

A Nonlinear Subgrid–Scale Model for Convection Dominated, Convection Diffusion Problems

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Abstract

We present a nonlinear subgrid–scale method for the stabilization of the Galerkin approximation to convection dominated, convection diffusion problems, establish existence and uniqueness results, and provide an a priori error estimate for the method.

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1 Introduction

In this report we present a new stabilizing method for the solution of convection dominated, convection diffusion problems. This new approach is a synthesis of the hierarchical subgrid model recently proposed and analyzed by Guermond in [11] and the nonlinear artificial diffusion method studied by Iliescu in [15]. These methods have the same stability and convergence properties as those for the streamline diffusion method [16].

We consider the modeling equation

$$\left. \begin{aligned} \mathcal{L}_\epsilon u &:= -\epsilon \Delta u + \mathbf{v} \cdot \nabla u + q u = f, & \text{in } \Omega, \\ u &= 0, & \text{on } \Gamma. \end{aligned} \right\} \quad (1.1)$$

This equation has been – and continues to be – well investigated as it is an important modeling equation in many applications, such as turbulent flow simulations, pollution dispersement, and computational meteorology.

It is well known that naive approximations to (1.1) contain nonphysical oscillations unless the discretization parameter h is chosen sufficiently small such that $\|\mathbf{v}\|h/\epsilon \sim O(1)$. From a Galerkin approximation point of view, these nonphysical oscillations are due to the fact that the approximation, u_h , is not stable in the Sobolev space H^1 as $\epsilon \rightarrow 0$.

In [11], in part analogous to the subgrid scale approach used in computational fluid dynamics, Guermond introduces an additional component into the bilinear form used in the approximation. This additional component is only defined on the added “subgrid-scale” elements. Combining the approaches of Iliescu and Guermond we introduce into the approximation scheme a nonlinear subgrid-scale model and analyze its properties.

In Section 2, we define the notation that we will use throughout the report. We then present the approximation scheme in Section 3. Finally, existence and uniqueness results and an a priori error estimate for the approximation are derived in Section 4.

2 Notation

In the following we assume (1.1) satisfies the following:

$$(A1) \quad 0 < \epsilon \ll 1,$$

$$(A2) \quad \mathbf{v} \in W^{1,\infty}(\Omega), \quad q \in L^\infty(\Omega), \quad \|\mathbf{v}\|_\infty + \|q\|_\infty = O(1),$$

$$(A3) \quad -\frac{1}{2}\nabla \cdot \mathbf{v} + q \geq 1.$$

As usual, $\|\cdot\|_0$ and $\|\cdot\|_1$ denote the Sobolev norms on $L^2(\Omega)$ and $H^1(\Omega)$, respectively, and $\|\cdot\|_\infty$ denotes the L^∞ norm on Ω .

Let $\Pi_h(\Omega)$ denote an edge-to-edge finite element triangulation of Ω . Also, let $S_h \subset H_0^1(\Omega)$ denote the conforming finite element space of piecewise linears on $\Pi_h(\Omega)$, and let $S_s \subset H_0^1(\Omega)$ denote the subgrid-space (e.g. cubic bubble functions) on $\Pi_h(\Omega)$. For a triangle T in $\Pi_h(\Omega)$, let h_T represent the local mesh parameter for T .

Let $B_\epsilon(\cdot, \cdot)$, $b(\cdot, \cdot)$, $b_{1,T}(\cdot, \cdot)$, $b_{sy}(\cdot, \cdot)$, $b_{sy,T}(\cdot, \cdot)$: $H_0^1(\Omega) \times H_0^1(\Omega) \longrightarrow \mathbb{R}$ denote respectively the following bilinear forms:

$$B_\epsilon(u, v) := \int_\Omega \epsilon \nabla u \cdot \nabla v + (\mathbf{v} \cdot \nabla u + qu) v \, dx, \quad (2.1)$$

$$b(u, v) := \int_\Omega (\mathbf{v} \cdot \nabla u + qu) v \, dx, \quad (2.2)$$

$$b_{1,T}(u, v) := \int_T \mathbf{v} \cdot \nabla u v \, dx, \quad T \in \Pi_h(\Omega), \quad (2.3)$$

$$b_{sy}(u, v) := \int_\Omega (q - \nabla \cdot \mathbf{v}/2) u v \, dx, \quad (2.4)$$

$$b_{sy,T}(u, v) := \int_T (q - \nabla \cdot \mathbf{v}/2) u v \, dx, \quad T \in \Pi_h(\Omega). \quad (2.5)$$

We define an energy norm, and a streamline-derivative seminorm associated with (1.1) as

$$\|u\| := \left\{ \epsilon \|\nabla u\|_0^2 + \|u\|_0^2 \right\}^{1/2}, \quad (2.6)$$

$$|u|_{1,\mathbf{v}} := \left(\int_{\Omega} (\mathbf{v} \cdot \nabla u)^2 dx \right)^{1/2}. \quad (2.7)$$

The notation $|\cdot|_{1,\mathbf{v},T}$, $\|\cdot\|_T$, is used to denote the streamline derivative seminorm and the L_2 norm restricted to the triangle T .

Note that assumptions (A1) - (A3), the definition (2.1), and integration by parts imply that

$$B_{\epsilon}(v, v) \geq \|v\|^2, \quad \forall v \in H_0^1(\Omega). \quad (2.8)$$

Definition of the Subgrid-Space

Given $U \in S_h \oplus S_s := X_h$, introduce $U_h := I_h U \in S_h$ as the (bounded) projection operator defined as the nodal interpolant of U in S_h , and $U_s := U - U_h \in S_s$. We assume that the usual inverse inequalities hold on S_h and S_s .

We further assume a decomposition of S_s into a direct sum satisfying

$$S_s = \bigoplus_{T \in \Pi_h} S_s^T,$$

where, for $f \in S_s^T$ we have that $\text{supp}(f) \subset T$.

Note the space of bubble functions satisfies this decomposition.

Definition of the Sub-Grid-Regularization-Operator

Let $AV_s(\cdot, \cdot) : S_s \times S_s \rightarrow \mathbb{R}$, be defined by:

$$AV_s(U, V) := \sum_{T \in \Pi_h} h_T \int_T d(|\nabla U|) \nabla U \cdot \nabla V \, dx, \quad (2.9)$$

where

(A4) $d(\cdot)$ is uniformly Lipschitz continuous from $\mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}^+$

(A5) $0 < d_{min} \leq d(\cdot) \leq d_{max}$,

(A6) $d(\cdot)$ is monotone nondecreasing.

A key inequality which is used to establish that the approximation satisfies the optimality error estimates for this problem is

$$\forall V \in X_h, \forall T \in \Pi_h \quad \sup_{\phi_s \in S_s^T} \frac{b_{1,T}(V_h, \phi_s)}{\|\phi_s\|_T} \geq c_a |V_h|_{1,\mathbf{v},T} - c_\delta b_{sy,T}(V, V)^{1/2}. \quad (2.10)$$

This local property of the approximating spaces is satisfied by some commonly used approximating elements.

Theorem 2.1 *Let S_h represent the space of continuous piecewise linears, and let S_s represent the space of cubic bubbles functions on Π_h . Then, the estimate (2.10) is satisfied.*

Proof: See Guermond [11] pg. 1303-1305. ■

3 Discrete Approximation

The approximation $U \in X_h \subset H_0^1(\Omega)$ is determined via:

$$B_\epsilon(U, v) + AV_s(U_s, v_s) = (f, v) \quad \forall v \in X_h. \quad (3.1)$$

Note that the true solution $u \in H_0^1(\Omega)$ satisfies

$$B_\epsilon(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega). \quad (3.2)$$

The space S_h represents the space of *resolvable modes* of the true solution u . The subgrid-scale space S_s is introduced to stabilize the approximation method. As such, the overall accuracy of the approximation U is controlled by the approximation property of u in S_h . Consequently, the error estimates we establish in this paper are given in terms of the mesh parameter h for S_h .

4 Existence, Uniqueness, and A Priori Error Estimate

In this section, following the approach presented in [15], we establish the existence and uniqueness of the approximation U given by (3.1), and we derive an a priori error estimate for U .

To show existence of a solution U to (3.1), we rewrite (3.1) as a fixed point problem in the following form:

$$u = Au, u \in X. \quad (4.1)$$

The Leray–Schauder Principle considers a family of fixed point problems associated with (4.1). Namely,

$$u = tAu, u \in X \quad 0 \leq t < 1. \quad (4.2)$$

Theorem 4.1 (Leray–Schauder Principle: [29], Pg.64) *Let X denote a Banach Space and $A : X \rightarrow X$ a compact operator. If there is a number $r > 0$ such that if u is a solution of (4.2) it satisfies $\|u\| \leq r$, then (4.1) has a solution.*

■

Lemma 4.1 (Existence of U) *Assume (A1)–(A6) are satisfied. Then, there exists a solution U to (3.1) satisfying:*

$$\|U\|_1 \leq \epsilon^{-1} \|f\|_{-1}, \quad (4.3)$$

where $\|\cdot\|_{-1}$ denotes the norm on $H^{-1}(\Omega)$.

Proof:

Notationally, we rewrite (3.1) in the form (4.1) via:

$$U = B_\epsilon^{-1}(f) - B_\epsilon^{-1}(AV_s(U)) := A(U).$$

To verify the existence of a bound r to U satisfying (4.2), we consider, for $0 < t \leq 1$,

$$B_\epsilon \left(\frac{1}{t}U, v \right) + AV_s(U_s, v_s) = (f, v), \quad v \in X_h.$$

From assumptions (A3) and (A4) we have with $v = U$,

$$\begin{aligned} \frac{1}{t}\epsilon \|\nabla U\|^2 + \frac{1}{t}\|U\|^2 &\leq \|f\|_{-1}\|U\|_1, \\ \implies \|U\|_1 &\leq \frac{t}{\min\{\epsilon, 1\}}\|f\|_{-1} \leq \epsilon^{-1}\|f\|_{-1}. \end{aligned}$$

Next, as X_h is finite dimensional, the compactness of A is equivalent to A being continuous.

To verify the continuity of A at U , let $y_1 := A(U)$ and $y_2 := A(V)$. Then,

$$B_\epsilon(y_1 - y_2, v) = -AV_s(U_s, v) + AV_s(V_s, v), \quad \forall v \in X_h.$$

For $v = y_1 - y_2$, and using the coercivity of $B_\epsilon(\cdot, \cdot)$, we obtain

$$\begin{aligned} \epsilon \|y_1 - y_2\|_1^2 &\leq |AV_s(U_s, y_1 - y_2) - AV_s(V_s, y_1 - y_2)| \\ &\leq \left| \sum_T h_T \int_T (d(|\nabla U_s|)\nabla U_s - d(|\nabla V_s|)\nabla V_s) \cdot \nabla(y_1 - y_2) dx \right| \\ &\leq \sum_T h_T \| (d(|\nabla U_s|)\nabla U_s - d(|\nabla V_s|)\nabla V_s) \|_T \| \nabla(y_1 - y_2) \|_T \\ &\leq \left(\sum_T h_T^2 \| (d(|\nabla U_s|)\nabla U_s - d(|\nabla V_s|)\nabla V_s) \|_T^2 \right)^{1/2} \| (y_1 - y_2) \|_1 \\ \text{i.e. } \epsilon \|y_1 - y_2\|_1 &\leq \left(\sum_T h_T^2 \| (d(|\nabla U_s|)\nabla U_s - d(|\nabla V_s|)\nabla V_s) \|_T^2 \right)^{1/2}. \end{aligned}$$

Now, we have that

$$\begin{aligned} \|d(|\nabla U_s|)\nabla U_s - d(|\nabla V_s|)\nabla V_s\| &= \|d(|\nabla U_s|)\nabla U_s - d(|\nabla V_s|)\nabla U_s + d(|\nabla V_s|)\nabla U_s - d(|\nabla V_s|)\nabla V_s\| \\ &= \| (d(|\nabla U_s|) - d(|\nabla V_s|)) \nabla U_s + d(|\nabla V_s|) (\nabla U_s - \nabla V_s) \| \\ &\leq c_d \| |\nabla U_s| - |\nabla V_s| \| \| \nabla U_s \| + d_{max} \| U_s - V_s \|_1 \\ &\leq c_d \| \nabla U \| \| \nabla U_s - \nabla V_s \| + d_{max} \| U_s - V_s \|_1 \\ &\leq c_{d,U} \| U_s - V_s \|_1. \end{aligned} \tag{4.4}$$

Thus, we obtain

$$\begin{aligned} \epsilon \|y_1 - y_2\|_1 &\leq c \left(\sum_T h_T^2 \|U_s - V_s\|_{1,T}^2 \right)^{1/2} \\ &\leq c \|U_s - V_s\|_1 \leq \|U - V\|_1 \end{aligned}$$

from which the continuity of A then follows.

Applying Theorem 4.1 the existence of U is established. ■

Next we establish the uniqueness of U under the assumption that $d(\cdot)$ is monotonically non-decreasing. This condition is a generalization of that established in [15].

Lemma 4.2 (*Uniqueness of U*) *Assume that (A1)–(A6) are satisfied. Then there exists a unique solution to (3.1).*

Proof:

Assume that U and V in X_h satisfy (3.1). Then, it follows that

$$AV_s(U_s, v_s) - AV_s(V_s, v_s) + B_\epsilon(U - V, v) = 0, \forall v \in X_h. \quad (4.5)$$

Choosing $v = U - V$, and using (2.8) we have

$$\begin{aligned} AV_s(U_s, U_s - V_s) - AV_s(V_s, U_s - V_s) + \epsilon \|\nabla(U - V)\|^2 + \|U - V\|^2 &\leq 0 \\ \sum_T h_T \int_T (d(|\nabla U_s|) \nabla U_s \cdot (\nabla U_s - \nabla V_s) - d(|\nabla V_s|) \nabla V_s \cdot (\nabla U_s - \nabla V_s)) dx & \\ + \epsilon \|\nabla(U - V)\|^2 + \|U - V\|^2 &\leq 0 \\ \sum_T h_T (d(|\nabla U_s|) \nabla U_s - d(|\nabla V_s|) \nabla V_s, \nabla U_s - \nabla V_s)_T & \\ + \epsilon \|\nabla(U - V)\|^2 + \|U - V\|^2 &\leq 0 \quad (4.6) \end{aligned}$$

Now for $d(\cdot)$ monotone non-decreasing we have that

$$\begin{aligned}
(d(|t_1|)t_1 - d(|t_2|)t_2, t_1 - t_2)_T &\geq (d(|t_1|)t_1 - d(|t_2|)t_2, t_1 - t_2) \\
&\quad - \left(\frac{1}{2}(d(|t_1|) + d(|t_2|))(t_1 - t_2), t_1 - t_2 \right) \\
&= \left(\frac{1}{2}(d(|t_1|) - d(|t_2|))(t_1 + t_2), t_1 - t_2 \right) \\
&= \frac{1}{2} \int_T (d(|t_1|) - d(|t_2|)) (|t_1|^2 - |t_2|^2) dx \\
&\geq 0.
\end{aligned} \tag{4.7}$$

Combining (4.5),(4.6),(4.7) we obtain that

$$\epsilon \|\nabla(U - V)\|^2 + \|U - V\|^2 \leq 0,$$

from which the uniqueness of the solution to (3.1) follows. ■

Having established the existence and uniqueness of U satisfying (3.1) we next give an a priori error estimate. Convergence of U to the true solution u under refinement follows immediately from the estimate.

Theorem 4.2 *For the solution u of (3.2) and its approximation U given by (3.1) we have the following estimates.*

$$\begin{aligned}
\epsilon^{1/2} \|\nabla(u - U)\| + \|u - U\| + \left(\sum_{T \in \Pi_h} h_T \|U_s\|_T^2 \right)^{1/2} &\leq \\
C \left\{ \inf_{w \in S_h} \sum_{T \in \Pi_h} \left[\epsilon \|\nabla(u - w)\|_T^2 + h_T |u - w|_{1,\mathbf{v},T}^2 + h_T^{-1} \|u - w\|_T^2 \right] \right\}^{1/2} &\tag{4.8}
\end{aligned}$$

$$\begin{aligned}
\left(\sum_{T \in \Pi_h} h_T |u - U|_{1,\mathbf{v},T}^2 \right)^{1/2} &\leq \\
C \left\{ \inf_{w \in S_h} \sum_{T \in \Pi_h} \left[\epsilon \|\nabla(u - w)\|_T^2 + h_T |u - w|_{1,\mathbf{v},T}^2 + h_T^{-1} \|u - w\|_T^2 \right] \right\}^{1/2} &\tag{4.9}
\end{aligned}$$

Proof:

Let w represent an arbitrary element of S_h and rewrite $u - U = \eta - \psi$, where $\eta = u - w$ and $\psi = U - w$. Noting that $U_s = \psi_s$, subtracting (3.1) from (3.2) we obtain

$$B_\epsilon(\psi, v) + AV_s(\psi_s, v_s) = B_\epsilon(\eta, v), \quad (4.10)$$

for all $v \in X_h$.

Choosing $v = \psi$, (4.10) becomes

$$\begin{aligned} \sum_{T \in \Pi_h} \epsilon \|\nabla \psi\|_{0,T}^2 + b_{sy}(\psi, \psi) + AV_s(\psi_s, \psi_s) &= \int_{\Omega} \epsilon \nabla \eta \cdot \nabla \psi \, dx + b(\eta, \psi) \\ &= \int_{\Omega} \epsilon \nabla \eta \cdot \nabla \psi \, dx + 2b_{sy}(\eta, \psi) - b(\psi, \eta) \\ &= \int_{\Omega} \epsilon \nabla \eta \cdot \nabla \psi \, dx + 2b_{sy}(\eta, \psi) \\ &\quad - \int_{\Omega} (\mathbf{v} \cdot \nabla \psi) \eta \, dx - \int_{\Omega} q \psi \eta \, dx \end{aligned} \quad (4.11)$$

We can bound the terms on the right-hand side of (4.11) in the following manner:

$$\begin{aligned} \int_{\Omega} q \psi \eta \, dx &\leq \|q\|_{\infty} \|\psi\| \|\eta\| \\ &\leq \|q\|_{\infty} \left(\int_{\Omega} (q - \frac{1}{2} \nabla \cdot \mathbf{v}) \psi^2 \, dx \right)^{1/2} \|\eta\| \\ &= \|q\|_{\infty} \left(\int_{\Omega} (\mathbf{v} \cdot \nabla \psi) \psi + q \psi^2 \, dx \right)^{1/2} \|\eta\| \\ &= \|q\|_{\infty} b_{sy}(\psi, \psi)^{1/2} \|\eta\| \\ &\leq \gamma_1 b_{sy}(\psi, \psi) + \frac{\|q\|_{\infty}^2}{4\gamma_1} \|\eta\|^2 \\ &\leq \gamma_1 b_{sy}(\psi, \psi) + \frac{\|q\|_{\infty}^2}{4\gamma_1} \sum_{T \in \Pi_h} \|\eta\|_T^2, \end{aligned} \quad (4.12)$$

for $\gamma_1 > 0$ a constant.

The next term is bounded as

$$2b_{sy}(\eta, \psi) \leq \gamma_2 b_{sy}(\psi, \psi) + \frac{1}{4\gamma_2} b_{sy}(\eta, \eta)$$

$$\leq \gamma_2 b_{sy}(\psi, \psi) + \frac{c}{4\gamma_2} \sum_{T \in \Pi_h} \|\eta\|_T^2. \quad (4.13)$$

Continuing,

$$\begin{aligned} \int_{\Omega} \epsilon \nabla \eta \cdot \nabla \psi \, dx &= \sum_{T \in \Pi_h} \epsilon \int_T \nabla \eta \cdot \nabla \psi \, dx \\ &\leq \sum_{T \in \Pi_h} \epsilon \|\nabla \eta\|_T \|\nabla \psi\|_T \\ &\leq \gamma_3 \sum_{T \in \Pi_h} \epsilon \|\nabla \psi\|_T^2 + \frac{1}{4\gamma_3} \sum_{T \in \Pi_h} \epsilon \|\nabla \eta\|_T^2. \end{aligned} \quad (4.14)$$

Thus, from (4.11) and (4.12)–(4.14) we obtain

$$\begin{aligned} \sum_{T \in \Pi_h} \epsilon (1 - \gamma_3) \|\nabla \psi\|_T^2 &+ (1 - \gamma_1 - \gamma_2) b_{sy}(\psi, \psi) + AV_s(\psi_s, \psi_s) \\ &\leq - \int_{\Omega} (\mathbf{v} \cdot \nabla \psi) \eta \, dx + \frac{1}{4\gamma_3} \sum_{T \in \Pi_h} \epsilon \|\nabla \eta\|_T^2 + \frac{c}{4\gamma_2} \sum_{T \in \Pi_h} \|\eta\|_T^2 \\ &\quad + \frac{\|q\|_{\infty}^2}{4\gamma_1} \sum_{T \in \Pi_h} \|\eta\|_T^2 \\ &\leq - \int_{\Omega} (\mathbf{v} \cdot \nabla \psi) \eta \, dx + \frac{1}{4\gamma_3} \sum_{T \in \Pi_h} \epsilon \|\nabla \eta\|_T^2 \\ &\quad + \left(\frac{\|q\|_{\infty}^2}{4\gamma_1} + \frac{c}{4\gamma_2} \right) \sum_{T \in \Pi_h} \|\eta\|_T^2. \end{aligned} \quad (4.15)$$

Finally, we consider the $\int_{\Omega} (\mathbf{v} \cdot \nabla \psi) \eta \, dx$ term.

$$\begin{aligned} \int_{\Omega} (\mathbf{v} \cdot \nabla \psi) \eta \, dx &= \sum_{T \in \Pi_h} \int_T (\mathbf{v} \cdot \nabla \psi) \eta \, dx \\ &\leq \sum_{T \in \Pi_h} |\psi|_{1, \mathbf{v}, T} \|\eta\|_T. \end{aligned} \quad (4.16)$$

To handle the $|\psi|_{1, \mathbf{v}, T}$ term we use (2.10). Since $\text{supp}(\phi_s) \subset T$,

$$b_{1, T}(\psi_h, \phi_s) = \int_T \mathbf{v} \cdot \nabla \psi_h \phi_s \, dx$$

$$\begin{aligned}
&= \int_T \epsilon \nabla \eta \cdot \nabla \phi_s \, dx + \int_T \mathbf{v} \cdot \nabla \eta \phi_s \, dx + \int_T q \eta \phi_s \, dx \\
&\quad - \int_T \epsilon \nabla \psi \cdot \nabla \phi_s \, dx - \int_T \mathbf{v} \cdot \nabla \psi_s \phi_s \, dx - \int_T q \psi \phi_s \, dx - AV_s(\psi_s, \phi_s) \\
&\leq \epsilon h_T^{-1} \|\nabla \eta\|_T \|\phi_s\|_T + \epsilon h_T^{-1} \|\nabla \psi\|_T \|\phi_s\|_T + |\eta|_{1, \mathbf{v}, T} \|\phi_s\|_T + c \|\eta\|_T \|\phi_s\|_T \\
&\quad + c \|\nabla \psi_s\|_T \|\phi_s\|_T + c b_{sy, T}(\psi, \psi)^{1/2} \|\phi_s\|_T + c d_{max} \|\nabla \psi_s\|_T \|\phi_s\|_T .
\end{aligned}$$

Hence, using (2.10),

$$\begin{aligned}
c_a |\psi_h|_{1, \mathbf{v}, T} &\leq \epsilon h_T^{-1} \|\nabla \eta\|_T + \epsilon h_T^{-1} \|\nabla \psi\|_T + |\eta|_{1, \mathbf{v}, T} + c \|\eta\|_T \\
&\quad + c \|\nabla \psi_s\|_T + c b_{sy, T}(\psi, \psi)^{1/2} .
\end{aligned} \tag{4.17}$$

Using $\psi = \psi_h + \psi_s$ and (4.17), the term $|\psi|_{1, \mathbf{v}, T} \|\eta\|_T$ is bound as

$$\begin{aligned}
|\psi|_{1, \mathbf{v}, T} \|\eta\|_T &\leq \epsilon h_T^{-1} \|\nabla \eta\|_T \|\eta\|_T + |\eta|_{1, \mathbf{v}, T} \|\eta\|_T + c \|\eta\|_T^2 \\
&\quad + \epsilon h_T^{-1} \|\nabla \psi\|_T \|\eta\|_T + c \|\nabla \psi_s\|_T \|\eta\|_T + c b_{sy, T}(\psi, \psi)^{1/2} \|\eta\|_T \\
&\leq \epsilon h_T^{-1/2} \|\nabla \eta\|_T h_T^{-1/2} \|\eta\|_T + |\eta|_{1, \mathbf{v}, T} \|\eta\|_T + c \|\eta\|_T^2 \\
&\quad + \epsilon^{1/2} \|\nabla \psi\|_T \epsilon^{1/2} h_T^{-1} \|\eta\|_T + h_T^{1/2} \|\nabla \psi_s\|_T c h_T^{-1/2} \|\eta\|_T \\
&\quad + b_{sy, T}(\psi, \psi)^{1/2} c \|\eta\|_T .
\end{aligned}$$

Next, summing the above terms over all triangles $T \in \Pi_h$ yields the estimate:

$$\begin{aligned}
\sum_{T \in \Pi_h} |\psi|_{1, \mathbf{v}, T} \|\eta\|_T &\leq \sum_{T \in \Pi_h} \left(\frac{1}{2} \epsilon^2 h_T^{-1} \|\nabla \eta\|_T^2 + \frac{1}{2} h_T^{-1} \|\eta\|_T^2 + \frac{1}{2} h_T |\eta|_{1, \mathbf{v}, T}^2 + c \|\eta\|_T^2 \right. \\
&\quad \left. + \frac{1}{2} h_T^{-1} \|\eta\|_T^2 + \frac{1}{4\mu_1} \epsilon h_T^{-2} \|\eta\|_T^2 + \frac{c^2}{4\mu_2} h_T^{-1} \|\eta\|_T^2 + \frac{c^2}{4\mu_3} \|\eta\|_T^2 \right) \\
&\quad + \sum_{T \in \Pi_h} \left(\epsilon \mu_1 \|\nabla \psi\|_T^2 + \mu_2 h_T \|\nabla \psi_s\|_T^2 + \mu_3 b_{sy, T}(\psi, \psi) \right) .
\end{aligned}$$

Thus, returning to (4.15), for suitably chosen values of μ_1, μ_2 , and μ_3 , and assuming $0 < \epsilon < h_T < 1$, we have that

$$\sum_{T \in \Pi_h} \epsilon \|\nabla \psi\|_T^2 + b_{sy}(\psi, \psi) + h_T \|\nabla \psi_s\|_T^2$$

$$\begin{aligned}
&\leq C \sum_{T \in \Pi_h} \left(\epsilon \|\nabla \eta\|_T^2 + \|\eta\|_T^2 + \epsilon^2 h_T^{-1} \|\nabla \eta\|_T^2 + h_T^{-1} \|\eta\|_T^2 + h_T |\eta|_{1,\mathbf{v},T}^2 + \epsilon h_T^{-2} \|\eta\|_T^2 \right) \\
&\leq C \sum_{T \in \Pi_h} \left(\epsilon \|\nabla \eta\|_T^2 + h_T |\eta|_{1,\mathbf{v},T}^2 + (1 + h_T^{-1} + \epsilon h_T^{-2}) \|\eta\|_T^2 \right). \tag{4.18}
\end{aligned}$$

Finally, estimate (4.8) follows the triangle inequality and applying the estimate (4.18).

To establish (4.9) note that from (4.17) we have

$$\begin{aligned}
c_a h_T^{1/2} |\psi_h|_{1,\mathbf{v},T} &\leq \epsilon h_T^{-1/2} \|\nabla \eta\|_T + \epsilon h_T^{-1/2} \|\nabla \psi\|_T + c h_T^{1/2} |\eta|_{1,\mathbf{v},T} + h_T^{1/2} \|\eta\|_T \\
&\quad + c h_T^{1/2} \|\nabla \psi_s\|_T + c h_T^{1/2} b_{sy,T}(\psi, \psi)^{1/2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{T \in \Pi_h} h_T |\psi_h|_{1,\mathbf{v},T}^2 &\leq C \sum_T \left\{ \epsilon^2 h_T^{-1} \|\nabla \eta\|_T^2 + \epsilon^2 h_T^{-1} \|\nabla \psi\|_T^2 + h_T |\eta|_{1,\mathbf{v},T}^2 \right. \\
&\quad \left. + h_T \|\eta\|_T^2 + h_T \|\nabla \psi_s\|_T^2 + h_T b_{sy,T}(\psi, \psi) \right\}.
\end{aligned}$$

Since we assume $0 < \epsilon < h_T < 1$,

$$\begin{aligned}
\sum_{T \in \Pi_h} h_T |\psi_h|_{1,\mathbf{v},T}^2 &\leq C \sum_T \left\{ \epsilon^2 h_T^{-1} \|\nabla \eta\|_T^2 + \epsilon \|\nabla \psi\|_T^2 + h_T |\eta|_{1,\mathbf{v},T}^2 \right. \\
&\quad \left. + h_T \|\eta\|_T^2 + h_T \|\nabla \psi_s\|_T^2 \right\} + c b_{sy}(\psi, \psi).
\end{aligned}$$

Now, using the triangle inequality as in obtaining the estimate (4.8), and the inequalities from (4.18) and (4.8), estimate (4.9) follows. ■

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