delta t}{Re_M} \|D(\mathbf{E}^n)\|^2 \\
& + \nu \sum_{n=1}^l \frac{\lambda_m^2 \Delta t}{\lambda_M} \|\mathbf{F}_u^n\|^2 + \sum_{n=1}^l \frac{\lambda_m \Delta t}{\lambda_M} \|\mathbf{F}^n\|^2 \leq \Delta t \sum_{n=1}^l \left\{ 2\alpha \left| c(Re \mathbf{u}_h^{n-1}, \mathbf{E}^n, \mathbf{E}^n) \right| + \left| b(\lambda \mathbf{u}_h^{n-1}, \mathbf{F}^n, \mathbf{F}^n) \right| \right. \\
& \quad \left. + 2\alpha\nu |(D(\mathbf{E}^n), \mathbf{F}_u^n)| + 2\alpha \left( 1 - \frac{Re_m}{Re_M} \right) |(D(\mathbf{E}^n), \mathbf{F}^n)| + \nu |(\mathbf{F}^n, \mathbf{F}_u^n)| \right\} \\
& \quad + \Delta t \sum_{n=1}^l \{ 2\alpha |\mathcal{F}_1(\mathbf{E}^n)| + |\mathcal{F}_2(\mathbf{F}^n)| \}. \tag{4.17}
\end{aligned}$$

We now estimate each term on the right hand side of (4.17). For  $c(\mathbf{u}_h^{n-1}, \mathbf{E}^n, \mathbf{E}^n)$  we have that

$$\begin{aligned}
2\alpha c(Re \mathbf{u}_h^{n-1}, \mathbf{E}^n, \mathbf{E}^n) & \leq 2\alpha Re_M \left| \left( \mathbf{u}_h^{n-1} \cdot \nabla \mathbf{E}^n, \mathbf{E}^n \right) \right| \\
& \leq 2\alpha Re_M \left\| \mathbf{u}_h^{n-1} \cdot \nabla \mathbf{E}^n \right\| \|\mathbf{E}^n\| \\
& \leq 2\alpha Re_M \left\| \mathbf{u}_h^{n-1} \right\|_\infty d^{\frac{1}{2}} \|\nabla \mathbf{E}^n\| \|\mathbf{E}^n\| \\
& \leq 4\alpha^2 Re_M^2 \epsilon_1 \|\nabla \mathbf{E}^n\|^2 + \frac{d K^2}{4\epsilon_1} \|\mathbf{E}^n\|^2, \text{ using (IH1).} \\
& \leq 4\alpha^2 Re_M^2 C_K^2 \epsilon_1 \|D(\mathbf{E}^n)\|^2 + \frac{d K^2}{4\epsilon_1} \|\mathbf{E}^n\|^2, \tag{4.18} \\
& \quad \text{(using Korn's lemma)}
\end{aligned}$$

Note that for  $\mathbf{v} = 0$  on  $\partial\Omega$ , applying Green's theorem we have

$$b(\mathbf{v}, \tau, \sigma) = -b(\mathbf{v}, \sigma, \tau) - (\nabla \cdot \mathbf{v} \tau, \sigma), \tag{4.19}$$

$$\Rightarrow b(\mathbf{v}, \tau, \tau) = -\frac{1}{2} (\nabla \cdot \mathbf{v} \tau, \tau). \tag{4.20}$$

Using (4.20),

$$\begin{aligned}
\left| b(\lambda \mathbf{u}_h^{n-1}, \mathbf{F}^n, \mathbf{F}^n) \right| & \leq \frac{\lambda_M}{2} \left| \left( \nabla \cdot \mathbf{u}_h^{n-1} \mathbf{F}^n, \mathbf{F}^n \right) \right| \\
& = \frac{\lambda_M}{2} \left| \left( \nabla \cdot (\mathbf{u}_h^{n-1} - \mathcal{U}^{n-1}) \mathbf{F}^n, \mathbf{F}^n \right) + \left( \nabla \cdot \mathcal{U}^{n-1} \mathbf{F}^n, \mathbf{F}^n \right) \right| \\
& \leq \frac{\lambda_M}{2} \left\| \nabla \cdot \mathbf{E}^{n-1} \right\|_\infty \|\mathbf{F}^n\|^2 + \frac{\lambda_M}{2} \left\| \nabla \cdot \mathcal{U}^{n-1} \right\|_\infty \|\mathbf{F}^n\|^2 \\
& \leq \frac{\lambda_M}{2} \left( d \left\| \nabla \cdot \mathbf{E}^{n-1} \right\|_\infty \|\mathbf{F}^n\|^2 + M \|\mathbf{F}^n\|^2 \right), \text{ using (4.5).}
\end{aligned}$$

Next, with  $\tilde{R} = (1 - Re_m/Re_M)$ ,

$$\begin{aligned}
2\alpha \tilde{R} |(D(\mathbf{E}^n), \mathbf{F}^n)| & \leq 2\alpha \tilde{R} \|D(\mathbf{E}^n)\| \|\mathbf{F}^n\| \\
& \leq 4\alpha^2 \tilde{R}^2 \epsilon_2 \|D(\mathbf{E}^n)\|^2 + \frac{1}{4\epsilon_2} \|\mathbf{F}^n\|^2.
\end{aligned}$$

Then

$$\begin{aligned} 2\alpha |(D(\mathbf{E}^n), \nu \mathbf{F}_u^n)| &\leq 2\alpha \|D(\mathbf{E}^n)\| \|\nu \mathbf{F}_u^n\| \\ &\leq 4\alpha^2 \epsilon_3 \|D(\mathbf{E}^n)\|^2 + \frac{\nu^2}{4\epsilon_3} \|\mathbf{F}_u^n\|^2. \end{aligned}$$

Also,

$$\begin{aligned} |(\mathbf{F}^n, \nu \mathbf{F}_u^n)| &= \nu |(\mathbf{F}^n, \mathbf{F}_u^n)| \\ &\leq \nu \|\mathbf{F}^n\| \|\mathbf{F}_u^n\| \\ &\leq \|\mathbf{F}^n\|^2 + \frac{\nu^2}{4} \|\mathbf{F}_u^n\|^2. \end{aligned}$$

Thus, for the first summation on the right hand side of (4.17), we have

$$\begin{aligned} \Delta t \sum_{n=1}^l \left\{ 2\alpha \left| c(\text{Re } \mathbf{u}_h^{n-1}, \mathbf{E}^n, \mathbf{E}^n) \right| + \left| b(\lambda \mathbf{u}_h^{n-1}, \mathbf{F}^n, \mathbf{F}^n) \right| + 2\alpha \nu |(D(\mathbf{E}^n), \mathbf{F}_u^n)| \right. \\ \left. - 2\alpha \tilde{R} |(D(\mathbf{E}^n), \mathbf{F}^n)| + \nu |(\mathbf{F}^n, \mathbf{F}_u^n)| \right\} \leq \Delta t \sum_{n=1}^l \left( 4\alpha^2 (C_K^2 \text{Re}_M^2 \epsilon_1 + \tilde{R}^2 \epsilon_2 + \epsilon_3) \|D(\mathbf{E}^n)\|^2 \right. \\ \left. + \Delta t \sum_{n=1}^l \left( \frac{\acute{d} K^2}{4\epsilon_1} \right) \|\mathbf{E}^n\|^2 + \Delta t \sum_{n=1}^l \left( \frac{\lambda_M}{2} \left( \acute{d} \|\nabla \mathbf{E}^{n-1}\|_\infty + M \right) + \frac{1}{4\epsilon_2} + 1 \right) \|\mathbf{F}^n\|^2 \right. \\ \left. + \Delta t \sum_{n=1}^l \left( \frac{\nu^2}{4\epsilon_3} + \frac{\nu^2}{4} \right) \|\mathbf{F}_u^n\|^2 \right) \end{aligned} \quad (4.21)$$

Next we consider  $\mathcal{F}_1(\mathbf{E}^n)$ .

$$\begin{aligned} |(p^n, \nabla \cdot \mathbf{E}^n)| &= |(p^n - \mathcal{P}^n, \nabla \cdot \mathbf{E}^n)| \\ &\leq \|p^n - \mathcal{P}^n\| \acute{d}^{\frac{1}{2}} \|\nabla \mathbf{E}^n\| \\ &\leq C_K^2 \epsilon_5 \|D(\mathbf{E}^n)\|^2 + \frac{\acute{d}}{4\epsilon_5} \|p^n - \mathcal{P}^n\|^2, \text{ (Korn's lemma)}. \end{aligned} \quad (4.22)$$

$$\begin{aligned} |(\text{Re } d_t \mathbf{\Lambda}^n, \mathbf{E}^n)| &\leq \text{Re}_M \|\mathbf{E}^n\| \|d_t \mathbf{\Lambda}^n\| \\ &\leq \text{Re}_M^2 \|\mathbf{E}^n\|^2 + \frac{1}{4} \|d_t \mathbf{\Lambda}^n\|^2. \end{aligned}$$

$$\begin{aligned} \left| c(\text{Re } \mathbf{u}_h^{n-1}, \mathbf{\Lambda}^n, \mathbf{E}^n) \right| &\leq \text{Re}_M \|\mathbf{E}^n\| \left\| \mathbf{u}_h^{n-1} \cdot \nabla \mathbf{\Lambda}^n \right\| \\ &\leq \text{Re}_M \|\mathbf{E}^n\| \left\| \mathbf{u}_h^{n-1} \right\|_\infty \acute{d}^{\frac{1}{2}} \|\nabla \mathbf{\Lambda}^n\| \\ &\leq \text{Re}_M^2 \|\mathbf{E}^n\|^2 + \frac{K^2 \acute{d}}{4} \|\nabla \mathbf{\Lambda}^n\|^2, \quad \text{using (IH1)}. \end{aligned} \quad (4.23)$$

$$2(1 - \alpha) |(D(\mathbf{\Lambda}^n), D(\mathbf{E}^n))| \leq (1 - \alpha) \epsilon_6 \|D(\mathbf{E}^n)\|^2 + \frac{1 - \alpha}{\epsilon_6} \|D(\mathbf{\Lambda}^n)\|^2. \quad (4.24)$$

$$\begin{aligned} |(\mathbf{\Gamma}^n, D(\mathbf{E}^n))| &\leq \|D(\mathbf{E}^n)\| \|\mathbf{\Gamma}^n\| \\ &\leq \epsilon_7 \|D(\mathbf{E}^n)\|^2 + \frac{1}{4\epsilon_7} \|\mathbf{\Gamma}^n\|^2. \end{aligned} \quad (4.25)$$

For the  $R_1(\mathbf{E}^n)$  terms we have:

$$|(Re d_t \mathbf{u}^n, \mathbf{E}^n) - (Re \mathbf{u}_t^n, \mathbf{E}^n)| \leq Re_M^2 \|\mathbf{E}^n\|^2 + \frac{1}{4} \|d_t \mathbf{u}^n - \mathbf{u}_t^n\|^2. \quad (4.26)$$

$$\begin{aligned} |c(Re \mathbf{u}_h^{n-1}, \mathbf{u}^n, \mathbf{E}^n) - c(Re \mathbf{u}^n, \mathbf{u}^n, \mathbf{E}^n)| &= |c(Re (\mathbf{u}_h^{n-1} - \mathcal{U}^{n-1}), \mathbf{u}^n, \mathbf{E}^n) + c(Re (\mathcal{U}^{n-1} - \mathbf{u}^{n-1}), \mathbf{u}^n, \mathbf{E}^n) \\ &\quad + c(Re (\mathbf{u}^{n-1} - \mathbf{u}^n), \mathbf{u}^n, \mathbf{E}^n)| \\ &\leq Re_M \|\mathbf{E}^{n-1} \cdot \nabla \mathbf{u}^n\| \|\mathbf{E}^n\| + Re_M \|\mathbf{\Lambda}^{n-1} \cdot \nabla \mathbf{u}^n\| \|\mathbf{E}^n\| \\ &\quad + Re_M \|(\mathbf{u}^n - \mathbf{u}^{n-1}) \cdot \nabla \mathbf{u}^n\| \|\mathbf{E}^n\| \\ &\leq Re_M \acute{d} M \|\mathbf{E}^{n-1}\| \|\mathbf{E}^n\| + Re_M \acute{d} M \|\mathbf{\Lambda}^{n-1}\| \|\mathbf{E}^n\| \\ &\quad + Re_M \acute{d} M \|(\mathbf{u}^n - \mathbf{u}^{n-1})\| \|\mathbf{E}^n\|, \text{ using (3.8)} \\ &\leq \frac{Re_M^2 \acute{d}^2 M^2}{2} \|\mathbf{E}^{n-1}\|^2 + \frac{3}{2} \|\mathbf{E}^n\|^2 + \frac{Re_M^2 \acute{d}^2 M^2}{2} \|\mathbf{\Lambda}^{n-1}\|^2 \\ &\quad + \frac{Re_M^2 \acute{d}^2 M^2}{2} \Delta t \int_{t^{n-1}}^{t^n} \|\mathbf{u}_t\|^2 dt. \end{aligned} \quad (4.27)$$

Combining (4.22)-(4.27) we have the following estimate for  $\mathcal{F}_1(\mathbf{E}^n)$ :

$$\begin{aligned} |2\alpha \mathcal{F}_1(\mathbf{E}^n)| &\leq 2\alpha (C_K^2 \epsilon_5 + (1 - \alpha) \epsilon_6 + \epsilon_7) \|D(\mathbf{E}^n)\|^2 + 2\alpha \left(3Re_M^2 + \frac{3}{2}\right) \|\mathbf{E}^n\|^2 \\ &\quad + \alpha Re_M^2 \acute{d}^2 M^2 \|\mathbf{E}^{n-1}\|^2 + 2\alpha \frac{\acute{d}}{4\epsilon_5} \|(p^n - \mathcal{P}^n)\|^2 + \alpha Re_M^2 \acute{d}^2 M^2 \|\mathbf{\Lambda}^{n-1}\|^2 \\ &\quad + 2\alpha \left(\frac{K^2 \acute{d}}{4} + \frac{1 - \alpha}{\epsilon_6}\right) \|\nabla \mathbf{\Lambda}^n\|^2 + \alpha \frac{1}{2} \|d_t \mathbf{\Lambda}^n\|^2 + \alpha \frac{1}{2\epsilon_7} \|\mathbf{\Gamma}^n\|^2 \\ &\quad + \alpha \frac{1}{2} \|d_t \mathbf{u}^n - \mathbf{u}_t^n\|^2 + \alpha Re_M^2 \acute{d}^2 M^2 \Delta t \int_{t^{n-1}}^{t^n} \|\mathbf{u}_t\|^2 dt. \end{aligned} \quad (4.28)$$

Next we consider the terms in  $\mathcal{F}_2(\mathbf{F}^n)$ .

$$|(\lambda d_t \mathbf{\Gamma}^n, \mathbf{F}^n)| \leq \lambda_M^2 \|\mathbf{F}^n\|^2 + \frac{1}{4} \|d_t \mathbf{\Gamma}^n\|^2. \quad (4.29)$$

$$\begin{aligned} |b(\lambda \mathbf{u}_h^{n-1}, \mathbf{\Gamma}^n, \tilde{\mathbf{F}}^n)| &= |b(\lambda \mathbf{u}_h^{n-1}, \mathbf{\Gamma}^n, \mathbf{F}^n) + b(\lambda \mathbf{u}_h^{n-1}, \mathbf{\Gamma}^n, \nu \mathbf{F}_u^n)| \\ &\leq \lambda_M \|\mathbf{u}_h^{n-1} \cdot \nabla \mathbf{\Gamma}^n\| \|\mathbf{F}^n\| + \lambda_M \|\mathbf{u}_h^{n-1} \cdot \nabla \mathbf{\Gamma}^n\| \|\nu \mathbf{F}_u^n\| \\ &\leq \lambda_M \acute{d}^{\frac{1}{2}} \|\mathbf{u}_h^{n-1}\|_\infty \|\nabla \mathbf{\Gamma}^n\| \|\mathbf{F}^n\| + \lambda_M \acute{d}^{\frac{1}{2}} \|\mathbf{u}_h^{n-1}\|_\infty \|\nabla \mathbf{\Gamma}^n\| \|\nu \mathbf{F}_u^n\| \\ &\leq \lambda_M^2 \|\mathbf{F}^n\|^2 + \nu^2 \|\mathbf{F}_u^n\|^2 + \frac{\acute{d} K^2}{2} \|\nabla \mathbf{\Gamma}^n\|^2. \end{aligned} \quad (4.30)$$

$$\begin{aligned} 2\alpha |(D(\mathbf{\Lambda}^n), \tilde{\mathbf{F}}^n)| &= 2\alpha |(D(\mathbf{\Lambda}^n), \mathbf{F}^n) + (D(\mathbf{\Lambda}^n), \nu \mathbf{F}_u^n)| \\ &\leq \|\mathbf{F}^n\|^2 + \nu^2 \|\mathbf{F}_u^n\|^2 + 2\alpha^2 \|\nabla \mathbf{\Lambda}^n\|^2. \end{aligned} \quad (4.31)$$

$$\begin{aligned} |(\mathbf{\Gamma}^n, \tilde{\mathbf{F}}^n)| &= |(\mathbf{\Gamma}^n, \mathbf{F}^n) + \nu (\mathbf{\Gamma}^n, \nu \mathbf{F}_u^n)| \\ &\leq \|\mathbf{F}^n\|^2 + \nu^2 \|\mathbf{F}_u^n\|^2 + \frac{1}{2} \|\mathbf{\Gamma}^n\|^2. \end{aligned} \quad (4.32)$$

For the terms making up  $R_2(\mathbf{F}^n)$  we have:

$$|(\lambda d_t \tau^n, \mathbf{F}^n) - (\lambda \tau_t^n, \mathbf{F}^n)| \leq \|\lambda \mathbf{F}^n\| \|d_t \tau^n - \tau_t^n\| \quad (4.33)$$

$$\leq \lambda_M^2 \|\mathbf{F}^n\|^2 + \frac{1}{4} \|d_t \tau^n - \tau_t^n\|^2. \quad (4.34)$$

$$\begin{aligned} |(\lambda \tau_t^n, \nu \mathbf{F}_u^n)| &= \left| (\lambda \tau_t^n, \nu \mathbf{u}_h^{n-1} \cdot \nabla \mathbf{F}^n) \right| \\ &= \left| B(\lambda \nu \mathbf{u}_h^{n-1}, \mathbf{F}^n, \tau_t^n) \right| \\ &\leq \left| B(\lambda \nu \mathbf{u}_h^{n-1}, \tau_t^n, \mathbf{F}^n) \right| + \left| (\nabla \cdot \mathbf{u}_h^{n-1} \lambda \nu \mathbf{F}^n, \tau_t^n) \right| \quad (\text{using (4.19)}) \\ &\leq \lambda_M \nu \left\| \mathbf{u}_h^{n-1} \cdot \nabla \tau_t^n \right\| \|\mathbf{F}^n\| + \left| (\nabla \cdot (\mathbf{u}_h^{n-1} - \mathcal{U}^{n-1}) \lambda \nu \mathbf{F}^n, \tau_t^n) \right| \\ &\quad + \left| (\nabla \cdot \mathcal{U}^{n-1} \lambda \nu \mathbf{F}^n, \tau_t^n) \right| \\ &\leq \lambda_M \nu \left\| \mathbf{u}_h^{n-1} \right\|_\infty d^{\frac{1}{2}} \|\nabla \tau_t^n\| \|\mathbf{F}^n\| + \lambda_M \nu \left\| \nabla \cdot (\mathbf{u}_h^{n-1} - \mathcal{U}^{n-1}) \right\|_\infty \|\mathbf{F}^n\| \|\tau_t^n\| \\ &\quad + \left\| \nabla \cdot \mathcal{U}^{n-1} \right\|_\infty \lambda_M \nu \|\mathbf{F}^n\| \|\tau_t^n\| \\ &\leq \lambda_M^2 \left( 2 + d \left\| \nabla \mathbf{E}^{n-1} \right\|_\infty \right) \|\mathbf{F}^n\|^2 + \frac{\nu^2}{4} \left( d^2 M^2 + d \left\| \nabla \mathbf{E}^{n-1} \right\|_\infty \right) \|\tau_t^n\|^2 \\ &\quad + \frac{\nu^2}{4} K^2 d \|\nabla \tau_t^n\|^2, \quad (\text{using (4.5) and (IH1)}) . \end{aligned} \quad (4.35)$$

$$\begin{aligned} |b(\lambda \mathbf{u}_h^{n-1}, \tau^n, \tilde{\mathbf{F}}^n) - b(\lambda \mathbf{u}^n, \tau^n, \tilde{\mathbf{F}}^n)| &= \left| (\lambda (\mathbf{u}_h^{n-1} - \mathbf{u}^n) \cdot \nabla \tau^n, \tilde{\mathbf{F}}^n) \right| \\ &\leq \left\| (\mathbf{u}_h^{n-1} - \mathbf{u}^n) \cdot \nabla \tau^n \right\| \|\lambda \tilde{\mathbf{F}}^n\| \\ &\leq \frac{\lambda_M^2}{2} \|\tilde{\mathbf{F}}^n\|^2 + \frac{1}{2} d^3 \|\nabla \tau^n\|_\infty^2 \|\mathbf{u}_h^{n-1} - \mathbf{u}^n\|^2 \\ &\leq \lambda_M^2 \|\mathbf{F}^n\|^2 + \lambda_M^2 \nu^2 \|\mathbf{F}_u^n\|^2 + \frac{1}{2} d^3 M^2 \left\| -\mathbf{E}^{n-1} - \mathbf{\Lambda}^{n-1} + \mathbf{u}^{n-1} - \mathbf{u}^n \right\|^2 \\ &\leq \lambda_M^2 \|\mathbf{F}^n\|^2 + \lambda_M^2 \nu^2 \|\mathbf{F}_u^n\|^2 + \frac{3}{2} d^3 M^2 \left\| \mathbf{E}^{n-1} \right\|^2 + \frac{3}{2} d^3 M^2 \left\| \mathbf{\Lambda}^{n-1} \right\|^2 \\ &\quad + \frac{3}{2} d^3 M^2 \Delta t \int_{t^{n-1}}^{t^n} \|\mathbf{u}_t\|^2 dt. \end{aligned} \quad (4.36)$$

In order to estimate the  $g_a$  terms in  $\mathcal{F}_2(\cdot)$  note that

$$\begin{aligned} \lambda \left( g_a \left( \tau_h^{n-1}, \nabla \mathbf{u}_h^{n-1} \right) - g_a \left( \tau^n, \nabla \mathbf{u}^n \right) \right) &= \lambda \left( g_a \left( \tau_h^{n-1}, \nabla (\mathbf{u}_h^{n-1} - \mathcal{U}^{n-1}) \right) + g_a \left( \tau_h^{n-1}, \nabla (\mathcal{U}^{n-1} - \mathbf{u}^{n-1}) \right) \right) \\ &\quad + g_a \left( \tau_h^{n-1}, \nabla (\mathbf{u}^{n-1} - \mathbf{u}^n) \right) + g_a \left( \tau_h^{n-1} - \mathcal{T}^{n-1}, \nabla \mathbf{u}^n \right) \\ &\quad + g_a \left( \mathcal{T}^{n-1} - \tau^{n-1}, \nabla \mathbf{u}^n \right) + g_a \left( \tau^{n-1} - \tau^n, \nabla \mathbf{u}^n \right) \\ &= \lambda \left( -g_a \left( \tau_h^{n-1}, \nabla \mathbf{E}^{n-1} \right) - g_a \left( \tau_h^{n-1}, \nabla \mathbf{\Lambda}^{n-1} \right) \right) \\ &\quad - g_a \left( \tau_h^{n-1}, \nabla (\mathbf{u}^n - \mathbf{u}^{n-1}) \right) - g_a \left( \mathbf{F}^{n-1}, \nabla \mathbf{u}^n \right) \\ &\quad - g_a \left( \mathbf{\Gamma}^{n-1}, \nabla \mathbf{u}^n \right) - g_a \left( \tau^n - \tau^{n-1}, \nabla \mathbf{u}^n \right). \end{aligned} \quad (4.37)$$

Bounding each of the terms on the right hand side of (4.37) we obtain

$$\left| (\lambda g_a \left( \tau_h^{n-1}, \nabla \mathbf{E}^{n-1} \right), \tilde{\mathbf{F}}^n) \right| \leq \left\| g_a \left( \tau_h^{n-1}, \nabla \mathbf{E}^{n-1} \right) \right\| \|\lambda \tilde{\mathbf{F}}^n\|$$

$$\begin{aligned}
&\leq 4\acute{d} \left\| \tau_h^{n-1} \right\|_{\infty} \left\| \nabla \mathbf{E}^{n-1} \right\| \lambda_M \left\| \tilde{\mathbf{F}}^n \right\| \\
&\leq \epsilon_8 \left\| \nabla \mathbf{E}^{n-1} \right\|^2 + \frac{8\acute{d}^2 K^2 \lambda_M^2}{\epsilon_8} \left\| \mathbf{F}^n \right\|^2 + \frac{8\acute{d}^2 K^2 \lambda_M^2}{\epsilon_8} \nu^2 \left\| \mathbf{F}_u^n \right\|^2, \quad (4.38)
\end{aligned}$$

$$\left| \left( \lambda g_a \left( \tau_h^{n-1}, \nabla \mathbf{\Lambda}^{n-1} \right), \tilde{\mathbf{F}}^n \right) \right| \leq 8\acute{d}^2 K^2 \left\| \nabla \mathbf{\Lambda}^{n-1} \right\|^2 + \lambda_M^2 \left\| \mathbf{F}^n \right\|^2 + \lambda_M^2 \nu^2 \left\| \mathbf{F}_u^n \right\|^2, \quad (4.39)$$

$$\left| \left( \lambda g_a \left( \tau_h^{n-1}, \nabla (\mathbf{u}^n - \mathbf{u}^{n-1}) \right), \tilde{\mathbf{F}}^n \right) \right| \leq 8\acute{d}^2 K^2 \Delta t \int_{t^{n-1}}^{t^n} \left\| \nabla u_t \right\|^2 dt + \lambda_M^2 \left\| \mathbf{F}^n \right\|^2 + \lambda_M^2 \nu^2 \left\| \mathbf{F}_u^n \right\|^2, \quad (4.40)$$

$$\left| \left( \lambda g_a \left( \mathbf{F}^{n-1}, \nabla \mathbf{u}^n \right), \tilde{\mathbf{F}}^n \right) \right| \leq 8\acute{d}^2 M^2 \left\| \mathbf{F}^{n-1} \right\|^2 + \lambda_M^2 \left\| \mathbf{F}^n \right\|^2 + \lambda_M^2 \nu^2 \left\| \mathbf{F}_u^n \right\|^2, \quad (4.41)$$

$$\left| \left( \lambda g_a \left( \mathbf{\Gamma}^{n-1}, \nabla \mathbf{u}^n \right), \tilde{\mathbf{F}}^n \right) \right| \leq 8\acute{d}^2 M^2 \left\| \mathbf{\Gamma}^{n-1} \right\|^2 + \lambda_M^2 \left\| \mathbf{F}^n \right\|^2 + \lambda_M^2 \nu^2 \left\| \mathbf{F}_u^n \right\|^2, \quad (4.42)$$

$$\left| \left( \lambda g_a \left( \tau^n - \tau^{n-1}, \nabla \mathbf{u}^n \right), \tilde{\mathbf{F}}^n \right) \right| \leq 8\acute{d}^2 M^2 \Delta t \int_{t^{n-1}}^{t^n} \left\| \tau_t \right\|^2 dt + \lambda_M^2 \left\| \mathbf{F}^n \right\|^2 + \lambda_M^2 \nu^2 \left\| \mathbf{F}_u^n \right\|^2. \quad (4.43)$$

Combining the estimates in (4.29)-(4.36), (4.38)-(4.43), we obtain the following estimate for  $\mathcal{F}_2(\mathbf{F}^n)$ :

$$\begin{aligned}
|\mathcal{F}_2(\mathbf{F}^n)| &\leq \epsilon_8 \left\| \nabla \mathbf{E}^{n-1} \right\|^2 + \nu^2 \left\| \mathbf{F}_u^n \right\|^2 \left( 6\lambda_M^2 + 3 + \frac{8\acute{d}^2 K^2 \lambda_M^2}{\epsilon_8} \right) \\
&\quad + \left\| \mathbf{F}^n \right\|^2 \left( 11\lambda_M^2 + 2 + \frac{8\acute{d}^2 K^2 \lambda_M^2}{\epsilon_8} + \acute{d} \left\| \nabla \mathbf{E}^{n-1} \right\|_{\infty} \right) \\
&\quad + \left\| \mathbf{E}^{n-1} \right\|^2 \left( \frac{3}{2} \acute{d}^3 M^2 \right) + \left\| \mathbf{F}^{n-1} \right\|^2 \left( 8\acute{d}^2 M^2 \right) \\
&\quad + 2\alpha^2 \left\| \nabla \mathbf{\Lambda}^n \right\|^2 + \left\| \nabla \mathbf{\Gamma}^n \right\|^2 \left( \frac{\acute{d} K^2}{2} \right) + \left\| \mathbf{\Gamma}^n \right\|^2 \left( \frac{1}{2} \right) + \left\| d_t \mathbf{\Gamma}^n \right\|^2 \left( \frac{1}{4} \right) \\
&\quad + \left\| \nabla \mathbf{\Lambda}^{n-1} \right\|^2 \left( 8\acute{d}^2 K^2 \right) + \left\| \mathbf{\Lambda}^{n-1} \right\|^2 \left( \frac{3}{2} \acute{d}^3 M^2 \right) + \left\| \mathbf{\Gamma}^{n-1} \right\|^2 \left( 8\acute{d}^2 M^2 \right) \\
&\quad + \frac{1}{4} \left\| d_t \tau^n - \tau_t^n \right\|^2 + \frac{\nu^2}{4} \left( \acute{d}^2 M^2 + \acute{d} \left\| \nabla \mathbf{E}^{n-1} \right\|_{\infty} \right) \left\| \tau_t^n \right\|^2 + \frac{\nu^2}{4} K^2 \acute{d} \left\| \nabla \tau_t^n \right\|^2 \\
&\quad + \frac{3}{2} \acute{d}^3 M^2 \Delta t \int_{t^{n-1}}^{t^n} \left\| \mathbf{u}_t \right\|^2 dt + 8\acute{d}^2 M^2 \Delta t \int_{t^{n-1}}^{t^n} \left\| \tau_t \right\|^2 dt + 8\acute{d}^2 K^2 \Delta t \int_{t^{n-1}}^{t^n} \left\| \nabla \mathbf{u}_t \right\|^2 dt. \quad (4.44)
\end{aligned}$$

Then, from (4.17) we have that

$$\begin{aligned}
&\alpha Re_m \left( \left\| \mathbf{E}^l \right\|^2 - \left\| \mathbf{E}^0 \right\|^2 \right) + \frac{\lambda_m}{2} \left( \left\| \mathbf{F}^l \right\|^2 - \left\| \mathbf{F}^0 \right\|^2 \right) + 4\alpha(1-\alpha) \sum_{n=1}^l \frac{Re_m \Delta t}{Re_M} \left\| D(\mathbf{E}^n) \right\|^2 + \nu \sum_{n=1}^l \frac{\lambda_m^2 \Delta t}{\lambda_M} \left\| \mathbf{F}_u^n \right\|^2 \\
&\quad + \sum_{n=1}^l \frac{\lambda_m \Delta t}{\lambda_M} \left\| \mathbf{F}^n \right\|^2 \leq \Delta t \sum_{n=1}^l \left\{ 4\alpha^2 \left( C_K^2 Re_M^2 \epsilon_1 + \tilde{R}^2 \epsilon_2 + \epsilon_3 \right) + 2\alpha \left( C_K^2 \epsilon_5 + (1-\alpha) \epsilon_6 + \epsilon_7 \right) \right\} \left\| D(\mathbf{E}^n) \right\|^2 \\
&\quad + \Delta t \sum_{n=1}^l C_K^2 \epsilon_8 \left\| D(\mathbf{E}^{n-1}) \right\|^2 + \Delta t \sum_{n=1}^l \left( \frac{\acute{d} K^2}{4\epsilon_1} + 2\alpha \left( 3Re_M^2 + \frac{3}{2} \right) \right) \left\| \mathbf{E}^n \right\|^2 \\
&\quad + \Delta t \sum_{n=1}^l \left( \frac{3\acute{d}^2 M^2}{2} + \alpha Re_M^2 \acute{d}^2 M^2 \right) \left\| \mathbf{E}^{n-1} \right\|^2 \\
&\quad + \Delta t \sum_{n=1}^l \left\{ \left( \frac{\lambda_M}{2} + 1 \right) \acute{d} \left\| \nabla \mathbf{E}^{n-1} \right\|_{\infty} + \frac{\lambda_M \acute{d} M}{2} + \frac{1}{4\epsilon_2} + 3 + 11\lambda_M^2 + \frac{8\acute{d}^2 K^2 \lambda_M^2}{\epsilon_8} \right\} \left\| \mathbf{F}^n \right\|^2
\end{aligned}$$

$$\begin{aligned}
& + \Delta t \sum_{n=1}^l 8d^2 M^2 \left\| \mathbf{F}^{n-1} \right\|^2 \\
& + \Delta t \sum_{n=1}^l \nu^2 \left( \frac{1}{4\epsilon_3} + \frac{1}{4} + 6\lambda_M^2 + 3 + \frac{8d^2 K^2 \lambda_M^2}{\epsilon_8} \right) \left\| \mathbf{F}_u^n \right\|^2 \\
& + \Delta t \sum_{n=1}^l \frac{2\alpha \dot{d}}{4\epsilon_5} \left\| p^n - \mathbf{P}^n \right\|^2 + \Delta t \sum_{n=1}^l \left( \alpha Re_M^2 \dot{d}^2 M^2 + \frac{3\dot{d}^3 M^2}{2} \right) \left\| \Lambda^{n-1} \right\|^2 \\
& + \Delta t \sum_{n=1}^l 2\alpha \left( \frac{K^2 \dot{d}}{4} + \frac{1-\alpha}{\epsilon_6} + 2\alpha^2 \right) \left\| \nabla \Lambda^n \right\|^2 + \Delta t \sum_{n=1}^l 8d^2 K^2 \left\| \nabla \Lambda^{n-1} \right\|^2 \\
& + \Delta t \sum_{n=1}^l \frac{\alpha}{2} \left\| d_t \Lambda^n \right\|^2 + \Delta t \sum_{n=1}^l \frac{1}{4} \left\| d_t \Gamma^n \right\|^2 + \Delta t \sum_{n=1}^l \left( \frac{\alpha}{2\epsilon_7} + \frac{1}{2} \right) \left\| \Gamma^n \right\|^2 \\
& + \Delta t \sum_{n=1}^l \frac{\dot{d} K^2}{2} \left\| \nabla \Gamma^n \right\|^2 + \Delta t \sum_{n=1}^l 8d^2 M^2 \left\| \Gamma^{n-1} \right\|^2 \\
& + \Delta t \sum_{n=1}^l \frac{\alpha}{2} \left\| d_t \mathbf{u}^n - \mathbf{u}_t^n \right\|^2 + \Delta t \sum_{n=1}^l \frac{1}{4} \left\| d_t \tau^n - \tau_t^n \right\|^2 \\
& + \Delta t \sum_{n=1}^l \frac{\nu^2}{4} \left( d^2 M^2 + \dot{d} \left\| \nabla \mathbf{E}^{n-1} \right\|_\infty \right) \left\| \tau_t^n \right\|^2 \\
& + \Delta t \sum_{n=1}^l \left( \alpha Re_M^2 \dot{d}^2 M^2 \Delta t + \frac{3}{2} \dot{d}^3 M^2 \Delta t \right) \int_{t^{n-1}}^{t^n} \left\| \mathbf{u}_t \right\|^2 dt \\
& + \Delta t \sum_{n=1}^l \frac{\nu^2}{4} K^2 \dot{d} \left\| \nabla \tau_t^n \right\|^2 + \Delta t \sum_{n=1}^l 8d^2 M^2 \Delta t \int_{t^{n-1}}^{t^n} \left\| \tau_t \right\|^2 dt \\
& + \Delta t \sum_{n=1}^l 8d^2 K^2 \Delta t \int_{t^{n-1}}^{t^n} \left\| \nabla \mathbf{u}_t \right\|^2 dt. \tag{4.45}
\end{aligned}$$

With the following choices:

$$\begin{aligned}
\epsilon_1 &= \frac{Re_m(1-\alpha)}{14C_K^2 Re_M^3 \alpha}, & \epsilon_2 &= \frac{Re_m(1-\alpha)}{14\tilde{R}^2 Re_M \alpha}, & \epsilon_3 &= \frac{Re_m(1-\alpha)}{14 Re_M \alpha}, \\
\epsilon_5 &= \frac{Re_m(1-\alpha)}{7C_K^2 Re_M}, & \epsilon_6 &= \frac{Re_m}{7Re_M}, & \epsilon_7 &= \frac{Re_m(1-\alpha)}{7Re_M}, \\
\epsilon_8 &= \frac{Re_m 2\alpha(1-\alpha)}{7Re_M C_K^2}, & \mathbf{u}_h^0 &= \mathcal{U}^0 (\Rightarrow \mathbf{E}^0 = \mathbf{0}), & \tau_h^0 &= \mathcal{T}^0 (\Rightarrow \mathbf{F}^0 = \mathbf{0}),
\end{aligned}$$

substituting into (4.45) yields

$$\begin{aligned}
& \alpha Re_m \left\| \mathbf{E}^l \right\|^2 + \frac{\lambda_m}{2} \left\| \mathbf{F}^l \right\|^2 + 2\alpha(1-\alpha) \frac{Re_m}{Re_M} \Delta t \sum_{n=1}^l \left\| D(\mathbf{E}^n) \right\|^2 \\
& + \left( \nu \frac{\lambda_m^2}{\lambda_M} - \nu^2 \left( \frac{7\tilde{R}^2 Re_M \alpha}{2Re_m(1-\alpha)} + \frac{28\dot{d}^2 C_K^2 K^2 \lambda_M^2 Re_M}{\alpha(1-\alpha) Re_m} + \frac{13}{4} + 6\lambda_M^2 \right) \right) \Delta t \sum_{n=1}^l \left\| \mathbf{F}_u^n \right\|^2
\end{aligned}$$

$$\begin{aligned}
+\frac{\lambda_m}{\lambda_M}\Delta t \sum_{n=1}^l \|\mathbf{F}^n\|^2 &\leq C_1\Delta t \sum_{n=1}^l \|\mathbf{E}^n\|^2 + \Delta t \sum_{n=1}^l \left( C_2 \left( 1 + \|\nabla \mathbf{E}^{n-1}\|_\infty \right) \right) \|\mathbf{F}^n\|^2 \\
&+ C_3\Delta t \sum_{n=1}^l \|\mathbf{\Lambda}^n\|^2 + C_4\Delta t \sum_{n=1}^l \|\nabla \mathbf{\Lambda}^n\|^2 + \frac{\alpha}{2}\Delta t \sum_{n=1}^l \|d_t \mathbf{\Lambda}^n\|^2 \\
&+ \frac{1}{4}\Delta t \sum_{n=1}^l \|d_t \mathbf{\Gamma}^n\|^2 + C_5\Delta t \sum_{n=1}^l \|\mathbf{\Gamma}^n\|^2 + C_6\Delta t \sum_{n=1}^l \|\nabla \mathbf{\Gamma}^n\|^2 \\
&+ \frac{\alpha}{2}\Delta t \sum_{n=1}^l \|d_t \mathbf{u}^n - \mathbf{u}_t^n\|^2 + \frac{1}{4}\Delta t \sum_{n=1}^l \|d_t \tau^n - \tau_t^n\|^2 \\
&+ \frac{2\alpha \acute{d}}{4\epsilon_5}\Delta t \sum_{n=1}^l \|p^n - \mathbf{P}^n\|^2 \\
&+ \Delta t \sum_{n=1}^l \frac{\nu^2}{4} \left( \acute{d}^2 M^2 + \acute{d} \|\nabla \mathbf{E}^{n-1}\|_\infty \right) \|\tau_t^n\|^2 \\
&+ |\Delta t|^2 \left( \alpha Re_M^2 \acute{d}^2 M^2 + \frac{3}{2} \acute{d}^3 M^2 \right) \|\mathbf{u}_t\|_{0,0}^2 + K^2 \acute{d} \frac{\nu^2}{4} \|\nabla \tau_t\|_{0,0}^2 \\
&+ 8\acute{d}^2 M^2 |\Delta t|^2 \|\tau_t\|_{0,0}^2 + 8\acute{d}^2 K^2 |\Delta t|^2 \|\nabla \mathbf{u}_t\|_{0,1}^2. \tag{4.46}
\end{aligned}$$

We now apply the interpolation properties of the approximating spaces to estimate the terms on the right hand side of (4.46). Using elements of order  $k$  for velocity, elements of order  $m$  for stress, and elements of order  $q$  for pressure, we have

$$\begin{aligned}
\sum_{n=1}^l \Delta t \|\nabla \mathbf{\Lambda}^n\|^2 + \sum_{n=1}^l \Delta t \|\nabla \mathbf{\Gamma}^n\|^2 &\leq C \left( h^{2k} \sum_{n=1}^l \Delta t \|\mathbf{u}^n\|_{k+1}^2 + h^{2m} \sum_{n=1}^l \Delta t \|\tau^n\|_{m+1}^2 \right) \\
&\leq C \left( h^{2k} \|\mathbf{u}\|_{0,k+1}^2 + h^{2m} \|\tau\|_{0,m+1}^2 \right), \tag{4.47}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^l \Delta t \|\mathbf{\Lambda}^n\|^2 + \sum_{n=1}^l \Delta t \|\mathbf{\Gamma}^n\|^2 + \sum_{n=1}^l \Delta t \|p - \mathcal{P}^n\|^2 \\
\leq C \left( h^{2k+2} \sum_{n=1}^l \Delta t \|\mathbf{u}^n\|_{k+1}^2 + h^{2m+2} \sum_{n=1}^l \Delta t \|\tau^n\|_{m+1}^2 + h^{2q+2} \sum_{n=1}^l \Delta t \|p^n\|_{q+1}^2 \right) \\
\leq C \left( h^{2k+2} \|\mathbf{u}\|_{0,k+1}^2 + h^{2m+2} \|\tau\|_{0,m+1}^2 + h^{2q+2} \|p\|_{0,q+1}^2 \right), \tag{4.48}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^l \Delta t \|d_t \mathbf{\Lambda}^n\|^2 &= \sum_{n=1}^l \Delta t \left\| \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} 1 \frac{\partial \mathbf{\Lambda}}{\partial t} dt \right\|^2 \\
&\leq \sum_{n=1}^l \Delta t \left( \frac{1}{\Delta t} \right)^2 \int_{\Omega} \left( \int_{t_{n-1}}^{t_n} 1 dt \right) \left( \int_{t_{n-1}}^{t_n} \left( \frac{\partial \mathbf{\Lambda}}{\partial t} \right)^2 dt \right) dx \\
&\leq Ch^{2k+2} \|\mathbf{u}_t\|_{0,k+1}^2, \tag{4.49}
\end{aligned}$$

and similarly,

$$\sum_{n=1}^l \Delta t \|d_t \mathbf{\Gamma}^n\|^2 \leq Ch^{2m+2} \|\tau_t\|_{0,m+1}^2. \tag{4.50}$$

Note that  $d_t \mathbf{u}^n - \mathbf{u}_t^n$  may be expressed as

$$d_t \mathbf{u}^n - \mathbf{u}_t^n = \frac{1}{2\Delta t} \int_{t_{n-1}}^{t_n} \mathbf{u}_{tt}(\cdot, t)(t_{n-1} - t) dt.$$

Also,

$$\begin{aligned} \left( \frac{1}{2\Delta t} \int_{t_{n-1}}^{t_n} \mathbf{u}_{tt}(\cdot, t)(t_{n-1} - t) dt \right)^2 &\leq \frac{1}{4|\Delta t|^2} \int_{t_{n-1}}^{t_n} \mathbf{u}_{tt}(\cdot, t)^2 dt \int_{t_{n-1}}^{t_n} (t_{n-1} - t)^2 dt \\ &= \frac{1}{12} \Delta t \int_{t_{n-1}}^{t_n} \mathbf{u}_{tt}^2(\cdot, t) dt. \end{aligned}$$

Therefore it follows that

$$\begin{aligned} \sum_{n=1}^l \Delta t \|d_t \mathbf{u}^n - \mathbf{u}_t^n\|^2 &\leq \sum_{n=1}^l \Delta t \int_{\Omega} \frac{1}{12} \Delta t \int_{t_{n-1}}^{t_n} \mathbf{u}_{tt}^2(\cdot, t) dt d\mathbf{x} \\ &= \frac{1}{12} |\Delta t|^2 \|\mathbf{u}_{tt}\|_{0,0}^2. \end{aligned} \quad (4.51)$$

Similarly, for  $d_t \tau^n - \tau_t^n$  we have

$$\sum_{n=1}^l \Delta t \|d_t \tau^n - \tau_t^n\|^2 \leq \frac{1}{12} |\Delta t|^2 \|\tau_{tt}\|_{0,0}^2. \quad (4.52)$$

In view of (4.47)-(4.52), our induction hypotheses (IH1),(IH2), and with  $\nu$  chosen such that

$$\nu \leq \frac{1}{2} \frac{\lambda_m^2}{\lambda_M} \left( \frac{7\tilde{R}^2 Re_M \alpha}{2Re_m(1-\alpha)} + \frac{28d^2 C_K^2 K^2 \lambda_M^2 Re_M}{\alpha(1-\alpha)Re_m} + \frac{13}{4} + 6\lambda_M^2 \right)^{-1} \quad (4.53)$$

from (4.46) we obtain

$$\begin{aligned} &\alpha Re_m \|\mathbf{E}^l\|^2 + \frac{\lambda_m}{2} \|\mathbf{F}^l\|^2 + 2\alpha(1-\alpha) \frac{Re_m}{Re_M} \Delta t \sum_{n=1}^l \|D(\mathbf{E}^n)\|^2 + \frac{\nu}{2} \frac{\lambda_m^2}{\lambda_M} \sum_{n=1}^l \Delta t \|\mathbf{F}_u^n\|^2 \\ &\leq C \sum_{n=1}^l \Delta t (\|\mathbf{E}^n\|^2 + \|\mathbf{F}^n\|^2) + C \sum_{n=1}^l \Delta t \|\nabla \mathbf{E}^{n-1}\|_{\infty} \|\mathbf{F}^n\|^2 + \nu^2 C \|\tau_t\|_{0,1}^2 \\ &\quad + \frac{\nu^2}{4} \sum_{n=1}^l \Delta t \|\nabla \mathbf{E}^{n-1}\|_{\infty} \|\tau_t^n\|^2 + C |\Delta t|^2 (\|\mathbf{u}\|_{0,1}^2 + \|\tau_t\|_{0,0}^2 + \|\mathbf{u}_{tt}\|_{0,0}^2 + \|\tau_{tt}\|_{0,0}^2) \\ &\quad + Ch^{2k+2} \|\mathbf{u}\|_{0,k+1}^2 + Ch^{2m+2} \|\tau\|_{0,m+1}^2 + Ch^{2q+2} \|p\|_{0,q+1}^2 + Ch^{2k} \|\mathbf{u}\|_{0,k+1}^2 \\ &\quad + Ch^{2k+2} \|\mathbf{u}_t\|_{0,k+1}^2 + Ch^{2m} \|\tau\|_{0,m+1}^2 + Ch^{2m+2} \|\tau_t\|_{0,m+1}^2, \end{aligned} \quad (4.54)$$

where the  $C$ 's denote constants independent of  $l, \Delta t, h, \nu$ . Applying Gronwall's lemma and (IH2) to (4.54), the estimate given in (4.6) follows. ■

**Step 2.** We show that the induction hypotheses, (IH1) and (IH2) are true.

**Verification of (IH1)**

Assume that (IH1) holds true for  $n = 1, 2, \dots, l-1$ . By interpolation properties, inverse estimates and (4.6), we have that

$$\begin{aligned}
\|\mathbf{u}_h^l\|_\infty &\leq \|\mathbf{u}_h^l - \mathbf{u}^l\|_\infty + \|\mathbf{u}^l\|_\infty \\
&\leq \|\mathbf{E}^l\|_\infty + \|\Lambda^l\|_\infty + M \\
&\leq Ch^{-\frac{d}{2}} \|\mathbf{E}^l\|_0 + Ch^{-\frac{d}{2}} \|\Lambda^l\|_0 + M \\
&\leq C \left( |\Delta t| h^{-\frac{d}{2}} + \nu h^{-\frac{d}{2}} + h^{k-\frac{d}{2}} + h^{m-\frac{d}{2}} + h^{q+1-\frac{d}{2}} + h^{k+1-\frac{d}{2}} \right) + M. \tag{4.55}
\end{aligned}$$

Note that the expression  $C \left( |\Delta t| h^{-\frac{d}{2}} + \nu h^{-\frac{d}{2}} + h^{k-\frac{d}{2}} + h^{m-\frac{d}{2}} + h^{q+1-\frac{d}{2}} + h^{k+1-\frac{d}{2}} \right)$  is independent of  $l$ . Hence, if we set  $k, m \geq \frac{d}{2}, q \geq \frac{d}{2} - 1$ , and choose  $h, \Delta t, \nu$  such that

$$h^{k-\frac{d}{2}}, h^{m-\frac{d}{2}}, h^{q+1-\frac{d}{2}} \leq \frac{1}{C}, \quad \Delta t, \nu \leq \frac{h^{\frac{d}{2}}}{C}, \tag{4.56}$$

then from (4.55)

$$\|\mathbf{u}_h^l\|_\infty \leq M + 6.$$

Similarly it follows that  $\|\tau_h^l\|_\infty \leq M + 6$ . ■

### Verification of (IH2)

Assume that (IH2) is true for  $n = 1, 2, \dots, l-1$ . Equations (4.6), (4.54), and Korn's inequality imply

$$\sum_{n=1}^l \Delta t \|\nabla \mathbf{E}^n\|_0^2 \leq C \left( h^{2k} + h^{2m} + h^{2q+2} + |\Delta t|^2 + \nu^2 \right). \tag{4.57}$$

Applying the inverse estimate and using the inequality

$$\sum_{n=1}^l a_n \leq \sqrt{l} \left( \sum_{n=1}^l a_n^2 \right)^{\frac{1}{2}},$$

from (4.57) we obtain

$$\begin{aligned}
\sum_{n=1}^l \Delta t \|\nabla \mathbf{E}^n\|_\infty &\leq Ch^{-\frac{d}{2}} \sum_{n=1}^l \Delta t \|\nabla \mathbf{E}^n\| \\
&\leq Ch^{-\frac{d}{2}} \sqrt{\Delta t} \sqrt{l} \left( \sum_{n=1}^l \Delta t \|\nabla \mathbf{E}^n\|_0^2 \right)^{\frac{1}{2}} \\
&\leq \tilde{C} \left( \Delta t h^{-\frac{d}{2}} + \nu h^{-\frac{d}{2}} + h^{k-\frac{d}{2}} + h^{m-\frac{d}{2}} + h^{q+1-\frac{d}{2}} \right),
\end{aligned}$$

where  $\tilde{C} = C\sqrt{T}$  is a constant independent of  $l, h, \Delta t$ , and  $\nu$ . Hence when

$$\nu, \Delta t \leq \frac{h^{\frac{d}{2}}}{5\tilde{C}}, \tag{4.58}$$

and

$$h^{k-\frac{d}{2}}, h^{m-\frac{d}{2}}, h^{q+1-\frac{d}{2}} \leq \frac{1}{5\tilde{C}},$$

(IH2) holds. ■

**Step 3.** We derive the error estimates in (4.1) and (4.2).

**Proof of the Theorem 4.1.**

Using estimates (4.6) and (approximation properties), we have

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{\infty,0}^2 + \|\tau - \tau_h\|_{\infty,0}^2 &\leq \|\mathbf{E}\|_{\infty,0}^2 + \|\Lambda\|_{\infty,0}^2 + \|\mathbf{F}\|_{\infty,0}^2 + \|\Gamma\|_{\infty,0}^2 \\ &\leq G(\Delta t, h, \nu) + C \left( h^{2k+2} \|\mathbf{u}\|_{\infty,k+1}^2 + h^{2m+2} \|\tau\|_{\infty,m+1}^2 \right). \end{aligned}$$

Note the restrictions on  $\nu$  and  $\Delta t$  from (4.53), (4.56), (4.58), (3.18), (3.19), and the hypothesis of Theorem 4.1.

To establish (4.2), from (4.6), (4.54), we have

$$\|\nabla \mathbf{E}\|_{0,0}^2 \leq C(T+1)G(\Delta t, h, \nu) \quad (4.59)$$

and

$$\|\mathbf{E}\|_{0,0}^2 + \|\mathbf{F}\|_{0,0}^2 \leq TG(\Delta t, h, \nu). \quad (4.60)$$

Hence

$$\|\mathbf{E}\|_{1,0}^2 + \|\mathbf{F}\|_{0,0}^2 \leq \tilde{C}G(\Delta t, h, \nu). \quad (4.61) \quad \blacksquare$$

## 5 Numerical Results

In this section, we present a numerical simulation of viscoelastic fluid flow involving two immiscible fluids. For a discussion on the numerical implementation of the continuum surface force model see [17].

Let  $\Omega := (0, 1) \times (0, 1)$ , and at  $t = 0$ , let

$$\Omega_1 := \left\{ (x, y) : \frac{(x-0.5)^2}{0.35^2} + \frac{(y-0.5)^2}{0.25^2} < 1 \right\}, \quad \mathcal{I} := \left\{ (x, y) : \frac{(x-0.5)^2}{0.35^2} + \frac{(y-0.5)^2}{0.25^2} = 1 \right\}$$

and  $\Omega_2 = \Omega \setminus (\Omega_1 \cup \mathcal{I})$ . Initially, both fluids are at rest,  $\mathbf{u}(\mathbf{x}, 0) = \mathbf{0}$ . We assume,  $Re_1 = Re_2 = 1.0$  and  $\lambda_1 = \lambda_2 = 0.1$ . It is common in polymer processing that two fluids have very similar properties so the above assumptions are reasonable. Also, the coefficient of interfacial tension is assumed to be constant,  $\sigma = 5.0$ .

From a minimum energy argument, we have that the interfacial forces will drive  $\Omega_1$  from its initial elliptical profile to a circular orientation.

In the computations we use for  $\nu$ , the SUPG coefficient,  $\nu = 0.6h$ , and take  $\Delta t = h/2$ . To approximate the velocity and pressure we use the Taylor-Hood approximation pair (continuous piecewise quadratics for velocity, continuous piecewise linears for pressure) and use a continuous piecewise linear approximation for the polymeric stress.

Presented in Figures 5.1, 5.2, 5.3, and 5.4 is the velocity field and the interface  $\mathcal{I}$  at times  $t = 0.00, 0.11, 0.55$ , and  $3.54$ , for the grid with  $h = 1/64$ .

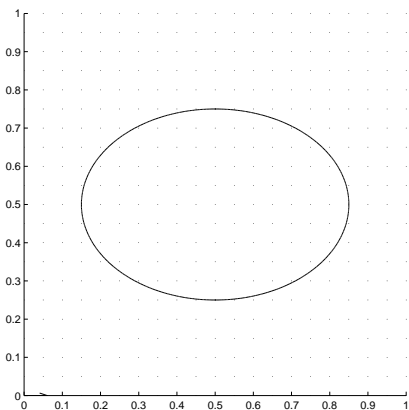


Figure 5.1: Initial velocity field

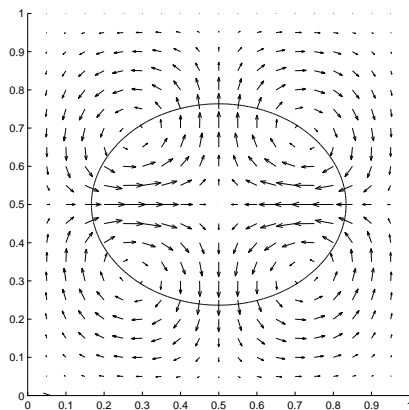


Figure 5.2: Velocity field after 10 time steps

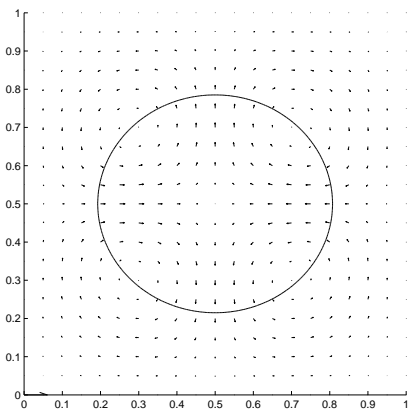


Figure 5.3: Velocity field after 50 time steps

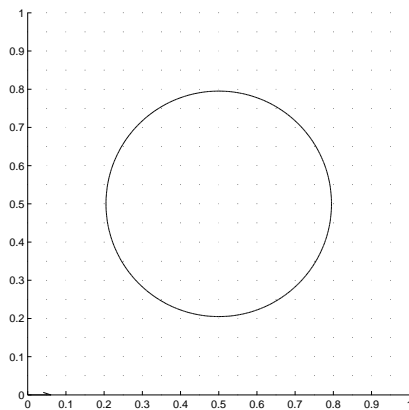


Figure 5.4: Velocity field after 320 time steps

In Table 5.1, we list  $\|\mathbf{u}_h\|_{0,1}$  and  $\|\tau_h\|_{0,0}$  at time  $T = 3.536$ , together with their experimental convergence rates. The experimental convergence rate for  $\|\mathbf{u}_h\|_{0,1}$  was computed as follows. From Theorem 4.1, the choice of approximating elements used,  $\nu = 0.6 h$ , and  $\Delta t = h/2$ , we have

$$\|\mathbf{u}_h\|_{0,1} - \|\mathbf{u}\|_{0,1} \leq \|\mathbf{u}_h - \mathbf{u}\|_{0,1} \leq C_v h. \quad (5.62)$$

Using  $\|\mathbf{u}_{1/64}\|_{0,1} - \|\mathbf{u}\|_{0,1} = C \frac{1}{64}$  and  $\|\mathbf{u}_{1/48}\|_{0,1} - \|\mathbf{u}\|_{0,1} = C \frac{1}{48}$ , we obtain an estimate for

$\|\mathbf{u}\|_{0,1} \sim \|\mathbf{u}_\infty\| = 0.455327$  and an estimate for  $C_v = 1.072320$ .

Using  $\|\mathbf{u}_\infty\|_{0,1}$  we then compute the experimental convergence rates for  $\|\mathbf{u}_h\|_{0,1}$  given in Table 5.1. The experimental convergence rates for  $\|\tau_h\|_{0,0}$  are computed analogously. From Theorem 4.1 and (5.62), we have that the theoretical asymptotic convergence rates for  $\|\mathbf{u}_h\|_{0,1}$  and  $\|\tau_h\|_{0,0}$  is 1.

$h$	$\ \mathbf{u}_h\ _{0,1}$	Exp. conv. rate	$\ \tau_h\ _{0,0}$	Exp. conv. rate
1/32	.422046	1.00	.463853	1.00
1/40	.428505	1.00	.458594	1.00
1/45	.431560	1.00	.456062	1.00
1/48	.432987	1.00	.454650	1.00
1/50	.434386	1.01	.454567	0.99
1/54	.436445	1.01	.453620	0.98
1/64	.438572	1.00	.449975	1.00
$\infty$	.455327		.435950	

Table 5.1: Experimental Rates of Convergence

## References

- [1] J. Baranger and A. Machmoum. Existence of approximate solutions and error bounds for viscoelastic fluid flow: Characteristics method. *Comput. Methods Appl. Mech. Engrg.*, 148:39–52, 1997.
- [2] J. Baranger and D. Sandri. Finite element approximation of viscoelastic fluid flow: Existence of approximate solutions and error bounds. I. Discontinuous constraints. *Numer. Math.*, 63:13–27, 1992.
- [3] J. Baranger and S. Wardi. Numerical analysis of a FEM for a transient viscoelastic flow. *Comput. Methods Appl. Mech. Engrg.*, 125:171–185, 1995.
- [4] G. K. Batchelor. *An Introduction to Fluid Dynamics*. Cambridge University Press, Cambridge, 1967.
- [5] R. B. Bird, R. C. Armstrong, and O. Hassager. *Dynamics of Polymeric Liquids*. John Wiley and Sons, Inc., 1987.
- [6] J. U. Brackbill, D. B. Kothe, and C. Zemach. A continuum method for modeling surface tension. *J. Comp. Phys.*, 100:335–354, 1992.
- [7] S. C. Brenner and L. R. Scott. *The Mathematical Theory of Finite Element Methods*. Springer-Verlag Publishing, 1994.
- [8] P. G. Ciarlet. *The Finite Element Methods for Elliptic Problems*. North-Holland Publishing, 1979.
- [9] Ph. Clément. Approximation by finite element functions using local regularization. *RAIRO Anal. Numer.*, 2:77–84, 1975.
- [10] V. J. Ervin and W. W. Miles. Approximation of time-dependent, viscoelastic fluid flow: SUPG approximation. To appear in *SIAM J. Numer. Anal.* (available at <http://www.math.clemson.edu/~vjervin/papers/erv021.pdf>), 2002.
- [11] C. Guillope and J. C. Saut. Existence results for the flow of viscoelastic fluids with a differential constitutive law. *Nonlinear Anal. Theory Methods Appl.*, 15:849–869, 1990.
- [12] J. G. Heywood and R. Rannacher. Finite element approximation of the nonstationary Navier–Stokes problem. Part IV: Error analysis for second-order time discretization. *SIAM J. Numer. Anal.*, 2:353–384, 1990.
- [13] B. Liu. The analysis of a finite element method with streamline diffusion for the compressible Navier-Stokes equations. *SIAM J. Numer. Anal.*, 1:1–16, 2000.
- [14] K. Najib and D. Sandri. On a decoupled algorithm for solving a finite element problem for the approximation of viscoelastic fluid flow. *Numer. Math.*, 72:223–238, 1995.
- [15] M. Renardy. *Mathematical Analysis of Viscoelastic Flows*. SIAM, Philadelphia, PA, 2000.

- [16] D. Sandri. Finite element approximation of viscoelastic fluid flow: Existence of approximate solutions and error bounds. Continuous approximation of the stress. *SIAM J. Numer. Anal.*, 31:362–377, 1994.
- [17] A. Tornberg and B. Engquist. A finite element based level-set method for multiphase flow applications. *Comput. Visual. Sci.*, 3:93–101, 2000.
- [18] D. F. Zhang and D. A. Zumbrunnen. Influences of fluidic interfaces on the formation of fine scale structures by chaotic mixing. *J. Fluids Eng.*, 118, 1996.