On Some Integrals Over A Unit Sphere

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Let $B_n \subset R^n$ be a unit ball and $S^{n-1} \subset R^n$ be a unit sphere. For $n = 2$, the unit ball is the interior of the unit circle mainly $B_2 = \{(x_1, x_2) \in R^2 : x_1^2 + x_2^2 \leq 1\}$ (see Figure 1a). The unit sphere in R^2 is the unit circle $S^1 = \{(x_1, x_2) \in R^2 : x_1^2 + x_2^2 = 1\}$ (see Figure 1b). Similarly for $n = 3$, the unit ball is given by $B_3 = \{(x_1, x_2, x_3) \in R^3 : x_1^2 + x_2^2 + x_3^2 \leq 1\}$ and the unit sphere is the spherical shell $S^2 = \{(x_1, x_2, x_3) \in R^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ (see Figure 2ab). Of course, in higher dimension, it is difficult to visualize these geometric objects however the mathematical definition still holds true. One can still define a point on the unit sphere as $\omega = (x_1, x_2, \ldots, x_n)$ as a unit vector on S^{n-1} with $|\omega|^2 = \sum_{j=1}^{n} x_j^2 = 1$.

Integrals over the unit sphere in R^n are common in solving problems in spherical coordinates where azimuthal symmetry is not present. For example, spherical harmonics in $n = 3$ are usually denoted by $Y_l^m(\theta, \phi)$ where normally θ is taken as the polar (colatitudinal) coordinate with $\theta \in [0, \pi]$, and ϕ as the azimuthal (longitudinal) coordinate with $\phi \in [0, 2\pi)$, see Arfken [**1**] or Pinsky2 [**9**]. In 3-D The spherical harmonics Y_l^m is usually the angular portion of the solution for a boundary value problems involving a

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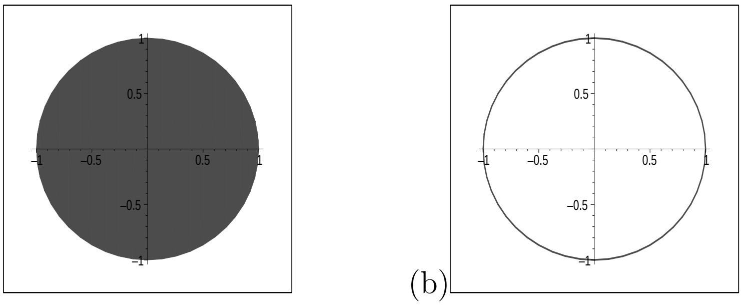
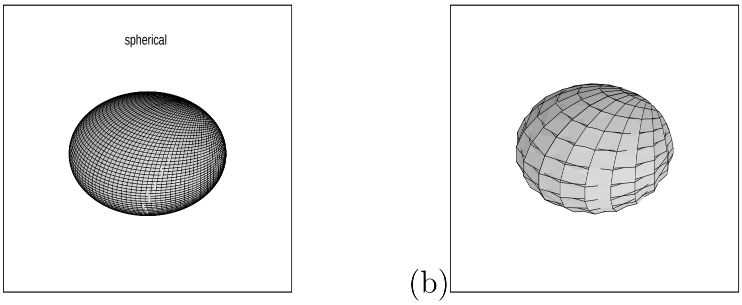


(a) (b)

Figure 1: (a) $x^2 + y^2 \leq 1$ (b) $x^2 + y^2 = 1$
(Using Maple)



(a) (b)

Figure 2: (a) $x^2 + y^2 + z^2 \leq 1$ (b) $x^2 + y^2 + z^2 = 1$
(Using Maple)

partial differential equation in spherical coordinates. Spherical harmonics are implemented in Mathematica as SphericalHarmonicY[l, m, theta, phi]. The normalized spherical harmonics actually form an orthonormal basis so any arbitrary real function $f(\theta, \phi)$ can be expanded in terms of the spherical harmonics and one can find Fourier series type expansions using these functions. The calculation of the coefficient of the expansion involves integrating the function over the unit sphere in S^2. The applications of spherical harmonics range from solving Laplace's equation in physics and engineering to solving radiative transport equations for biological tissues. See Griffiths [**2**] for some examples in electro statics involving Laplace's equations and see Ishimaru [**3**] for some examples in scattering of photons in random media involving radiative transport equations. The motivation for our note comes from using spherical harmonics for solving the photon transport equation in biological tissue with spatially varying refractive index, see Ferwerda [**4**] or Khan [**5**]. The radiative transport equation (RTE) for a medium with a spatially varying refractive index describes the density of particles (photons) which travel at time t though the point y in the direction $\omega \in S^{n-1}$. One can derive simpler deterministic models from the RTE by expanding the density in spherical harmonics and retaining a limited number of terms. This expansion also requires integrating the radiative transport equation over S^{n-1}. Now that we have stated our motivation, the rest of the paper will deal with integrals over the unit sphere.

Common Integrals Over A Unit Sphere. There are several common integrals over a unit sphere in R^n. The simplest is the integral of a constant over a unit sphere:

$$\int_{S^{n-1}} d\omega = |S^{n-1}| = \frac{2(\pi)^{n/2}}{\Gamma(n/2)} \tag{1}$$

where $d\omega$ is the surface measure on S^{n-1} and

$$\Gamma(m) = \int_0^\infty y^{m-1} e^{-y} dy$$

is the Gamma function which satisfies $\Gamma(m+1) = m\Gamma(m)$ if $m > 0$ and $\Gamma(m+1) = m!$ if m is an integer with $\Gamma(2) = \Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$. For $n = 2$, $d\omega$ is given as the incremental length $d\theta$ on the unit circle and if we integrate around the unit circle we get $|S^1| = 2\pi$ which is the circumference of the circle. For $n = 3$, $d\omega$ is the incremental area $d\theta d\phi$ on the spherical shell and if we integrate around the spherical shell we get $|S^2| = 4\pi$ which is the area of the spherical shell. These values of course corresponds to equation (1) as one can easily see from the properties of the Γ function just mentioned.

The next two integrals are slightly more complicated.

Proposition 1 $\int_{S^{n-1}}(a \cdot \omega)\omega d\omega = \frac{|S^{n-1}|}{n}a$ *for any vector* a *in* R^n*.*

Proof: Let $\omega = (x_1, x_2, \ldots, x_n)^T \in S^{n-1}$ so that $|\omega| = 1$ and $x_1 \neq -1$, then one can construct an orthogonal basis $\{\omega, \omega^2, \ldots, \omega^n\} \subset S^{n-1}$ in R^n (see Cullen [**6**]). Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis in R^n and denote $G \in SO(n)$ (special orthogonal group) which permutes the standard basis as follows:$Ge_1 = e_2, Ge_2 = e_3, \ldots, Ge_n = e_1$. Then if we let $X = (\omega, \omega^2, \ldots, \omega^n)$ and $H = XGX^{-1}$. Then obviously $H \in SO(n)$ and $H\omega = \omega^2, H\omega^2 = \omega^3, \ldots, H\omega^n = \omega$. Therefore $\{\omega, H\omega, H^2\omega, \ldots, H^{n-1}\omega\}$ forms an orthogonal basis in R^n. Hence, any vector a can be written as $a = \sum_{k=0}^{n-1}(a \cdot H^k\omega)H^k\omega$ and

$$\int_{S^{n-1}}(a \cdot \omega)\omega d\omega = \frac{1}{n}\sum_{k=0}^{n-1}\int_{S^{n-1}}(a \cdot H^k\omega)H^k\omega dH^k\omega = \frac{1}{n}\int_{S^{n-1}} a d\omega = \frac{|S^{n-1}|}{n}a. \quad (2)$$

Proposition 2 $\int_{S^{n-1}}(a \cdot \omega)(b \cdot \omega)d\omega = \frac{|S^{n-1}|}{n}(a \cdot b)$ *for any vectors* a *and* b *in* R^n*.*

Proof: Let $\{\omega, H\omega, \ldots, H^{n-1}\omega\} \subset S^{n-1}$ be a basis in R^n as in the last proof. Hence any two vectors a and b can be written as $a = \sum_{k=0}^{n-1}(a \cdot H^k\omega)H^k\omega$ and $b = \sum_{k=0}^{n-1}(b \cdot H^k\omega)H^k\omega$. If we further note that $a \cdot b = \sum_{k=0}^{n-1}(a \cdot H^k\omega)(b \cdot H^k\omega)$, we get

$$\int_{S^{n-1}}(a \cdot \omega)(b \cdot \omega)d\omega = \frac{1}{n}\sum_{k=0}^{n-1}\int_{S^{n-1}}(a \cdot H^k\omega)(b \cdot H^k\omega)dH^k\omega$$

$$= \frac{1}{n}\int_{S^{n-1}}(a\cdot b)d\omega = \frac{|S^{n-1}|}{n}(a\cdot b). \tag{3}$$

Uncommon Integrals Over A Unit Sphere. The integrals explained below arise in finding the spherical harmonic expansions for the radiative transport equation with spatially varying refractive indices, see Ferwerda [**4**] and Khan [**5**]. Let $\omega^{-1} = (x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1})$, we will now investigate how to evaluate integrals of this form,

$$\int_{S^{n-1}}(a\cdot\omega^{-1})\omega d\omega. \tag{4}$$

First, we note that these integrals are only defined in the Cauchy principal value sense. Let us first discuss the simplest of the cases $n = 2, 3$ to clarify the meaning of these integrals.

Proposition 3 *$\int_{S^1}(a\cdot\omega^{-1})\omega d\omega = |S^1|a$ for any vector a in R^2.*

Proof: $\omega^{-1} = \frac{1}{\cos(\phi)}\hat{x} + \frac{1}{\sin(\phi)}\hat{y}$, where $\hat{x}$,$\hat{y}$ are unit vectors in the xy plane. Therefore,

$$(a\cdot\omega^{-1})\omega = \left(a_x + \frac{x_1}{x_2}a_y\right)\hat{x} + \left(\frac{x_2}{x_1}a_x + a_y\right)\hat{y} \tag{5}$$

where $x_1 = \cos(\phi)$ and $x_2 = \sin(\phi)$. Now if we compute $\frac{x_1}{x_2}$ and $\frac{x_2}{x_1}$ and simplify the right hand side of equation (4) we get,

$$(a\cdot\omega^{-1})\omega = a + a_y\cot(\phi)\hat{x} + a_x\tan(\phi)\hat{y}. \tag{6}$$

Therefore,

$$\int_{2\pi}(a\cdot\omega^{-1})\omega d\omega = \int_{2\pi} a d\omega + \int_{2\pi} a_y\cot(\phi)d\omega\hat{x} + \int_{2\pi} a_x\tan(\phi)d\omega\hat{y}. \tag{7}$$

Since Cauchy's principal value integrals (see Brown and Churchill [**7**]) of $\cot(\phi)$ and $\tan(\phi)$ over all of 2π radians vanish, we get,

$$\int_{2\pi}(a\cdot\omega^{-1})\omega d\omega = 2\pi a. \tag{8}$$

Proposition 4 *$\int_{S^2}(a \cdot \omega^{-1})\omega d\omega = |S^2| a$ for any vector a in R^3.*

Proof: $\omega^{-1} = \frac{1}{\sin(\theta)\cos(\phi)}\hat{x} + \frac{1}{\sin(\theta)\sin(\phi)}\hat{y} + \frac{1}{cos(\theta)}\hat{z}$, where $\hat{x}$,$\hat{y}$, $\hat{z}$ are unit vectors in the xyz space. Therefore,

$$\begin{aligned}(a \cdot \omega^{-1})\omega = \left(a_x + \frac{x_1}{x_2}a_y\right. &+ \left.\frac{x_1}{x_3}a_z\right)\hat{x} + \left(\frac{x_2}{x_1}a_x + a_y + \frac{x_2}{x_3}a_z\right)\hat{y} \\ &+ \left(\frac{x_3}{x_1}a_x + \frac{x_3}{x_2}a_y + a_z\right)\hat{z} \qquad (9)\end{aligned}$$

where $x_1 = \sin(\theta)\cos(\phi)$, $x_2 = \sin(\theta)\sin(\phi)$, and $x_3 = \cos(\theta)$. Now if we compute $\frac{x_1}{x_2}$,$\frac{x_1}{x_3}$,$\frac{x_2}{x_1}$, $\frac{x_2}{x_3}$,$\frac{x_3}{x_1}$, and $\frac{x_3}{x_2}$ and simplify the right hand side of equation (9) we get,

$$\begin{aligned}(a \cdot \omega^{-1})\omega &= a + (a_y\cot(\phi) + a_z\tan(\theta)\cos(\phi))\hat{x} \\ &+ (a_x\tan(\phi) + a_z\tan(\theta)\sin(\phi))\hat{y} + (a_x\cot(\theta)\sec(\phi) + a_y\cot(\theta)\csc(\phi))\hat{z}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{4\pi}(a \cdot \omega^{-1})\omega d\omega &= \int_{4\pi} a d\omega + \int_{4\pi}(a_y\cot(\phi) + a_z\tan(\theta)\cos(\phi))d\omega\hat{x} \\ &+ \int_{4\pi}(a_x\tan(\phi) + a_z\tan(\theta)\sin(\phi))d\omega\hat{y} \\ &+ \int_{4\pi}(a_x\cot(\theta)\sec(\phi) + a_y\cot(\theta)\csc(\phi))d\omega\hat{z} \qquad (10)\end{aligned}$$

Since Cauchy's principal value integrals of $\cot(\phi)$, $\tan(\theta)\cos(\phi)$,$\tan(\phi)$,$\tan(\theta)\sin(\phi)$, $\cot(\theta)\sec(\phi)$, and $\cot(\theta)\csc(\phi)$ over all of 4π steradians vanish, we get,

$$\int_{4\pi}(a \cdot \omega^{-1})\omega d\omega = 4\pi a. \qquad (11)$$

Proposition 5 *$\int_{S^1}(a \cdot \omega^{-1})(b \cdot \omega)d\omega = |S^1|(a \cdot b)$ for any vectors a and b in R^2.*

Proof: $\omega^{-1} = \frac{1}{\cos(\phi)}\hat{x} + \frac{1}{\sin(\phi)}\hat{y}$, where $\hat{x}$,$\hat{y}$ are unit vectors in the x-y plane. Therefore,

$$(a \cdot \omega^{-1})(b \cdot \omega) = a \cdot b + \frac{x_2}{x_1}a_x b_y + \frac{x_1}{x_2}a_y b_x \qquad (12)$$

where $x_1 = \cos(\phi)$ and $x_2 = \sin(\phi)$. Now if we compute $\frac{x_2}{x_1}$ and $\frac{x_1}{x_2}$ and simplify the right hand side of equation (12) we get,

$$(a \cdot \omega^{-1})(b \cdot \omega) = a \cdot b + a_x b_y \tan(\phi) + a_y b_x \cot(\phi). \tag{13}$$

Therefore,

$$\int_{2\pi} (a \cdot \omega^{-1})(b \cdot \omega) d\omega = \int_{2\pi} (a \cdot b) d\omega + \int_{2\pi} a_x b_y \tan(\phi) d\omega + \int_{2\pi} a_y b_x \cot(\phi) d\omega.$$

Since Cauchy's principal value integrals of $\tan(\phi)$ and $\cot(\phi)$ over all of 2π radians vanish, we get,

$$\int_{2\pi} (a \cdot \omega^{-1})(b \cdot \omega) d\omega = 2\pi(a \cdot b). \tag{14}$$

Proposition 6 *$\int_{S^2} (a \cdot \omega^{-1})(b \cdot \omega) d\omega = |S^2|(a \cdot b)$ for any vectors a and b in R^3.*

Proof: $\omega^{-1} = \frac{1}{\sin(\theta)\cos(\phi)}\hat{x} + \frac{1}{\sin(\theta)\sin(\phi)}\hat{y} + \frac{1}{cos(\theta)}\hat{z}$, where $\hat{x}$,$\hat{y}$, $\hat{z}$ are unit vectors in the xyz space. Therefore,

$$\begin{aligned}(a \cdot \omega^{-1})(b \cdot \omega) = a \cdot b + \frac{x_2}{x_1} a_x b_y \quad &+ \quad \frac{x_3}{x_1} a_x b_z + \frac{x_1}{x_2} a_y b_x + \frac{x_3}{x_2} a_y b_z \\ &+ \quad \frac{x_1}{x_3} a_z b_x + \frac{x_2}{x_3} a_z b_y \end{aligned} \tag{15}$$

where $x_1 = \sin(\theta)\cos(\phi)$, $x_2 = \sin(\theta)\sin(\phi)$, and $x_3 = \cos(\theta)$. Now if we compute $\frac{x_2}{x_1}$,$\frac{x_3}{x_1}$,$\frac{x_1}{x_2}$, $\frac{x_3}{x_2}$,$\frac{x_1}{x_3}$, and $\frac{x_2}{x_3}$ and simplify the right hand side of equation (15) we get,

$$\begin{aligned}(a \cdot \omega^{-1})(b \cdot \omega) \quad &= \quad (a \cdot b) + a_x b_y \tan(\phi) + a_x b_z \cot(\theta)\sec(\phi) + a_y b_x \cot(\phi) \\ &+ \quad a_y b_z \cot(\theta)\csc(\phi) + a_z b_x \tan(\theta)\cos(\phi) + a_z b_y \tan(\theta)\sin(\phi).\end{aligned}$$

Since Cauchy's principal value integrals of $\tan(\phi)$, $\cot(\theta)\sec(\phi)$,$\cot(\phi)$, $\cot(\theta)\csc(\phi)$, $\tan(\theta)\cos(\phi)$, and $\tan(\theta)\sin(\phi)$ over all of 4π steradians vanish, we get,

$$\int_{4\pi} (a \cdot \omega^{-1})(b \cdot \omega) d\omega = 4\pi(a \cdot b). \tag{16}$$

A General Approach. The results of Proposition 3,4,5 and 6 can be generalized as follows. We consider the bilinear function

$$F(a,b) = \int_{S^{n-1}} (a \cdot \omega^{-1})(b \cdot \omega) d\omega$$

which is the integral of the product of two linear functions, thus

$$F(a,b) = \sum_{i,j} a_i b_j \int_{S^{n-1}} (x_j/x_i) d\omega,$$

where each integral is interpreted as a Cauchy principal value. As above, we will use the notation that the vector $\omega = (x_1, \ldots, x_n)$.

There are basically two cases to consider, $i = j$ and $i \neq j$. The first case is obvious since we are integrating the function 1, which gives the total measure of S^{n-1}. To do the second, we can assume, without loss of generality, that $i = 1, j = 2$. Now we take a system of n-dimensional spherical coordinates $\theta_1, \ldots, \theta_{n-1}$, so that $x_1 = \cos\theta_1, x_2 = \sin\theta_1 \cos\theta_2, \ldots, x_n = \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-1}$ where $0 \leq \theta_j \leq \pi$ for $j = 1, \ldots, n-2$ and $0 \leq \theta_{n-1} \leq 2\pi$ for $j = 2, \ldots, n-1$. The Jacobian of this mapping is $\sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2}$ (see Pinsky1 [**8**]).

If $n > 3$ we have to evaluate the integral

$$\int_{S^{n-1}} (x_2/x_1) d\omega = \text{const.} \int_0^\pi \frac{\sin^{n-1}\theta_1}{\cos\theta_1} d\theta_1 \int_0^\pi \sin^{n-3}\theta_2 \cos\theta_2 \, d\theta_2.$$

The inner integral on θ_2 is already zero, so that we don't need to deal with the Cauchy principal value integral in θ_2. If $n = 3$, the inner integral is taken on the interval $[0, 2\pi]$ and is clearly equal to zero also. If $n = 2$ we have only one integral, which is a Cauchy principal value, equal to zero by the previous discussion.

Therefore we see that $F(a,b) = |S^{n-1}| \sum_{j=1}^n a_j b_j = |S^{n-1}|(a \cdot b)$.

Further reading and open problem. Propositions 1 and 2 can also be proved using Fubini's theorem (see Folland [**10**]) from real analysis. For example if we let $f(\omega) = \prod_{j=1}^{n} x_j^{\alpha_j}$ where $\alpha_j \geq 0$ is a positive integer. Then $\int_{S^{n-1}} f d\omega = 0$ if any of the α_j is odd and $\int_{S^{n-1}} f d\omega = \frac{2\Gamma(\beta_1)\cdots\Gamma(\beta_n)}{\Gamma(\beta_1+\cdots+\beta_n)}$ where $\beta_j = \frac{\alpha_j+1}{2}$ if all α_j are even. The open problem is to generalize this for the ω^{-1} case.

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