# On derivation of the radiative transfer equation and its diffusion approximation for scattering media with spatially varying refractive indices

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## ABSTRACT

Traditionally, the radiative transfer equation and its diffusion approximation are derived for a medium with a spatially constant refractive index (Ishimaru 1978 Wave Propagation and Scattering in Random Medial vol 1 (New York: Academic)). In this paper, we derive the radiative transport equation and its diffusion approximation relevant to optical tomography for a medium with a spatially varying refractive index.

**Keywords:** Radiative transport, optical tomography, diffusion approximation, biomedical imaging.

## 1 INTRODUCTION

Recent interest in the radiative transfer equation (RTE) for a medium with spatially varying refractive index is getting more attention [11, 9]. Media with spatially varying refractive index are among us in the form of biological tissues and the atmosphere, just to mention two examples [19, 17]. In optical tomography, the diffusion approximation is useful because biological tissue is a highly scattering medium where this approximation is valid. For a detailed discussion on the potential applications in optical imaging of biological tissue, see [9].

Recently Ferwerda [5] attempted to derive the RTE for a medium with a spatially varying refractive index. However, a closer look in the literature reveals that Ryzhik et al. [18] had derived the general form of the transport equation for the energy density of waves in a random media. Bekefi [4], Kravtsov and Orlov [13], and Apresyan and Kravtsov [2] had also derived various versions of a ray refractive index equation for a non-absorption and non-scattering media with spatially varying refractive indices. In fact, there have been other attempts to derive the RTE for a few special cases for a medium with a spatially varying refractive index as early as the time when Chandrasekhar derived his equation for a constant refractive index [7, 14]. Even though the RTE relevant to optical tomography for a medium with a spatially varying refractive index is only a special case of the general result, it is still not widely known to applied and interdisciplinary practitioners for optics applications [9]. Therefore, the aim of this paper is to present a simple derivation of the RTE and its diffusion approximation relevant to optical tomography accessible to a wider audience. If the reader wishes he may skip directly to the main results, mainly equation (43) for the RTE and equation (61) for its diffusion approximation for a medium with a spatially varying refractive index.

In this paper, we derive the RTE and its diffusion approximation relevant to optical tomography for a medium with spatially varying refractive index. The outline of the paper is as follows. In section 2, we derive the RTE. In section 3, we state the corresponding diffusion approximation. In section 4, we discuss conclusions and future work.

## 2 DERIVATION OF THE RTE FOR SPATIALLY VARYING REFRACTIVE INDEX

The fundamental quantity of interest in radiative transfer is the spectral density of the radiance or spectral radiance or simply radiance $L_\omega(\mathbf{r}, \mathbf{\Omega}, t)$ (which is sometimes called the specific intensity) which is defined as the amount of energy which at position $\mathbf{r}$ flows per second through a unit area perpendicular to the unit vector $\mathbf{\Omega}$ in the frequency interval $(\omega, \omega + d\omega)$. $L_\omega$ is measured in W $\mathrm{sr}^{-1}\mathrm{m}^{-2}\mathrm{Hz}^{-1}$ or in erg $\mathrm{s}^{-1}\mathrm{sr}^{-1}\mathrm{cm}^{-2}\mathrm{Hz}^{-1}$. The above definition is the most general definition of radiance which considers both statistical and wave aspects [18, 2]. The radiant flux $\mathbf{F}_\omega$, the total quantities $\mathbf{F}$ and $L$, and the energy density $\varphi(\mathbf{r})$ are defined as

$$\begin{aligned} \mathbf{F}_\omega(\mathbf{r},t) &= \int_{4\pi} L_\omega(\mathbf{r},\mathbf{\Omega},t)\mathbf{\Omega}d\mathbf{\Omega} & (1)\\ \mathbf{F}(\mathbf{r},t) &= \int_{-\infty}^{\infty} \mathbf{F}_\omega(\mathbf{r},t)d\omega, & (2)\\ L(\mathbf{r},\mathbf{\Omega},t) &= \int_{-\infty}^{\infty} L_\omega(\mathbf{r},\mathbf{\Omega},t)d\omega & (3)\\ \varphi(\mathbf{r},t) &= \frac{1}{c}\int_{4\pi} L(\mathbf{r},\mathbf{\Omega},t)d\mathbf{\Omega}. & (4) \end{aligned}$$

For the most general case, in an anisotropic, inhomogenous, and dispersive media, we also need to consider that beam propagates with a group velocity $\mathbf{v}_g$, the magnitude of the velocity $v_g$ may depend on frequency and direction of the wave $\mathbf{k}$. In the general case of an anisotropic medium, the direction of the group velocity no longer coincides with the wave vector $\mathbf{k}$ and the frequency $\omega$ is related to $\mathbf{k}$ via the dispersion relations $\omega = \omega(\mathbf{k},\mathbf{r})$. Therefore in general the energy density $\varphi$ and flux $\mathbf{F}$ are defined as

$$\begin{aligned} \mathbf{F}(\mathbf{r},t) &= \int_B L_\omega(\mathbf{r},\mathbf{\Omega},t)\mathbf{\Omega}d\omega d\mathbf{\Omega} & (5)\\ \varphi(\mathbf{r},t) &= \int_B \frac{1}{v_g} L_\omega(\mathbf{r},\mathbf{\Omega},t)d\omega d\mathbf{\Omega}. & (6) \end{aligned}$$

It should be emphasized that for anisotropic media, $\mathbf{\Omega} = \mathbf{v}_g/v_g$ rather than $\mathbf{\Omega} = \mathbf{k}/k$ is the unit vector of ray direction. Furthermore, the integration is carried out over the domain $B$ in direction and frequency, $(\omega,\mathbf{\Omega})$ space, corresponding to the propagation of waves. Using quantum approach, one can obtain a relationship between the group velocity and the spatially varying index of refraction $n(\omega,\mathbf{r})$ as in [4]:

$$n^2(\omega,\mathbf{r}) = \frac{v_g c^2}{\omega_0^2}\left|\frac{d\mathbf{k}}{d\omega d\mathbf{\Omega}}\right| = \frac{v_g c^2}{\omega^2} J \qquad (7)$$

where $J = |d\mathbf{k}/d\omega d\mathbf{\Omega}|$ is the Jacobian of the transformation from $\mathbf{k} = (k_x, k_y, k_z)$ space to $(\omega,\mathbf{\Omega})$ space where $\mathbf{\Omega}$ is the solid angle around the direction $\mathbf{v}_g/v_g$. The main idea behind the following derivation is: (i) start from energy balance using physical principles, and (ii) find derivatives along the ray using eikonal equations.

## 2.1 Energy Balance

Now from energy balance over a domain $B = \Delta\omega\Delta\mathbf{\Omega}$ we get:

$$\begin{aligned} &\underbrace{\frac{\partial}{\partial t}\int_B \frac{1}{v_g} L_\omega(\mathbf{r},\mathbf{\Omega},t)d\omega d\mathbf{\Omega}}_{\varphi(\mathbf{r},t)} \\ &+ \nabla\cdot \underbrace{\int_B L_\omega(\mathbf{r},\mathbf{\Omega},t)\mathbf{\Omega}d\omega d\mathbf{\Omega}}_{\mathbf{F}(\mathbf{r},t)} = \\ &- \underbrace{\int_B (\mu_a+\mu_s)L_\omega(\mathbf{r},\mathbf{\Omega},t)d\omega d\mathbf{\Omega}}_{\text{absorption}} \\ &+ \underbrace{\int_B \mu_s \int_{4\pi} f(\mathbf{\Omega},\mathbf{\Omega}')L_\omega(\mathbf{r},\mathbf{\Omega}',t)d\mathbf{\Omega}'d\omega d\mathbf{\Omega}}_{\text{scattering}} \\ &+ \underbrace{\int_B \epsilon_\omega(\mathbf{r},\mathbf{\Omega},t)d\omega d\mathbf{\Omega}}_{\text{source}} \qquad (8) \end{aligned}$$

where $\partial/\partial t$ and $\nabla$ are taken along the ray, $\mu_a$ is the absorption coefficient, $\mu_s$ is the scattering coefficient, $f(\mathbf{\Omega},\mathbf{\Omega}')$ is the scattering function (also called the phase function) which gives the probability that an energy packet travelling in direction $\mathbf{\Omega}'$ is scattered into direction $\mathbf{\Omega}$, $\epsilon_\omega(\mathbf{r},\mathbf{\Omega},t)$ is a source distribution per unit volume per unit frequency, and $f$ is normalized according to

$$\int_{4\pi} f(\mathbf{\Omega},\mathbf{\Omega}')d\mathbf{\Omega}' = 1. \qquad (9)$$

The right hand side of equation (8) can be dealt easily if we assume that the optical parameters $\mu_a$ and $\mu_s$ are independent of $\omega$, $\mathbf{\Omega}$, and $t$. The left hand side of equation (8) requires further analysis because this is the term which requires the calculation of the derivative along the geometric ray satisfying the eikonal equation of geometric optics.

## 2.2 Eikonal Equation

First we will consider the simple time independent isotropic case. In this case, the fundamental equation of geometric optics is a nonlinear partial differential equation

$$(\nabla\Psi)^2 = n^2 \qquad (10)$$

where $\Psi$ has come to be known as the eikonal and the respective equation as the eikonal equation. The eikonal equation is a nonlinear, partial differential equation belonging to the Hamilton-Jacobi variety. In the general case the Hamilton-Jacobi equation has the form

$$H\left(\frac{\partial\Psi}{\partial q_1}, \frac{\partial\Psi}{\partial q_2}, \ldots, \frac{\partial\Psi}{\partial q_n}; q_1, q_2, \ldots, q_n\right) = 0 \qquad (11)$$

where $\Psi = \Psi(q_1, q_2, \ldots, q_n)$ is the function to be determined, $q_j$ are arbitrary coordinates ($j = 1, 2, \ldots, n$), and $p_j = \partial\Psi/\partial q_i$ are the associated "momenta". A nonlinear first order partial differential equations such as Hamilton-Jacobi can be solved using the method of characteristics in a straight forward manner [10, 13]. If we let $q_j$, $j = 1, 2, 3$, as cartesian coordinates and $H = H(\mathbf{p}, \mathbf{r})$, then the PDEs characteristics satisfy

$$\begin{aligned} \frac{d\mathbf{r}}{d\tau} &= \frac{\partial H}{\partial \mathbf{p}}, & (12)\\ \frac{d\mathbf{p}}{d\tau} &= -\frac{\partial H}{\partial \mathbf{r}}, & (13)\\ \frac{d\Psi}{d\tau} &= \frac{\partial H}{\partial \mathbf{p}}, & (14)\\ \mathbf{p} &= -\nabla\Psi & (15) \end{aligned}$$

where the solution $\mathbf{r}(\tau), \mathbf{p}(\tau)$, and $\Psi(\tau)$ are called the characteristics of the system in the phase space $\{p_j, q_j\}$. The parameter $\tau$ varying along the ray can readily be related to the arc length $s$ mainly,

$$d\tau = \frac{ds}{|\partial H/\partial \mathbf{p}|}. \quad (16)$$

From this, the fundamental equation of geometric optics can be derived using either of the two Hamiltonians:

$$\begin{aligned} H_1 &= \frac{1}{2}\left[\mathbf{p}^2 - n^2(\mathbf{r})\right] = 0 & (17)\\ H_2 &= p - n(\mathbf{r}) = 0 & (18) \end{aligned}$$

where $p = \sqrt{\mathbf{p}^2}$ in which case $|\partial H/\partial \mathbf{p}| = 1$ and therefore the parameter $\tau$ is equal to the arc length $s$. The first Hamiltonian leads to

$$\begin{aligned} \frac{d\mathbf{r}}{d\tau} &= \mathbf{p}, & (19)\\ \frac{d\mathbf{p}}{d\tau} &= \frac{1}{2}\nabla n^2(\mathbf{r}) & (20) \end{aligned}$$

and the second Hamiltonian leads to

$$\begin{aligned} \frac{d\mathbf{r}}{ds} &= \frac{\mathbf{p}}{n}, & (21)\\ \frac{d\mathbf{p}}{ds} &= \frac{1}{2}\nabla n(\mathbf{r}). & (22) \end{aligned}$$

If we let $\mathbf{\Omega} = \mathbf{p}/p = \mathbf{p}/n$, being tangent to the ray, we get from equations (21) and (21),

$$\begin{aligned} \frac{d\mathbf{r}}{ds} &= \mathbf{\Omega}, & (23)\\ \frac{d\mathbf{\Omega}}{ds} &= \frac{1}{n}\nabla n - \frac{1}{n}(\nabla n \cdot \mathbf{\Omega})\mathbf{\Omega} & (24) \end{aligned}$$

where $dn/ds = \mathbf{\Omega} \cdot \nabla n$. This is exactly the starting point of Ferwerda's [5] derivation which implies that he did not consider the most general space-time eikonal equations as shown below.

## 2.3 Space Time Eikonal

The Hamiltonian for the space time case is (see [13]):

$$H(\omega, \mathbf{k}, \mathbf{r}) = \mathbf{k}^2 - \frac{\omega^2}{c^2}n^2(\omega, \mathbf{r}) = 0 \quad (25)$$

where $n^2(\omega, \mathbf{r})$ is usually defined as $\epsilon(\omega, \mathbf{r})$ which is the fourier transform of the permittivity $\tilde{\epsilon}(t - t', \mathbf{r})$ in the medium, $\mathbf{k} = \nabla\Psi$ and $\omega = -\partial\Psi/\partial t$. This equation still belongs to Hamilton-Jacobi variety. This is the eikonal equation in 8-D space $(\mathbf{r}, \mathbf{k}, t, \omega)$. Using the method of characteristics, we get a set of characteristics parametrized by $\zeta$ as $(\mathbf{r}(\zeta), \mathbf{k}(\zeta), t(\zeta), \omega(\zeta))$ and changing from the ray parameter $t(\zeta)$ to $s$,

$$s = \int \left|\frac{d\omega(\mathbf{k}, \mathbf{r})}{d\mathbf{k}}\right| dt(\zeta) = \int v_g dt(\zeta) \quad (26)$$

that has the sense of the arc length of the spatial ray projection, one gets the following relations (see [13, 2]):

$$\begin{aligned} \frac{d\mathbf{r}}{ds} &= \mathbf{\Omega}, & (27)\\ \frac{d\mathbf{k}}{ds} &= -\frac{1}{v_g}\frac{\partial\omega(\mathbf{k}, \mathbf{r})}{\partial \mathbf{r}}, & (28)\\ \frac{dt}{ds} &= \frac{1}{v_g}, & (29)\\ \frac{d\omega}{ds} &= \frac{1}{v_g}\frac{\partial\omega(\mathbf{k}, \mathbf{r})}{\partial t} & (30) \end{aligned}$$

where $\mathbf{v}_g = \partial\omega(\mathbf{k}, \mathbf{r})/\partial \mathbf{k}$ is the group velocity, and $\mathbf{\Omega} = \mathbf{v}_g/v_g$ is the unit vector which may not necessarity coincide with the vector $\mathbf{k}/k$ in the non-isotropic case. The derivations for the eikonal equations can also be easily worked out using the methods of differential geometry and Frenet frames in tensor notation [15, 16]. However, we did not follow the differential geometry approach in order to keep our derivation accessible for a broader audience.

## 2.4 RTE For Spatially Varying Refractive Index

In geometric optics, it is more convenient to the use the wave vector $\mathbf{k} = (k_x, k_y, k_z)$ and the elementary volume $d\omega d\mathbf{\Omega} = J^{-1}d\mathbf{k}$ (see [2] for details). Using this notation we get,

$$\begin{aligned} &\int_{\Delta\mathbf{k}_0}\left[\frac{\partial}{\partial t}\left(\frac{1}{v_g}L_\omega(\mathbf{r}, \mathbf{\Omega}, t)J^{-1}\right)\right.\\ &+ \left.\nabla\cdot\left(L_\omega(\mathbf{r}, \mathbf{\Omega}, t)\mathbf{\Omega}J^{-1}\right)\right]d\mathbf{k} =\\ &- \int_{\Delta\mathbf{k}_0}(\mu_a + \mu_s)L_\omega(\mathbf{r}, \mathbf{\Omega}, t)d\mathbf{k}\\ &+ \int_{\Delta\mathbf{k}_0}\mu_s\int_{4\pi}f(\mathbf{\Omega}, \mathbf{\Omega}')L_\omega(\mathbf{r}, \mathbf{\Omega}, t)d\mathbf{\Omega}'d\mathbf{k}\\ &+ \int_{\Delta\mathbf{k}_0}\epsilon_\omega(\mathbf{r}, \mathbf{\Omega}, t)d\mathbf{k} \qquad (31) \end{aligned}$$

where the derivatives $\partial/\partial t$ and $\nabla$ are taken along the ray. Now if we further note that

$$\frac{\partial}{\partial t}\left(\frac{1}{v_g}\right) = \frac{1}{v_g}\frac{\partial}{\partial t}\left[\ln\left(\frac{1}{v_g}\right)\right], \quad (32)$$

$$\nabla\cdot\boldsymbol{\Omega} = \frac{1}{v_g}\nabla\cdot\mathbf{v}_g + \boldsymbol{\Omega}\cdot\nabla\ln\left(\frac{1}{v_g}\right), \quad (33)$$

$$\frac{1}{v_g}\nabla\cdot\mathbf{v}_g = \frac{d}{ds}\left[\ln j(\mathbf{r},\mathbf{r}_0)\right] \quad (34)$$

where $j(\mathbf{r},\mathbf{r}_0)$ is the jacobian of the transformation from ray coordinates to space coordinates (equation (34) is a consequence of Liouville's formula in phase space from classical mechanics [6]) and the total derivative along the ray is:

$$\begin{aligned} D_s &= \frac{d}{ds} = \frac{dt}{ds}\frac{\partial}{\partial t} + \frac{d\mathbf{r}}{ds}\frac{\partial}{\partial\mathbf{r}} + \frac{d\boldsymbol{\Omega}}{ds}\frac{\partial}{\partial\boldsymbol{\Omega}} + \frac{d\omega}{ds}\frac{\partial}{\partial\omega} \\ &= \frac{1}{v_g}\frac{\partial}{\partial t} + \boldsymbol{\Omega}\cdot\nabla + \frac{d\boldsymbol{\Omega}}{ds}\cdot\nabla_{\boldsymbol{\Omega}} + \frac{d\omega}{ds}\frac{\partial}{\partial\omega} \quad (35)\end{aligned}$$

where we have simplified the expressions for $dt/ds$ and $d\mathbf{r}/ds$ using the characteristic equations for the space-time eikonal and in place of wave vector $\mathbf{k}$, we have used $\boldsymbol{\Omega} = \mathbf{v}_g/v_g$ and $\omega$. Then the balance equation (31) transforms into,

$$\begin{aligned} &\int_{\Delta\mathbf{k}_0} D_s\left(L_\omega J_0^{-1}\right)d\mathbf{k} \\ &+ \int_{\Delta\mathbf{k}_0}\left[D_s\ln\left(\frac{1}{v_g}\right)\right]\left(L_\omega J_0^{-1}\right)d\mathbf{k} \\ &+ \int_{\Delta\mathbf{k}_0}\left[D_s\ln(j)\right]\left(L_\omega J_0^{-1}\right)d\mathbf{k} \\ &+ \int_{\Delta\mathbf{k}_0}\left[D_s\ln\left(\frac{c^2}{\omega^2}\right)\right]\left(L_\omega J_0^{-1}\right)d\mathbf{k} = \\ &- \int_{\Delta\mathbf{k}_0}(\mu_a+\mu_s)\left(L_\omega J_0^{-1}\right)d\mathbf{k} \\ &+ \int_{\Delta\mathbf{k}_0}\mu_s\int_{4\pi} f(\boldsymbol{\Omega},\boldsymbol{\Omega}')\left(L_\omega J_0^{-1}\right)d\boldsymbol{\Omega}'d\mathbf{k} \\ &+ \int_{\Delta\mathbf{k}_0}\epsilon_\omega(\mathbf{r},\boldsymbol{\Omega},t)\left(J_0^{-1}\right)d\mathbf{k}. \quad (36)\end{aligned}$$

where $J_0^{-1} = |d\omega d\boldsymbol{\Omega}/d\mathbf{k}_0|$, $D_s$ is the total derivative along a ray and $d\boldsymbol{\Omega}/ds$ and $d\omega/dt$ follows from the characteristic equations for space-time eikonal,

$$\frac{d\boldsymbol{\Omega}}{ds} = \frac{1}{v_g}\left(\mathbf{1}-\boldsymbol{\Omega}\otimes\boldsymbol{\Omega}\right)\frac{d\mathbf{v}_g}{ds}, \quad (37)$$

$$\frac{d\omega}{ds} = \frac{1}{v_g}\frac{d\omega}{dt}, \quad (38)$$

where $\mathbf{1}$ is a unit tensor and $\otimes$ is a tensor product, and $\mathbf{1}-\boldsymbol{\Omega}\otimes\boldsymbol{\Omega}$ is the projection operator. We added the derivative of $\ln(c^2/\omega^2)$ in equation (36) which is equal to zero because frequency along the ray remains invariant. Now if we combine a few of the terms in equation (36), we arrive at,

$$\begin{aligned} &\int_{\Delta\mathbf{k}_0} J_0^{-1}D_sL_\omega d\mathbf{k} \\ &+ \int_{\Delta\mathbf{k}_0} J_0^{-1}\left[D_s\ln\left(\frac{c^2}{\omega^2 v_g}J_0^{-1}j\right)\right]L_\omega d\mathbf{k} = \\ &- \int_{\Delta\mathbf{k}_0} J_0^{-1}(\mu_a+\mu_s)L_\omega d\mathbf{k} \\ &+ \int_{\Delta\mathbf{k}_0} J_0^{-1}\mu_s\int_{4\pi} f(\boldsymbol{\Omega},\boldsymbol{\Omega}')L_\omega(\mathbf{r},\boldsymbol{\Omega},t)d\boldsymbol{\Omega}'d\mathbf{k} \\ &+ \int_{\Delta\mathbf{k}_0} J_0^{-1}\epsilon_\omega(\mathbf{r},\boldsymbol{\Omega},t)d\mathbf{k}. \quad (39)\end{aligned}$$

Now because of Liouville's theorem [6] which leads to the invariance of the phase volume implies that $J_0^{-1}j = J^{-1}$ and since

$$n^2(\omega,\mathbf{r}) = \frac{v_gc^2}{\omega^2}\left|\frac{d\mathbf{k}}{d\omega d\boldsymbol{\Omega}}\right| = \frac{v_gc^2}{\omega^2}J \quad (40)$$

we get,

$$\ln\left(\frac{1}{n^2(\omega,\mathbf{r})}\right) = \ln\left(\frac{c^2}{v_g\omega^2}J_0^{-1}j\right). \quad (41)$$

With this simplification and noting that $J_0^{-1}d\mathbf{k} = d\omega d\boldsymbol{\Omega}$ we arrive at the following transport equation

$$\begin{aligned} &\int_B D_sL_\omega d\omega d\boldsymbol{\Omega} \\ &+ \int_B D_s\ln\left(\frac{1}{n^2}\right)L_\omega d\omega d\boldsymbol{\Omega} = \\ &- \int_B(\mu_a+\mu_s)L_\omega(\mathbf{r},\boldsymbol{\Omega},t)d\omega d\boldsymbol{\Omega} \\ &+ \int_B\mu_s\int_{4\pi} f(\boldsymbol{\Omega},\boldsymbol{\Omega}')L_\omega(\mathbf{r},\boldsymbol{\Omega},t)d\boldsymbol{\Omega}'d\omega d\boldsymbol{\Omega} \\ &+ \int_B\epsilon_\omega(\mathbf{r},\boldsymbol{\Omega},t)d\omega d\boldsymbol{\Omega}. \quad (42)\end{aligned}$$

If we further assume that $n(\mathbf{r})$ is only a function of $\mathbf{r}$ and does not depend on $(\boldsymbol{\Omega},t,\omega)$, $v_g = c/n$, and the equation (42) is satisfied in the strong sense [10, 20] i.e. left hand side of the integrand is equal to the right hand side of the integrand, we get for $L(\mathbf{r},\boldsymbol{\Omega},t)$ independent of $\omega$, the following RTE for a medium with spatially varying refractive index:

$$\begin{aligned} &\frac{n}{c}\frac{\partial L}{\partial t}\boldsymbol{\Omega}\cdot\nabla L + \frac{1}{n}\nabla n\cdot\nabla_{\boldsymbol{\Omega}}L - \frac{2}{n}(\boldsymbol{\Omega}\cdot\nabla n)L = \\ &-(\mu_a+\mu_s)L + \mu_s\int_{4\pi} f(\boldsymbol{\Omega},\boldsymbol{\Omega}')L(\mathbf{r},\boldsymbol{\Omega}',t)d\boldsymbol{\Omega}' \\ &+ \epsilon(\mathbf{r},\boldsymbol{\Omega},t). \quad (43)\end{aligned}$$

From straight forward manipulation of the simple eikonal equation (24) which does not include the general space-time case, Ferwerda [5] derived the following RTE:

$$\begin{aligned} &\frac{n}{c}\frac{\partial L}{\partial t} + \boldsymbol{\Omega}\cdot\nabla L + \frac{1}{n}\nabla n\cdot\nabla_{\boldsymbol{\Omega}}L + (\nabla\cdot\boldsymbol{\Omega})L = \\ &-(\mu_a+\mu_s)L + \mu_s\int_{4\pi} f(\boldsymbol{\Omega},\boldsymbol{\Omega}')L(\mathbf{r},\boldsymbol{\Omega}',t)d\boldsymbol{\Omega}' \\ &+ \epsilon(\mathbf{r},\boldsymbol{\Omega},t) \quad (44)\end{aligned}$$

where the expression for $\nabla \cdot \mathbf{\Omega}$ is given by:

$$\nabla \cdot \mathbf{\Omega} = \frac{1}{n}\left[\mathbf{\Omega}^{-1} \cdot \nabla n\right] - \frac{3}{n}\left(\nabla n \cdot \mathbf{\Omega}\right) \tag{45}$$

where $\mathbf{\Omega}^{-1} = \left(\mathbf{\Omega}_x^{-1}, \mathbf{\Omega}_y^{-1}, \mathbf{\Omega}_z^{-1}\right)$. The derived RTE for a medium with a spatially varying refractive index (43) differs from equation (44) in that the $\nabla \cdot \mathbf{\Omega}$ term is equivalent to $-2(\mathbf{\Omega} \cdot \nabla n)L/n$ instead of equation (45). The reason for this difference is that Ferwerda did not consider the most general case of space time eikonal. However, even though Ferwerda's reasoning is erroneous, it does involve an interesting expression for $\nabla \cdot \mathbf{\Omega}$ equation (45) which leads to some interesting mathematical surface integrals [12].

# 3 DIFFUSION APPROXIMATION

Here we will use the following identities which can be easily derived using vector calculus [11]:

$$\int_{4\pi} \mathbf{\Omega}_i d\mathbf{\Omega} = 0 \tag{46}$$

$$\int_{4\pi} \mathbf{\Omega}_i \mathbf{\Omega}_j d\mathbf{\Omega} = \frac{4\pi}{3}\delta_{ij} \tag{47}$$

$$\int_{4\pi} \mathbf{\Omega}_i \mathbf{\Omega}_j \mathbf{\Omega}_k d\mathbf{\Omega} = 0 \tag{48}$$

for all $i, j, k \in x, y, z$. Following the same idea, one can establish the following using Cauchy's principle value theorem [1]:

$$\int_{4\pi} \mathbf{\Omega}_i^{-1} \mathbf{\Omega}_j d\mathbf{\Omega} = 4\pi\delta_{ij} \tag{49}$$

$$\int_{4\pi} \mathbf{\Omega}_i \mathbf{\Omega}_j \mathbf{\Omega}_k d\mathbf{\Omega} = 0 \tag{50}$$

for all $i, j, k \in x, y, z$. Now if we let

$$L(\mathbf{r}, \mathbf{\Omega}, t) \approx \Phi(\mathbf{r}, t) + \frac{3}{4\pi}(\mathbf{F} \cdot \mathbf{\Omega}) \tag{51}$$

where

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi}\int_{4\pi} L(\mathbf{r}, \mathbf{\Omega}, t) d\mathbf{\Omega} \tag{52}$$

$$\mathbf{F}(\mathbf{r}, t) = \int_{4\pi} L(\mathbf{r}, \mathbf{\Omega}, t)\mathbf{\Omega} d\mathbf{\Omega} = F(\mathbf{r}, t)\mathbf{\Omega}_f. \tag{53}$$

Equation (51) may be regarded as the first two terms of a Taylor's expansion of $L$ in terms of the powers of $\mathbf{\Omega} \cdot \mathbf{\Omega}_f$ and therefore the second term in (51) must be considerably smaller than the first: $\Phi(\mathbf{r}, t) \gg |\mathbf{F} \cdot \mathbf{\Omega}|$ (see [8]). Equation (51), the starting point of the diffusion approximation, is a reliable approximation if the radiance does not differ too much from an isotropic distribution. This will be the case when the probability of scattering of the photons is much greater than the probability of being absorbed. In terms of the scattering coefficient $\mu_s$ and the absorption coefficient $\mu_a$, this implies: $\mu_a \ll \mu_s$.

Using the same argument as in [11], if we integrate both of the equation (43) with respect to $\mathbf{\Omega}$ over all $4\pi$ steradians, we get

$$\frac{4\pi n}{c}\frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{F} + \frac{2}{n}\nabla n \cdot \mathbf{F} = -4\pi\mu_a\Phi + E \tag{54}$$

where

$$E(\mathbf{r}, t) = \int_{4\pi} \epsilon(\mathbf{r}, \mathbf{\Omega}, t) d\mathbf{\Omega}. \tag{55}$$

If we now plug (51) into the equation (43), multiply each term by $\mathbf{\Omega}$, and integrate over all $4\pi$ steradians, we get using the same arguments as in [11]:

$$\frac{n}{c}\frac{\partial \mathbf{F}}{\partial t} + \frac{4\pi}{3}\nabla\Phi - \frac{8\pi}{3n}\nabla n\Phi = -(\mu_a + \mu_s)\mathbf{F} + \bar{f}\mu_s\mathbf{F} + \mathbf{H} \tag{56}$$

where the mean scattering cosine $\bar{f}$ and $\mathbf{H}(\mathbf{r}, t)$ is given by,

$$\bar{f} = \int_{4\pi} f(\mathbf{\Omega}, \mathbf{\Omega}')(\mathbf{\Omega} \cdot \mathbf{\Omega}') d\mathbf{\Omega}' \tag{57}$$

$$\mathbf{H}(\mathbf{r}, t) = \int_{4\pi} \epsilon(\mathbf{r}, \mathbf{\Omega}, t)\mathbf{\Omega} d\mathbf{\Omega}. \tag{58}$$

Equations (54) and (56) form a closed system for $\Phi$ and $\mathbf{F}$ which are called the $P_1$-approximation to the radiative transfer equation (43) for a medium with a spatially varying refractive index. To obtain the diffusion approximation, we go a step further, assuming that $L(\mathbf{r}, \mathbf{\Omega}, t)$ is almost stationary in the sense that $\partial\mathbf{F}/\partial t$ is negligible in equation (56). In that case, we have approximately,

$$\frac{4\pi}{3}\nabla\Phi - \frac{8\pi}{3n}\nabla n\Phi = -(\mu_a + \mu_s')\mathbf{F} + \mathbf{H} \tag{59}$$

where $\mu_s' = (1 - \bar{f})\mu_s$. Equation (59) is called Fick's law corresponding to the RTE (43). We note that the assumption $\partial\mathbf{F}/\partial t = 0$ commonly used in the literature (see [3]) is clearly erroneous in the fully time-dependent case since $L(\mathbf{r}, \mathbf{\Omega}, t)$ is time dependent (and hence so is $\mathbf{F}$), but is usually justified by assuming that $\mathbf{F}$ is dominated by an exponentially decaying term $e^{-c\mu_a t}$ and $\mu_a \ll \mu_s'$ which is true for highly scattering media where the diffusion approximation is valid. If we solve for $\mathbf{F}$ in equation (59) we get,

$$\mathbf{F} = -4\pi D\nabla\Phi + \frac{8\pi}{n}D\nabla n\Phi + 3D\mathbf{H} \tag{60}$$

where $D = 1/3(\mu_a + \mu_s')$. Now substituting $\mathbf{F}$ from equation (60) into equation (54), we arrive at the diffusion approximation:

$$\frac{n}{c}\frac{\partial \Phi}{\partial t} - \nabla \cdot (D\nabla\Phi) + 2\nabla \cdot \left(\frac{D}{n}\nabla n\Phi\right)$$

$$
\begin{aligned}
- \quad & \frac{2D}{n}\nabla n \cdot \nabla\Phi + \frac{4D}{n^2}\nabla n \cdot \nabla n\Phi + \mu_a\Phi \\
= \quad & \frac{1}{4\pi}E - \frac{3D}{2\pi n}\nabla n \cdot \mathbf{H} - \frac{3}{4\pi}\nabla \cdot (D\mathbf{H}). \qquad (61)
\end{aligned}
$$

# 4 CONCLUSIONS

In summary, we have derived the RTE for spatially varying refractive index and its diffusion approximation. The diffusion approximation derived in equations (61) reduces to the familiar diffusion approximation for a medium with constant refractive index, as it should. In the case of DC optical tomography, the time-independent radiative transfer equation can be used and equation (61) simplifies to

$$
\begin{aligned}
\nabla \cdot (D\nabla\Phi) + 2\nabla \cdot \left(\frac{D}{n}\nabla n\Phi\right) \quad & - \quad \frac{2D}{n}\nabla n \cdot \nabla\Phi \\
\frac{4D}{n^2}\nabla n \cdot \nabla n\Phi + \mu_a\Phi \quad & = \quad \frac{1}{4\pi}E \qquad (62)
\end{aligned}
$$

where we further assumed the source $\epsilon$ to be independent of $\mathbf{\Omega}$ and hence that $\mathbf{H}$ is zero. Equation (62) after some simplification becomes

$$
\begin{aligned}
- \quad & \nabla \cdot (D\nabla\Phi) + \frac{2}{n}\nabla \cdot (D\nabla n)\,\Phi \\
+ \quad & \frac{2D}{n^2}\nabla n \cdot \nabla n\Phi + \mu_a\Phi = \frac{1}{4\pi}E. \qquad (63)
\end{aligned}
$$

We are currently investigating whether there are any analytic solutions to equation (63) in special cases to give us further insight into the differences between the constant and spatially varying refractive index case discussed in this paper.

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