On derivation of the radiative transfer equation and its spherical harmonics approximation for scattering media with spatially varying refractive indices

T. Khan* and A. Thomas
Department of Mathematical Sciences
Clemson University, Clemson, SC 29634-0975

ABSTRACT

Traditionally, the radiative transfer equation and its P_N or spherical harmonics approximation are derived for a medium with a spatially constant refractive index (Ishimaru 1978 Wave Propagation and Scattering in Random Medial vol 1 (New York: Academic)). In this report, we derive the radiative transport equation and its P_N approximation relevant to optical tomography for a medium with a spatially varying refractive index. We find the analytical solution of the coupled system of partial differential equations corresponding to the P_N approximation in a spherically symmetric geometry. We compute the eigenvalues of the system and compare them with the eigenvalues for the spatially constant refractive index case. We show that the P_N model with spatially varying refractive index for photon transport is substantially different than the spatially constant model.

Keywords: Radiative transport, optical tomography, P_N approximation, spherical harmonics expansion, refractive index, inverse problems, and biomedical imaging.

1 INTRODUCTION

Recent interest in the radiative transfer equation (RTE) for a medium with spatially varying refractive index is getting more attention [14, 10, 19, 18]. Media with spatially varying refractive index are among us in the form of biological tissues and the atmosphere, just to mention two examples [26, 23]. For a detailed discussion on the potential applications in optical imaging of biological tissue, see Jiang [10]. For a general introduction to optical tomography, we refer to the following [3, 11, 30, 16, 4] and the references therein.

Recently several authors including Ferwerda [7], Khan et al. [13], Tualle et al. [27], Marti-Lopez et al. [19] attempted to derive the RTE for a medium with a spatially varying refractive index. However, a closer look in the literature reveals that Ryzhik et al. [24]

*Corresponding author (Email: khan@clemson.edu)

had derived the general form of the transport equation for the energy density of waves in a random media. Bekefi [5], Kravtsov and Orlov [17], and Apresyan and Kravtsov [2] had also derived various versions of a ray refractive index equation for a non-absorption and non-scattering media with spatially varying refractive indices. In fact, there have been other attempts to derive the RTE for a few special cases for a medium with a spatially varying refractive index as early as the time when Chandrasekhar derived his equation for a constant refractive index [9, 20]. In fact, the RTE relevant to optical tomography for a medium with a spatially varying refractive index is only a special case of the general result. However the appropriate photon transport model for a medium with spatially varying refractive index is still not widely known to applied and interdisciplinary practitioners for optics applications [10]. The aim of this report is twofold. The first goal is to present a simple derivation of the RTE and derive its P_N approximation relevant to optical tomography accessible to a wider audience. The second is to compare the spatially varying refractive index model with the spatially constant model to conclude if there are substantial differences between these two models for photon transport phenomena in biological tissue.

The outline of the report is as follows. In section 2, we derive the RTE for a medium with spatially varying refractive index. In section 3, we derive the corresponding P_N approximation. In section 4, we find the analytical solution of the coupled system of partial differential equations corresponding to the P_N approximation in a spherically symmetric geometry. In section 5, we compare the eigenvalues for the spatially varying and spatially constant refractive index cases. In section 6, we discuss conclusions and future work.

2 DERIVATION OF THE RTE FOR SPATIALLY VARYING REFRACTIVE INDEX

The fundamental quantity of interest in radiative transfer is the spectral density of the radiance or spectral radiance or simply radiance $L_\omega(\mathbf{r}, \mathbf{\Omega}, t)$ (which is sometimes called the specific intensity) which is defined as the amount of energy which at position $\mathbf{r}$ flows per second through a unit area perpendicular to the unit vector $\mathbf{\Omega}$ in the frequency interval $(\omega, \omega + d\omega)$. L_ω is measured in W sr^{-1}m^{-2}Hz^{-1} or in erg s^{-1}sr^{-1}cm^{-2}Hz^{-1}. The above definition is the most general definition of radiance which considers both statistical and wave aspects [24, 2]. The radiant flux $\mathbf{F}_\omega$, the total quantities $\mathbf{F}$ and L, and the energy density $\Phi(\mathbf{r})$ are defined as

$$\mathbf{F}_\omega(\mathbf{r}, t) = \int_{S^2} L_\omega(\mathbf{r}, \mathbf{\Omega}, t)\mathbf{\Omega} d\mathbf{\Omega}, \tag{1}$$

$$\mathbf{F}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \mathbf{F}_\omega(\mathbf{r}, t) d\omega, \tag{2}$$

$$L(\mathbf{r}, \mathbf{\Omega}, t) = \int_{-\infty}^{\infty} L_\omega(\mathbf{r}, \mathbf{\Omega}, t) d\omega, \tag{3}$$

$$\Phi(\mathbf{r}, t) = \frac{1}{c}\int_{S^2} L(\mathbf{r}, \mathbf{\Omega}, t) d\mathbf{\Omega}. \tag{4}$$

For the most general case, in an anisotropic, inhomogenous, and dispersive media, we also need to consider that beam propagates with a group velocity $\mathbf{v}_g$, the magnitude of the velocity

v_g may depend on frequency and direction of the wave $\mathbf{k}$. In the general case of an anisotropic medium, the direction of the group velocity no longer coincides with the wave vector $\mathbf{k}$ and the frequency ω is related to $\mathbf{k}$ via the dispersion relations $\omega = \omega(\mathbf{k}, \mathbf{r})$. Therefore in general the energy density Φ and flux $\mathbf{F}$ are defined as

$$\mathbf{F}(\mathbf{r}, t) = \int_B L_\omega(\mathbf{r}, \mathbf{\Omega}, t)\mathbf{\Omega} d\omega d\mathbf{\Omega}, \tag{5}$$

$$\Phi(\mathbf{r}, t) = \int_B \frac{1}{v_g} L_\omega(\mathbf{r}, \mathbf{\Omega}, t) d\omega d\mathbf{\Omega}. \tag{6}$$

It should be emphasized that for anisotropic media, $\mathbf{\Omega} = \mathbf{v}_g/v_g$ rather than $\mathbf{\Omega} = \mathbf{k}/k$ is the unit vector of ray direction. Furthermore, the integration is carried out over the domain B in direction and frequency, $(\omega, \mathbf{\Omega})$ space, corresponding to the propagation of waves. Using quantum approach, one can obtain a relationship between the group velocity and the spatially varying index of refraction $n(\omega, \mathbf{r})$ as in [5]:

$$n^2(\omega, \mathbf{r}) = \frac{v_g c^2}{\omega_0^2} \left| \frac{d\mathbf{k}}{d\omega d\mathbf{\Omega}} \right| = \frac{v_g c^2}{\omega^2} J \tag{7}$$

where $J = |d\mathbf{k}/d\omega d\mathbf{\Omega}|$ is the Jacobian of the transformation from $\mathbf{k} = (k_x, k_y, k_z)$ space to $(\omega, \mathbf{\Omega})$ space where $\mathbf{\Omega}$ is the solid angle around the direction $\mathbf{v}_g/v_g$. The main idea behind the following derivation is: (i) start from energy balance using physical principles, and (ii) find derivatives along the ray using eikonal equations.

2.1 Energy Balance

Now from energy balance over a domain $B = \Delta\omega\Delta\mathbf{\Omega}$ we get:

$$\begin{aligned}
& \frac{\partial}{\partial t} \underbrace{\int_B \frac{1}{v_g} L_\omega(\mathbf{r}, \mathbf{\Omega}, t) d\omega d\mathbf{\Omega}}_{\Phi(\mathbf{r},t)} \\
+ \quad & \nabla \cdot \underbrace{\int_B L_\omega(\mathbf{r}, \mathbf{\Omega}, t)\mathbf{\Omega} d\omega d\mathbf{\Omega}}_{\mathbf{F}(\mathbf{r},t)} = \\
- \quad & \underbrace{\int_B (\mu_a + \mu_s) L_\omega(\mathbf{r}, \mathbf{\Omega}, t) d\omega d\mathbf{\Omega}}_{\text{absorption}} \\
+ \quad & \underbrace{\int_B \mu_s \int_{4\pi} f(\mathbf{\Omega}, \mathbf{\Omega}') L_\omega(\mathbf{r}, \mathbf{\Omega}', t) d\mathbf{\Omega}' d\omega d\mathbf{\Omega}}_{\text{scattering}} \\
+ \quad & \underbrace{\int_B \epsilon_\omega(\mathbf{r}, \mathbf{\Omega}, t) d\omega d\mathbf{\Omega}}_{\text{source}}
\end{aligned} \tag{8}$$

where $\partial/\partial t$ and ∇ are taken along the ray, μ_a is the absorption coefficient, μ_s is the scattering coefficient, $f(\mathbf{\Omega}, \mathbf{\Omega}')$ is the scattering function (also called the phase function) which gives the probability that an energy packet travelling in direction $\mathbf{\Omega}'$ is scattered into direction $\mathbf{\Omega}$, $\epsilon_\omega(\mathbf{r}, \mathbf{\Omega}, t)$ is a source distribution per unit volume per unit frequency, and f is normalized according to

$$\int_{S^2} f(\mathbf{\Omega}, \mathbf{\Omega}') d\mathbf{\Omega}' = 1. \tag{9}$$

The right hand side of equation (8) can be dealt easily if we assume that the optical parameters μ_a and μ_s are independent of ω, $\mathbf{\Omega}$, and t. The left hand side of equation (8) requires further analysis because this is the term which requires the calculation of the derivative along the geometric ray satisfying the eikonal equation of geometric optics.

2.2 Eikonal Equation

First we will consider the simple time independent isotropic case. In this case, the fundamental equation of geometric optics is a nonlinear partial differential equation

$$(\nabla\Psi)^2 = n^2 \tag{10}$$

where Ψ has come to be known as the eikonal and the respective equation as the eikonal equation. The eikonal equation is a nonlinear, partial differential equation belonging to the Hamilton-Jacobi variety. In the general case the Hamilton-Jacobi equation has the form

$$H\left(\frac{\partial\Psi}{\partial q_1}, \frac{\partial\Psi}{\partial q_2}, \ldots, \frac{\partial\Psi}{\partial q_n}; q_1, q_2, \ldots, q_n\right) = 0 \tag{11}$$

where $\Psi = \Psi(q_1, q_2, \ldots, q_n)$ is the function to be determined, q_j are arbitrary coordinates $(j = 1, 2, \ldots, n)$, and $p_j = \partial\Psi/\partial q_i$ are the associated "momenta". A nonlinear first order partial differential equations such as Hamilton-Jacobi can be solved using the method of characteristics in a straight forward manner [12, 17]. If we let q_j, $j = 1, 2, 3$, as cartesian coordinates and $H = H(\mathbf{p}, \mathbf{r})$, then the PDEs characteristics satisfy

$$\frac{d\mathbf{r}}{d\tau} = \frac{\partial H}{\partial \mathbf{p}}, \tag{12}$$

$$\frac{d\mathbf{p}}{d\tau} = -\frac{\partial H}{\partial \mathbf{r}}, \tag{13}$$

$$\frac{d\Psi}{d\tau} = \frac{\partial H}{\partial \mathbf{p}}, \tag{14}$$

$$\mathbf{p} = -\nabla\Psi \tag{15}$$

where the solution $\mathbf{r}(\tau), \mathbf{p}(\tau)$, and $\Psi(\tau)$ are called the characteristics of the system in the phase space $\{p_j, q_j\}$. The parameter τ varying along the ray can readily be related to the arc length s mainly,

$$d\tau = \frac{ds}{|\partial H/\partial \mathbf{p}|}. \tag{16}$$

From this, the fundamental equation of geometric optics can be derived using either of the two Hamiltonians:

$$H_1 = \frac{1}{2}\left[\mathbf{p}^2 - n^2(\mathbf{r})\right] = 0 \tag{17}$$

$$H_2 = p - n(\mathbf{r}) = 0 \tag{18}$$

where $p = \sqrt{\mathbf{p}^2}$ in which case $|\partial H/\partial \mathbf{p}| = 1$ and therefore the parameter τ is equal to the arc length s. The first Hamiltonian leads to

$$\frac{d\mathbf{r}}{d\tau} = \mathbf{p}, \tag{19}$$

$$\frac{d\mathbf{p}}{d\tau} = \frac{1}{2}\nabla n^2(\mathbf{r}) \tag{20}$$

and the second Hamiltonian leads to

$$\frac{d\mathbf{r}}{ds} = \frac{\mathbf{p}}{n}, \tag{21}$$

$$\frac{d\mathbf{p}}{ds} = \frac{1}{2}\nabla n(\mathbf{r}). \tag{22}$$

If we let $\mathbf{\Omega} = \mathbf{p}/p = \mathbf{p}/n$, being tangent to the ray, we get from equations (21) and (21),

$$\frac{d\mathbf{r}}{ds} = \mathbf{\Omega}, \tag{23}$$

$$\frac{d\mathbf{\Omega}}{ds} = \frac{1}{n}\nabla n - \frac{1}{n}(\nabla n \cdot \mathbf{\Omega})\mathbf{\Omega} \tag{24}$$

where $dn/ds = \mathbf{\Omega} \cdot \nabla n$. This is exactly the starting point of Ferwerda's [7] derivation which implies that he did not consider the most general space-time eikonal equations as shown below.

2.3 Space Time Eikonal

The Hamiltonian for the space time case is (see [17]):

$$H(\omega, \mathbf{k}, \mathbf{r}) = \mathbf{k}^2 - \frac{\omega^2}{c^2} n^2(\omega, \mathbf{r}) = 0 \tag{25}$$

where $n^2(\omega, \mathbf{r})$ is usually defined as $\epsilon(\omega, \mathbf{r})$ which is the fourier transform of the permittivity $\tilde{\epsilon}(t-t', \mathbf{r})$ in the medium, $\mathbf{k} = \nabla\Psi$ and $\omega = -\partial\Psi/\partial t$. This equation still belongs to Hamilton-Jacobi variety. This is the eikonal equation in 8-D space $(\mathbf{r}, \mathbf{k}, t, \omega)$. Using the method of characteristics, we get a set of characteristics parametrized by ζ as $(\mathbf{r}(\zeta), \mathbf{k}(\zeta), t(\zeta), \omega(\zeta))$ and changing from the ray parameter $t(\zeta)$ to s,

$$s = \int \left|\frac{d\omega(\mathbf{k}, \mathbf{r})}{d\mathbf{k}}\right| dt(\zeta) = \int v_g dt(\zeta) \tag{26}$$

that has the sense of the arc length of the spatial ray projection, one gets the following relations (see [17, 2]):

$$\frac{d\mathbf{r}}{ds} = \mathbf{\Omega}, \tag{27}$$

$$\frac{d\mathbf{k}}{ds} = -\frac{1}{v_g}\frac{\partial\omega(\mathbf{k},\mathbf{r})}{\partial\mathbf{r}}, \tag{28}$$

$$\frac{dt}{ds} = \frac{1}{v_g}, \tag{29}$$

$$\frac{d\omega}{ds} = \frac{1}{v_g}\frac{\partial\omega(\mathbf{k},\mathbf{r})}{\partial t} \tag{30}$$

where $\mathbf{v}_g = \partial\omega(\mathbf{k},\mathbf{r})/\partial\mathbf{k}$ is the group velocity, and $\mathbf{\Omega} = \mathbf{v}_g/v_g$ is the unit vector which may not necessarity coincide with the vector $\mathbf{k}/k$ in the non-isotropic case. The derivations for the eikonal equations can also be easily worked out using the methods of differential geometry and Frenet frames in tensor notation [21, 22]. However, we did not follow the differential geometry approach in order to keep our derivation accessible for a broader audience.

2.4 RTE For Spatially Varying Refractive Index

In geometric optics, it is more convenient to the use the wave vector $\mathbf{k} = (k_x, k_y, k_z)$ and the elementary volume $d\omega d\mathbf{\Omega} = J^{-1}d\mathbf{k}$ (see [2] for details). Using this notation we get,

$$\begin{aligned}
& \int_{\Delta\mathbf{k}_0}\left[\frac{\partial}{\partial t}\left(\frac{1}{v_g}L_\omega(\mathbf{r},\mathbf{\Omega},t)J^{-1}\right)\right. \\
+ & \left.\nabla\cdot\left(L_\omega(\mathbf{r},\mathbf{\Omega},t)\mathbf{\Omega}J^{-1}\right)\right]d\mathbf{k} = \\
- & \int_{\Delta\mathbf{k}_0}(\mu_a+\mu_s)L_\omega(\mathbf{r},\mathbf{\Omega},t)d\mathbf{k} \\
+ & \int_{\Delta\mathbf{k}_0}\mu_s\int_{4\pi}f(\mathbf{\Omega},\mathbf{\Omega}')L_\omega(\mathbf{r},\mathbf{\Omega},t)d\mathbf{\Omega}'d\mathbf{k} \\
+ & \int_{\Delta\mathbf{k}_0}\epsilon_\omega(\mathbf{r},\mathbf{\Omega},t)d\mathbf{k}
\end{aligned} \tag{31}$$

where the derivatives $\partial/\partial t$ and ∇ are taken along the ray. Now if we further note that

$$\frac{\partial}{\partial t}\left(\frac{1}{v_g}\right) = \frac{1}{v_g}\frac{\partial}{\partial t}\left[\ln\left(\frac{1}{v_g}\right)\right], \tag{32}$$

$$\nabla\cdot\mathbf{\Omega} = \frac{1}{v_g}\nabla\cdot\mathbf{v}_g + \mathbf{\Omega}\cdot\nabla\ln\left(\frac{1}{v_g}\right), \tag{33}$$

$$\frac{1}{v_g}\nabla\cdot\mathbf{v}_g = \frac{d}{ds}\left[\ln j(\mathbf{r},\mathbf{r}_0)\right] \tag{34}$$

where $j(\mathbf{r},\mathbf{r}_0)$ is the Jacobian of the transformation from ray coordinates to space coordinates (equation (34) is a consequence of Liouville's formula in phase space from classical mechanics

[8]) and the total derivative along the ray is:

$$
\begin{aligned}
D_s &= \frac{d}{ds} = \frac{dt}{ds}\frac{\partial}{\partial t} + \frac{d\mathbf{r}}{ds}\frac{\partial}{\partial \mathbf{r}} + \frac{d\mathbf{\Omega}}{ds}\frac{\partial}{\partial \mathbf{\Omega}} + \frac{d\omega}{ds}\frac{\partial}{\partial \omega} \\
&= \frac{1}{v_g}\frac{\partial}{\partial t} + \mathbf{\Omega}\cdot\nabla + \frac{d\mathbf{\Omega}}{ds}\cdot\nabla_{\mathbf{\Omega}} + \frac{d\omega}{ds}\frac{\partial}{\partial \omega}
\end{aligned} \tag{35}
$$

where we have simplified the expressions for dt/ds and $d\mathbf{r}/ds$ using the characteristic equations for the space-time eikonal and in place of wave vector $\mathbf{k}$, we have used $\mathbf{\Omega} = \mathbf{v}_g/v_g$ and ω. Then the balance equation (31) transforms into,

$$
\begin{aligned}
& \int_{\Delta\mathbf{k}_0} D_s\left(L_\omega J_0^{-1}\right) d\mathbf{k} \\
+ & \int_{\Delta\mathbf{k}_0} \left[D_s \ln\left(\frac{1}{v_g}\right)\right]\left(L_\omega J_0^{-1}\right) d\mathbf{k} \\
+ & \int_{\Delta\mathbf{k}_0} \left[D_s \ln(j)\right]\left(L_\omega J_0^{-1}\right) d\mathbf{k} \\
+ & \int_{\Delta\mathbf{k}_0} \left[D_s \ln\left(\frac{c^2}{\omega^2}\right)\right]\left(L_\omega J_0^{-1}\right) d\mathbf{k} = \\
- & \int_{\Delta\mathbf{k}_0} (\mu_a + \mu_s)\left(L_\omega J_0^{-1}\right) d\mathbf{k} \\
+ & \int_{\Delta\mathbf{k}_0} \mu_s \int_{4\pi} f(\mathbf{\Omega}, \mathbf{\Omega}')\left(L_\omega J_0^{-1}\right) d\mathbf{\Omega}' d\mathbf{k} \\
+ & \int_{\Delta\mathbf{k}_0} \epsilon_\omega(\mathbf{r}, \mathbf{\Omega}, t)\left(J_0^{-1}\right) d\mathbf{k}.
\end{aligned} \tag{36}
$$

where $J_0^{-1} = |d\omega d\mathbf{\Omega}/d\mathbf{k}_0|$, D_s is the total derivative along a ray and $d\mathbf{\Omega}/ds$ and $d\omega/dt$ follows from the characteristic equations for space-time eikonal,

$$
\frac{d\mathbf{\Omega}}{ds} = \frac{1}{v_g}\left(\mathbf{1} - \mathbf{\Omega}\otimes\mathbf{\Omega}\right)\frac{d\mathbf{v}_g}{ds}, \tag{37}
$$

$$
\frac{d\omega}{ds} = \frac{1}{v_g}\frac{d\omega}{dt}, \tag{38}
$$

where $\mathbf{1}$ is a unit tensor and $\otimes$ is a tensor product, and $\mathbf{1} - \mathbf{\Omega}\otimes\mathbf{\Omega}$ is the projection operator. We added the derivative of $\ln(c^2/\omega^2)$ in equation (36) which is equal to zero because frequency along the ray remains invariant. Now if we combine a few of the terms in equation (36), we arrive at,

$$
\begin{aligned}
& \int_{\Delta\mathbf{k}_0} J_0^{-1} D_s L_\omega d\mathbf{k} \\
+ & \int_{\Delta\mathbf{k}_0} J_0^{-1}\left[D_s \ln\left(\frac{c^2}{\omega^2 v_g} J_0^{-1} j\right)\right] L_\omega d\mathbf{k} = \\
- & \int_{\Delta\mathbf{k}_0} J_0^{-1}(\mu_a + \mu_s) L_\omega d\mathbf{k}
\end{aligned}
$$

$$+ \int_{\Delta \mathbf{k}_0} J_0^{-1} \mu_s \int_{4\pi} f(\mathbf{\Omega}, \mathbf{\Omega}') L_\omega(\mathbf{r}, \mathbf{\Omega}, t) d\mathbf{\Omega}' d\mathbf{k}$$

$$+ \int_{\Delta \mathbf{k}_0} J_0^{-1} \epsilon_\omega(\mathbf{r}, \mathbf{\Omega}, t) d\mathbf{k}. \tag{39}$$

Now because of Liouville's theorem [8] which leads to the invariance of the phase volume implies that $J_0^{-1} j = J^{-1}$ and since

$$n^2(\omega, \mathbf{r}) = \frac{v_g c^2}{\omega^2} \left| \frac{d\mathbf{k}}{d\omega d\mathbf{\Omega}} \right| = \frac{v_g c^2}{\omega^2} J \tag{40}$$

we get,

$$\ln \left(\frac{1}{n^2(\omega, \mathbf{r})} \right) = \ln \left(\frac{c^2}{v_g \omega^2} J_0^{-1} j \right). \tag{41}$$

With this simplification and noting that $J_0^{-1} d\mathbf{k} = d\omega d\mathbf{\Omega}$ we arrive at the following transport equation

$$\int_B D_s L_\omega d\omega d\mathbf{\Omega}$$

$$+ \int_B D_s \ln \left(\frac{1}{n^2} \right) L_\omega d\omega d\mathbf{\Omega} =$$

$$- \int_B (\mu_a + \mu_s) L_\omega(\mathbf{r}, \mathbf{\Omega}, t) d\omega d\mathbf{\Omega}$$

$$+ \int_B \mu_s \int_{4\pi} f(\mathbf{\Omega}, \mathbf{\Omega}') L_\omega(\mathbf{r}, \mathbf{\Omega}, t) d\mathbf{\Omega}' d\omega d\mathbf{\Omega}$$

$$+ \int_B \epsilon_\omega(\mathbf{r}, \mathbf{\Omega}, t) d\omega d\mathbf{\Omega}. \tag{42}$$

If we further assume that $n(\mathbf{r})$ is only a function of $\mathbf{r}$ and does not depend on $(\mathbf{\Omega}, t, \omega)$, $v_g = c/n$, and the equation (42) is satisfied in the strong sense [12, 28] i.e. left hand side of the integrand is equal to the right hand side of the integrand, we get for $L(\mathbf{r}, \mathbf{\Omega}, t)$ independent of ω, the following RTE for a medium with spatially varying refractive index:

$$\frac{n}{c} \frac{\partial L}{\partial t} + \mathbf{\Omega} \cdot \nabla L + \frac{1}{n} \nabla n \cdot \nabla_{\mathbf{\Omega}} L - \frac{2}{n} (\mathbf{\Omega} \cdot \nabla n) L =$$

$$-(\mu_a + \mu_s) L + \mu_s \int_{4\pi} f(\mathbf{\Omega}, \mathbf{\Omega}') L(\mathbf{r}, \mathbf{\Omega}', t) d\mathbf{\Omega}'$$

$$+ \epsilon(\mathbf{r}, \mathbf{\Omega}, t). \tag{43}$$

From straight forward manipulation of the simple eikonal equation (24) which does not include the general space-time case, Ferwerda [7] derived the following RTE:

$$\frac{n}{c} \frac{\partial L}{\partial t} + \mathbf{\Omega} \cdot \nabla L + \frac{1}{n} \nabla n \cdot \nabla_{\mathbf{\Omega}} L + (\nabla \cdot \mathbf{\Omega}) L =$$

$$-(\mu_a+\mu_s)L \quad + \quad \mu_s\int_{4\pi} f(\mathbf{\Omega},\mathbf{\Omega}')L(\mathbf{r},\mathbf{\Omega}',t)d\mathbf{\Omega}' + \epsilon(\mathbf{r},\mathbf{\Omega},t) \tag{44}$$

where the expression for $\nabla\cdot\mathbf{\Omega}$ is given by:

$$\nabla\cdot\mathbf{\Omega} = \frac{1}{n}\left[\mathbf{\Omega}^{-1}\cdot\nabla n\right] - \frac{3}{n}\left(\nabla n\cdot\mathbf{\Omega}\right) \tag{45}$$

where $\mathbf{\Omega}^{-1} = \left(\mathbf{\Omega}_x^{-1},\mathbf{\Omega}_y^{-1},\mathbf{\Omega}_z^{-1}\right)$. The derived RTE for a medium with a spatially varying refractive index (43) differs from equation (44) in that the $\nabla\cdot\mathbf{\Omega}$ term is equivalent to $-2(\mathbf{\Omega}\cdot\nabla n)L/n$ instead of equation (45). The reason for this difference is that Ferwerda did not consider the most general case of space time eikonal. However, even though Ferwerda's reasoning is erroneous, it does involve an interesting expression for $\nabla\cdot\mathbf{\Omega}$ equation (45) which leads to some interesting mathematical surface integrals [15].

3 P_N OR SPHERICAL HARMONICS APPROXIMATION

We recall from the previous section, the radiative transfer equation with spatially refractive index,

$$\frac{n}{c}\frac{\partial L}{\partial t} + \mathbf{\Omega}\cdot\nabla L + \frac{1}{n}\nabla n\cdot\nabla_{\mathbf{\Omega}}L - \frac{2}{n}(\mathbf{\Omega}\cdot\nabla n)L = -(\mu_a+\mu_s)L + \mu_s\int_{S^2} f(\mathbf{\Omega}\cdot\mathbf{\Omega}')L(\mathbf{r},\mathbf{\Omega}',t)d\mathbf{\Omega}' + \epsilon(\mathbf{r},\mathbf{\Omega},t). \tag{46}$$

We use the spherical harmonic expansion of L and ϵ,

$$L(\mathbf{r},\mathbf{\Omega},t) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}(\mathbf{r},t)Y_{\ell,m}(\mathbf{\Omega}) \tag{47}$$

and

$$\epsilon(\mathbf{r},\mathbf{\Omega},t) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\epsilon_{\ell,m}(\mathbf{r},t)Y_{\ell,m}(\mathbf{\Omega}) \tag{48}$$

where $((2\ell+1)/4\pi)^{1/2}$ is the normalization factor. The phase function f can also be expressed using the addition theorem (87) as,

$$\begin{aligned} f(\mathbf{\Omega}'\cdot\mathbf{\Omega}) &= \sum_{\ell=0}^{\infty}\frac{2\ell+1}{4\pi}f_\ell P_\ell(\mathbf{\Omega}'\cdot\mathbf{\Omega}) \\ &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_\ell Y^*_{\ell,m}(\mathbf{\Omega}')Y_{\ell,m}(\mathbf{\Omega}). \end{aligned} \tag{49}$$

Substituting these expansions into (46) we get

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \left(\frac{2\ell+1}{4\pi}\right)^{1/2}\left[\left[\frac{n}{c}\frac{\partial}{\partial t}+\mathbf{\Omega}\cdot\nabla\right.\right.$$
$$+\left.\frac{1}{n}\nabla n\cdot\nabla_{\mathbf{\Omega}}-\frac{2}{n}(\mathbf{\Omega}\cdot\nabla n)+\mu_t\right]\psi_{\ell,m}Y_{\ell,m}-\epsilon_{\ell,m}Y_{\ell,m}$$
$$-\left.\mu_s\int_{S^2}d\mathbf{\Omega}'\psi_{\ell,m}Y_{\ell,m}(\mathbf{\Omega}')\sum_{\ell'=0}^{\infty}\sum_{m'=-\ell'}^{\ell'}f_{\ell'}Y^*_{\ell',m'}(\mathbf{\Omega}')Y_{\ell',m'}(\mathbf{\Omega})\right]=0 \tag{50}$$

where $\mu_t=\mu_s+\mu_a$ is the transport coefficient. The integral over $\mathbf{\Omega}'$ can be calculated using the orthogonality relation for the spherical harmonics (86). Then equation (46) becomes

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \left(\frac{2\ell+1}{4\pi}\right)^{1/2}\left[\left[\frac{n}{c}\frac{\partial}{\partial t}+\mathbf{\Omega}\cdot\nabla\right.\right.$$
$$+\left.\left.\frac{1}{n}\nabla n\cdot\nabla_{\mathbf{\Omega}}-\frac{2}{n}(\mathbf{\Omega}\cdot\nabla n)+\mu_t^n\right]\psi_{\ell,m}-\epsilon_{\ell,m}\right]Y_{\ell,m}=0 \tag{51}$$

where $\mu_t^\ell=\mu_s(1-f_\ell)+\mu_a$ is the reduced transport coefficient. If we multiply equation (51) by $Y^*_{p,q}$ and integrate over $\mathbf{\Omega}$ we can use the orthogonality relation (86) in all terms except the terms with $\mathbf{\Omega}\cdot\nabla$, $(\nabla n\cdot\nabla_{\mathbf{\Omega}})/n$, and $-(2\mathbf{\Omega}\cdot\nabla n)/n$. Therefore we arrive at

$$\left(\frac{2p+1}{4\pi}\right)^{1/2}\frac{n}{c}\frac{\partial}{\partial t}\psi_{p,q}+\left(\frac{2p+1}{4\pi}\right)^{1/2}\mu_t^p\psi_{p,q}$$
$$+\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_{S^2}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\left[\mathbf{\Omega}\cdot\nabla+\frac{1}{n}\nabla n\cdot\nabla_{\mathbf{\Omega}}-\frac{2}{n}(\mathbf{\Omega}\cdot\nabla n)\right]\psi_{\ell,m}Y_{\ell,m}Y^*_{p,q}d\mathbf{\Omega}$$
$$=\left(\frac{2p+1}{4\pi}\right)^{1/2}\epsilon_{p,q}. \tag{52}$$

3.1 Calculating $\mathbf{\Omega}\cdot\nabla$ term

We begin by writing $\mathbf{\Omega}$ as $(\mathbf{\Omega}_x,\mathbf{\Omega}_y,\mathbf{\Omega}_z)$ which may be represented in spherical coordinates as $\mathbf{\Omega}=(\sin\vartheta\cos\varphi,\ \sin\vartheta\sin\varphi,\ \cos\vartheta)$. Then

$$\mathbf{\Omega}\cdot\nabla=\mathbf{\Omega}_x\frac{\partial}{\partial x}+\mathbf{\Omega}_y\frac{\partial}{\partial y}+\mathbf{\Omega}_z\frac{\partial}{\partial z}.$$

Now we multiply each term by $Y^*_{p,q}(\mathbf{\Omega})$ and use the recurrence relations for spherical harmonic functions given in (92)-(94) to explicitly find each coordinate of $\mathbf{\Omega}Y^*_{p,q}(\mathbf{\Omega})$:

$$\mathbf{\Omega}_xY^*_{p,q}(\mathbf{\Omega}) = \sin\vartheta\cos\varphi\, Y^*_{p,q}(\mathbf{\Omega})$$

$$
\begin{aligned}
&= \sin\vartheta \frac{1}{2}(e^{i\varphi} + e^{-i\varphi})\, Y^*_{p,q}(\mathbf{\Omega}) \\
&= \sin\vartheta \frac{1}{2}\left[(e^{-i\varphi} + e^{i\varphi})\, Y_{p,q}(\mathbf{\Omega})\right]^* \\
&= \left[\frac{1}{2}\sin\vartheta\, e^{i\varphi} Y_{p,q}(\mathbf{\Omega}) + \frac{1}{2}\sin\vartheta\, e^{-i\varphi} Y_{p,q}(\mathbf{\Omega}))\right]^* \\
&= -\frac{1}{2}\left(\frac{(p+q+1)(p+q+2)}{(2p+1)(2p+3)}\right)^{1/2} Y^*_{p+1,q+1}(\mathbf{\Omega}) \\
&\quad +\frac{1}{2}\left(\frac{(p-q)(p-q-1)}{(2p-1)(2p+1)}\right)^{1/2} Y^*_{p-1,q+1}(\mathbf{\Omega}) \\
&\quad +\frac{1}{2}\left(\frac{(p-q+1)(p-q+2)}{(2p+1)(2p+3)}\right)^{1/2} Y^*_{p+1,q-1}(\mathbf{\Omega}) \\
&\quad -\frac{1}{2}\left(\frac{(p+q)(p+q-1)}{(2p-1)(2p+1)}\right)^{1/2} Y^*_{p-1,q-1}(\mathbf{\Omega}),
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{\Omega}_y Y^*_{p,q}(\mathbf{\Omega}) &= \sin\vartheta \sin\varphi\, Y^*_{p,q}(\mathbf{\Omega}) \\
&= \sin\vartheta \frac{1}{2i}(e^{i\varphi} - e^{-i\varphi})\, Y^*_{p,q}(\mathbf{\Omega}) \\
&= \sin\vartheta \frac{1}{2i}\left[(e^{-i\varphi} - e^{i\varphi})\, Y_{p,q}(\mathbf{\Omega})\right]^* \\
&= -\frac{1}{2i}\left[\sin\vartheta\, e^{i\varphi} Y_{p,q}(\mathbf{\Omega})\right]^* + \frac{1}{2i}\left[\sin\vartheta\, e^{-i\varphi} Y_{p,q}(\mathbf{\Omega}))\right]^* \\
&= \frac{1}{2i}\left(\frac{(p+q+1)(p+q+2)}{(2p+1)(2p+3)}\right)^{1/2} Y^*_{p+1,q+1}(\mathbf{\Omega}) \\
&\quad -\frac{1}{2i}\left(\frac{(p-q)(p-q-1)}{(2p-1)(2p+1)}\right)^{1/2} Y^*_{p-1,q+1}(\mathbf{\Omega}) \\
&\quad +\frac{1}{2i}\left(\frac{(p-q+1)(p-q+2)}{(2p+1)(2p+3)}\right)^{1/2} Y^*_{p+1,q-1}(\mathbf{\Omega}) \\
&\quad -\frac{1}{2i}\left(\frac{(p+q)(p+q-1)}{(2p-1)(2p+1)}\right)^{1/2} Y^*_{p-1,q-1}(\mathbf{\Omega}),
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{\Omega}_z Y^*_{p,q}(\mathbf{\Omega}) &= \cos\vartheta\, Y^*_{p,q}(\mathbf{\Omega}) \\
&= \left[\cos\vartheta\, Y_{p,q}(\mathbf{\Omega})\right]^* \\
&= \left(\frac{(p-q+1)(p+q+1)}{(2p+1)(2p+3)}\right)^{1/2} Y^*_{p+1,q}(\mathbf{\Omega}) \\
&\quad +\left(\frac{(p-q)(p+q)}{(2p-1)(2p+1)}\right)^{1/2} Y^*_{p-1,q}(\mathbf{\Omega}).
\end{aligned}
$$

Thus by the orthogonality of the spherical harmonic functions

$$
\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\int_{S^2} Y_{\ell,m} Y^*_{p,q} \mathbf{\Omega}\cdot\nabla\psi_{\ell,m} d\mathbf{\Omega} \; =
$$

$$
\begin{aligned}
&-\frac{1}{2}\left(\frac{2p+3}{4\pi}\right)^{1/2}\left(\frac{(p+q+1)(p+q+2)}{(2p+1)(2p+3)}\right)^{1/2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\psi_{p+1,q+1}\\
&+\frac{1}{2}\left(\frac{2p-1}{4\pi}\right)^{1/2}\left(\frac{(p-q)(p-q-1)}{(2p-1)(2p+1)}\right)^{1/2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\psi_{p-1,q+1}\\
&+\frac{1}{2}\left(\frac{2p+3}{4\pi}\right)^{1/2}\left(\frac{(p-q+1)(p-q+2)}{(2p+1)(2p+3)}\right)^{1/2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\psi_{p+1,q-1}\\
&-\frac{1}{2}\left(\frac{2p-1}{4\pi}\right)^{1/2}\left(\frac{(p+q)(p+q-1)}{(2p-1)(2p+1)}\right)^{1/2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\psi_{p-1,q-1}\\
&+\left(\frac{2p+3}{4\pi}\right)^{1/2}\left(\frac{(p-q+1)(p+q+1)}{(2p+1)(2p+3)}\right)^{1/2}\frac{\partial}{\partial z}\psi_{p+1,q}\\
&+\left(\frac{2p-1}{4\pi}\right)^{1/2}\left(\frac{(p-q)(p+q)}{(2p-1)(2p+1)}\right)^{1/2}\frac{\partial}{\partial z}\psi_{p-1,q}. \qquad (53)
\end{aligned}
$$

3.2 Calculating $(\nabla n\cdot\nabla_{\mathbf{\Omega}})/n$ term

First note that

$$
\nabla_{\mathbf{\Omega}}L = \left(\cos\vartheta\cos\varphi\frac{\partial L}{\partial\vartheta}-\frac{\sin\varphi}{\sin\vartheta}\frac{\partial L}{\partial\varphi}\right)\hat{x}+\left(\cos\vartheta\sin\varphi\frac{\partial L}{\partial\vartheta}+\frac{\cos\varphi}{\sin\vartheta}\frac{\partial L}{\partial\varphi}\right)\hat{y}-\sin\vartheta\frac{\partial L}{\partial\vartheta}\hat{z}
$$

according to [14]. Therefore we get

$$
\begin{aligned}
\nabla_{\mathbf{\Omega}}L = &\left(\cos\vartheta\cos\varphi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}\frac{\partial}{\partial\vartheta}Y_{\ell,m}\right.\\
&\left.-\frac{\sin\varphi}{\sin\vartheta}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}\frac{\partial}{\partial\varphi}Y_{\ell,m}\right)\hat{x}\\
&+\left(\cos\vartheta\sin\varphi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}\frac{\partial}{\partial\vartheta}Y_{\ell,m}\right.\\
&\left.+\frac{\cos\varphi}{\sin\vartheta}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}\frac{\partial}{\partial\varphi}Y_{\ell,m}\right)\hat{y}\\
&-\sin\vartheta\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}\frac{\partial}{\partial\vartheta}Y_{\ell,m}\hat{z}.
\end{aligned}
$$

Making the substitutions (89) and (91) for the derivatives of $Y_{\ell,m}$ in terms of ϑ and φ respectively yields

$$
\begin{aligned}
&\frac{1}{n}\nabla n\cdot\nabla_{\mathbf{\Omega}}L\\
= &\frac{1}{n}\left(\cos\vartheta\cos\varphi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}\left[|m|(\cot\vartheta)Y_{\ell,m}\right.\right.
\end{aligned}
$$

$$+\ e^{-i\sigma_m\varphi}\rho(\ell,m)Y_{\ell,m+\sigma_m}\big] - \frac{\sin\varphi}{\sin\vartheta}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}(im)Y_{\ell,m}\Bigg)\frac{\partial n}{\partial x}$$
$$+\frac{1}{n}\Bigg(\cos\vartheta\sin\varphi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}\left[|m|(\cot\vartheta)Y_{\ell,m}\right.$$
$$+\ e^{-i\sigma_m\varphi}\rho(\ell,m)Y_{\ell,m+\sigma_m}\big] + \frac{\cos\varphi}{\sin\vartheta}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}(im)Y_{\ell,m}\Bigg)\frac{\partial n}{\partial y}$$
$$-\frac{1}{n}\left(\sin\vartheta\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}\left[|m|(\cot\vartheta)Y_{\ell,m}+e^{-i\sigma_m\varphi}\rho(\ell,m)Y_{\ell,m+\sigma_m}\right]\right)\frac{\partial n}{\partial z}.$$

We proceed further by multiplying $\nabla_{\mathbf{\Omega}}L$ with $Y^*_{p,q}$ and integrating over S^2 and changing the integration variable to ϑ and φ for convenience,

$$\int_{S^2}\frac{1}{n}\nabla n\cdot\nabla_{\mathbf{\Omega}}L(\mathbf{\Omega})Y^*_{p,q}(\mathbf{\Omega})d\mathbf{\Omega}$$
$$= \int_0^{2\pi}\int_0^{\pi}\frac{1}{n}\nabla n\cdot\nabla_{\mathbf{\Omega}}L(\vartheta,\varphi)Y^*_{p,q}(\vartheta,\varphi)\sin\vartheta d\vartheta d\varphi$$

$$= \int_0^{2\pi}\int_0^{\pi}\frac{1}{n}\Bigg(\cos\vartheta\cos\varphi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}|m|(\cot\vartheta)Y_{\ell,m}Y^*_{p,q}\sin\vartheta$$
$$+\cos\vartheta\cos\varphi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}e^{-i\sigma_m\varphi}\rho(\ell,m)Y_{\ell,m+\sigma_m}Y^*_{p,q}\sin\vartheta$$
$$-\frac{\sin\varphi}{\sin\vartheta}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}(im)Y_{\ell,m}Y^*_{p,q}\sin\vartheta\Bigg)d\vartheta d\varphi\,\frac{\partial n}{\partial x}$$
$$+\int_0^{2\pi}\int_0^{\pi}\frac{1}{n}\Bigg(\cos\vartheta\sin\varphi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}|m|(\cot\vartheta)Y_{\ell,m}Y^*_{p,q}\sin\vartheta$$
$$+\cos\vartheta\sin\varphi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}e^{-i\sigma_m\varphi}\rho(\ell,m)Y_{\ell,m+\sigma_m}Y^*_{p,q}\sin\vartheta$$
$$+\cos\varphi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}(im)Y_{\ell,m}Y^*_{p,q}\Bigg)d\vartheta d\varphi\,\frac{\partial n}{\partial y}$$
$$-\int_0^{2\pi}\int_0^{\pi}\frac{1}{n}\Bigg(\sin\vartheta\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}|m|(\cot\vartheta)Y_{\ell,m}Y^*_{p,q}\sin\vartheta$$
$$+\sin\vartheta\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}e^{-i\sigma_m\varphi}\rho(\ell,m)Y_{\ell,m+\sigma_m}Y^*_{p,q}\sin\vartheta\Bigg)d\vartheta d\varphi\,\frac{\partial n}{\partial z}$$

$$= \frac{1}{n}\Bigg(\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}|m|\int_0^{2\pi}\int_0^{\pi}(\cos^2\vartheta\cos\varphi)Y_{\ell,m}Y^*_{p,q}d\vartheta d\varphi$$

$$+\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}\rho(\ell,m)\int_0^{2\pi}\int_0^{\pi}(\cos\vartheta\cos\varphi\sin\vartheta)e^{-i\sigma_m\varphi}Y_{\ell,m+\sigma_m}Y^*_{p,q}d\vartheta d\varphi$$

$$-\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}(im)\int_0^{2\pi}\int_0^{\pi}\sin\varphi Y_{\ell,m}Y^*_{p,q}d\vartheta d\varphi\Bigg)\ \frac{\partial n}{\partial x}$$

$$+\frac{1}{n}\left(\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}|m|\int_0^{2\pi}\int_0^{\pi}(\cos^2\vartheta\sin\varphi)Y_{\ell,m}Y^*_{p,q}d\vartheta d\varphi\right.$$

$$+\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}\rho(\ell,m)\int_0^{2\pi}\int_0^{\pi}(\cos\vartheta\sin\varphi\sin\vartheta)e^{-i\sigma_m\varphi}Y_{\ell,m+\sigma_m}Y^*_{p,q}d\vartheta d\varphi$$

$$+\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}(im)\int_0^{2\pi}\int_0^{\pi}\cos\varphi Y_{\ell,m}Y^*_{p,q}d\vartheta d\varphi\Bigg)\ \frac{\partial n}{\partial y}$$

$$-\frac{1}{n}\left(\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}|m|\int_0^{2\pi}\int_0^{\pi}(\cot\vartheta\sin^2\vartheta)Y_{\ell,m}Y^*_{p,q}d\vartheta d\varphi\right.$$

$$+\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}\rho(\ell,m)\int_0^{2\pi}\int_0^{\pi}(\sin^2\vartheta)e^{-i\sigma_m\varphi}Y_{\ell,m+\sigma_m}Y^*_{p,q}d\vartheta d\varphi\Bigg)\ \frac{\partial n}{\partial z}$$

$$= \frac{1}{n}\left(\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}|m|I_1(\ell,m;p,q)\right.$$

$$+\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}\rho(\ell,m)I_2(\ell,m;p,q)$$

$$-\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}(im)I_3(\ell,m;p,q)\Bigg)\ \frac{\partial n}{\partial x}$$

$$+\frac{1}{n}\left(\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}|m|I_4(\ell,m;p,q)\right.$$

$$+\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}\rho(\ell,m)I_5(\ell,m;p,q)$$

$$+\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}(im)I_6(\ell,m;p,q)\Bigg)\ \frac{\partial n}{\partial y}$$

$$-\frac{1}{n}\left(\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}|m|I_7(\ell,m;p,q)\right.$$

$$+\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}\rho(\ell,m)I_8(\ell,m;p,q)\Bigg)\ \frac{\partial n}{\partial z} \tag{54}$$

where,

$$
\begin{aligned}
I_1(\ell,m) &:= \int_0^{2\pi}\int_0^{\pi}(\cos^2\vartheta\cos\varphi)Y_{\ell,m}Y^*_{p,q}d\vartheta d\varphi \\
I_2(\ell,m) &:= \int_0^{2\pi}\int_0^{\pi}(\cos\vartheta\cos\varphi\sin\vartheta)e^{-i\sigma_m\varphi}Y_{\ell,m+\sigma_m}Y^*_{p,q}d\vartheta d\varphi \\
I_3(\ell,m) &:= \int_0^{2\pi}\int_0^{\pi}\sin\varphi Y_{\ell,m}Y^*_{p,q}d\vartheta d\varphi \\
I_4(n,m) &:= \int_0^{2\pi}\int_0^{\pi}(\cos^2\vartheta\sin\varphi)Y_{\ell,m}Y^*_{p,q}d\vartheta d\varphi \\
I_5(\ell,m) &:= \int_0^{2\pi}\int_0^{\pi}(\cos\vartheta\sin\varphi\sin\vartheta)e^{-i\sigma_m\varphi}Y_{\ell,m+\sigma_m}Y^*_{p,q}d\vartheta d\varphi \\
I_6(\ell,m) &:= \int_0^{2\pi}\int_0^{\pi}\cos\varphi Y_{\ell,m}Y^*_{p,q}d\vartheta d\varphi \\
I_7(\ell,m) &:= \int_0^{2\pi}\int_0^{\pi}(\cot\vartheta\sin^2\vartheta)Y_{\ell,m}Y^*_{p,q}d\vartheta d\varphi \\
I_8(\ell,m) &:= \int_0^{2\pi}\int_0^{\pi}(\sin^2\vartheta)e^{-i\sigma_m\varphi}Y_{\ell,m+\sigma_m}Y^*_{p,q}d\vartheta d\varphi.
\end{aligned}
$$

The above integrals I_1 through I_8 are computed in the Appendix E.

3.3 Calculating $-(2\mathbf{\Omega}\cdot\nabla n)/n$ term

Let n be a function of x, y, z mainly $n(x,y,z)$, then $\nabla n = (\frac{\partial n}{\partial x}, \frac{\partial n}{\partial y}, \frac{\partial n}{\partial z})$ and

$$\mathbf{\Omega}\cdot\nabla n = \mathbf{\Omega}_x\frac{\partial n}{\partial x} + \mathbf{\Omega}_y\frac{\partial n}{\partial y} + \mathbf{\Omega}_z\frac{\partial n}{\partial z}.$$

Then

$$
\begin{aligned}
&\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\int_{S^2}-\frac{2}{n}\mathbf{\Omega}\cdot\nabla n Y_{\ell,m}Y^*_{p,q}\psi_{\ell,m}d\mathbf{\Omega} = \\
&\quad \frac{1}{n}\left(\frac{2p+3}{4\pi}\right)^{1/2}\left(\frac{(p+q+1)(p+q+2)}{(2p+1)(2p+3)}\right)^{1/2}\left(\frac{\partial n}{\partial x}+i\frac{\partial n}{\partial y}\right)\psi_{p+1,q+1} \\
&\quad -\frac{1}{n}\left(\frac{2p-1}{4\pi}\right)^{1/2}\left(\frac{(p-q)(p-q-1)}{(2p-1)(2p+1)}\right)^{1/2}\left(\frac{\partial n}{\partial x}+i\frac{\partial n}{\partial y}\right)\psi_{p-1,q+1} \\
&\quad -\frac{1}{n}\left(\frac{2p+3}{4\pi}\right)^{1/2}\left(\frac{(p-q+1)(p-q+2)}{(2p+1)(2p+3)}\right)^{1/2}\left(\frac{\partial n}{\partial x}-i\frac{\partial n}{\partial y}\right)\psi_{p+1,q-1} \\
&\quad +\frac{1}{n}\left(\frac{2p-1}{4\pi}\right)^{1/2}\left(\frac{(p+q)(p+q-1)}{(2p-1)(2p+1)}\right)^{1/2}\left(\frac{\partial n}{\partial x}-i\frac{\partial n}{\partial y}\right)\psi_{p-1,q-1} \\
&\quad -\frac{2}{n}\left(\frac{2p+3}{4\pi}\right)^{1/2}\left(\frac{(p-q+1)(p+q+1)}{(2p+1)(2p+3)}\right)^{1/2}\frac{\partial n}{\partial z}\psi_{p+1,q}
\end{aligned}
$$

$$-\frac{2}{n}\left(\frac{2p-1}{4\pi}\right)^{1/2}\left(\frac{(p-q)(p+q)}{(2p-1)(2p+1)}\right)^{1/2}\frac{\partial n}{\partial z}\psi_{p-1,q}. \tag{55}$$

3.4 The Coupled System of Partial Differential Equations

Now if we plug in equations (53), (54), and (55) into equation (52) we obtain the following infinite system of coupled partial differential equations as an alternate representation for (46):

$$\begin{aligned}
&\left(\frac{2p+1}{4\pi}\right)^{1/2}\frac{n}{c}\frac{\partial}{\partial t}\psi_{p,q}+\left(\frac{2p+1}{4\pi}\right)^{1/2}\mu_t^p\psi_{p,q}\\
&-\frac{1}{2}\left(\frac{2p+3}{4\pi}\right)^{1/2}\left(\frac{(p+q+1)(p+q+2)}{(2p+1)(2p+3)}\right)^{1/2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\psi_{p+1,q+1}\\
&+\frac{1}{2}\left(\frac{2p-1}{4\pi}\right)^{1/2}\left(\frac{(p-q)(p-q-1)}{(2p-1)(2p+1)}\right)^{1/2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\psi_{p-1,q+1}\\
&+\frac{1}{2}\left(\frac{2p+3}{4\pi}\right)^{1/2}\left(\frac{(p-q+1)(p-q+2)}{(2p+1)(2p+3)}\right)^{1/2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\psi_{p+1,q-1}\\
&-\frac{1}{2}\left(\frac{2p-1}{4\pi}\right)^{1/2}\left(\frac{(p+q)(p+q-1)}{(2p-1)(2p+1)}\right)^{1/2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\psi_{p-1,q-1}\\
&+\left(\frac{2p+3}{4\pi}\right)^{1/2}\left(\frac{(p-q+1)(p+q+1)}{(2p+1)(2p+3)}\right)^{1/2}\frac{\partial}{\partial z}\psi_{p+1,q}\\
&+\left(\frac{2p-1}{4\pi}\right)^{1/2}\left(\frac{(p-q)(p+q)}{(2p-1)(2p+1)}\right)^{1/2}\frac{\partial}{\partial z}\psi_{p-1,q}\\
&+\frac{1}{n}\left(\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}|m|I_1(\ell,m;p,q)\right.\\
&+\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}\rho(\ell,m)I_2(\ell,m;p,q)\\
&\left.-\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}(im)I_3(\ell,m;p,q)\right)\frac{\partial n}{\partial x}\\
&+\frac{1}{n}\left(\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}|m|I_4(\ell,m;p,q)\right.\\
&+\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}\rho(\ell,m)I_5(\ell,m;p,q)\\
&\left.+\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}(im)I_6(\ell,m;p,q)\right)\frac{\partial n}{\partial y}\\
&-\frac{1}{n}\left(\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}|m|I_7(\ell,m;p,q)\right.
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{4\pi}\right)^{1/2}\psi_{\ell,m}\rho(\ell,m)I_8(\ell,m;p,q)\Bigg)\ \frac{\partial n}{\partial z}\\
&+\frac{1}{n}\left(\frac{2p+3}{4\pi}\right)^{1/2}\left(\frac{(p+q+1)(p+q+2)}{(2p+1)(2p+3)}\right)^{1/2}\left(\frac{\partial n}{\partial x}+i\frac{\partial n}{\partial y}\right)\psi_{p+1,q+1}\\
&-\frac{1}{n}\left(\frac{2p-1}{4\pi}\right)^{1/2}\left(\frac{(p-q)(p-q-1)}{(2p-1)(2p+1)}\right)^{1/2}\left(\frac{\partial n}{\partial x}+i\frac{\partial n}{\partial y}\right)\psi_{p-1,q+1}\\
&-\frac{1}{n}\left(\frac{2p+3}{4\pi}\right)^{1/2}\left(\frac{(p-q+1)(p-q+2)}{(2p+1)(2p+3)}\right)^{1/2}\left(\frac{\partial n}{\partial x}-i\frac{\partial n}{\partial y}\right)\psi_{p+1,q-1}\\
&+\frac{1}{n}\left(\frac{2p-1}{4\pi}\right)^{1/2}\left(\frac{(p+q)(p+q-1)}{(2p-1)(2p+1)}\right)^{1/2}\left(\frac{\partial n}{\partial x}-i\frac{\partial n}{\partial y}\right)\psi_{p-1,q-1}\\
&-\frac{2}{n}\left(\frac{2p+3}{4\pi}\right)^{1/2}\left(\frac{(p-q+1)(p+q+1)}{(2p+1)(2p+3)}\right)^{1/2}\frac{\partial n}{\partial z}\psi_{p+1,q}\\
&-\frac{2}{n}\left(\frac{2p-1}{4\pi}\right)^{1/2}\left(\frac{(p-q)(p+q)}{(2p-1)(2p+1)}\right)^{1/2}\frac{\partial n}{\partial z}\psi_{p-1,q}\\
&=\left(\frac{2p+1}{4\pi}\right)^{1/2}\epsilon_{p,q}. \qquad (56)
\end{aligned}
$$

Now we will introduce some notations to simplify equation (56). Let

$$
\begin{aligned}
\alpha_p^q &:= \left(\frac{(p+q)(p+q+1)}{2p+1}\right)^{1/2}\\
\beta_p^q &:= \left(\frac{(p+q)(p+q-1)}{2p+1}\right)^{1/2}\\
\xi_p^q &:= \left(\frac{(p-q+1)(p+q+1)}{2p+1}\right)^{1/2}\\
\eta_p^q &:= \left(\frac{(p-q)(p+q)}{2p+1}\right)^{1/2}.
\end{aligned}
$$

Using these notations defined above and factoring out coefficients and simplifying we get within the P_N approximation,

$$
\begin{aligned}
&(2p+1)^{1/2}\left(\frac{n}{c}\frac{\partial}{\partial t}+\mu_t^p\right)\psi_{p,q}+\eta_p^q\left[\frac{\partial}{\partial z}-\frac{2}{n}\frac{\partial n}{\partial z}\right]\psi_{p-1,q}\\
&+\beta_p^q\left[\frac{1}{n}\left(\frac{\partial n}{\partial x}-i\frac{\partial n}{\partial y}\right)-\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\right]\psi_{p-1,q-1}\\
&-\beta_p^{-q}\left[\frac{1}{n}\left(\frac{\partial n}{\partial x}+i\frac{\partial n}{\partial y}\right)-\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\right]\psi_{p-1,q+1}+\xi_p^q\left[\frac{\partial}{\partial z}-\frac{2}{n}\frac{\partial n}{\partial z}\right]\psi_{p+1,q}\\
&-\alpha_p^{-q+1}\left[\frac{1}{n}\left(\frac{\partial n}{\partial x}-i\frac{\partial n}{\partial y}\right)-\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\right]\psi_{p+1,q-1}\\
&+\alpha_p^{q+1}\left[\frac{1}{n}\left(\frac{\partial n}{\partial x}+i\frac{\partial n}{\partial y}\right)-\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\right]\psi_{p+1,q+1}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\ell=0}^{N}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{n^2}\right)^{1/2}\left[\frac{\partial n}{\partial x}\left[|m|I_1(\ell,m;p,q)+\rho(\ell,m)I_2(\ell,m;p,q)\right.\right.\\
&-\left.(im)I_3(\ell,m;p,q)\right]+\frac{\partial n}{\partial y}\left[|m|I_4(\ell,m;p,q)+\rho(\ell,m)I_5(\ell,m;p,q)\right.\\
&+\left.\left.(im)I_6(\ell,m;p,q)\right]-\frac{\partial n}{\partial z}\left[|m|I_7(\ell,m;p,q)+\rho(\ell,m)I_8(\ell,m;p,q)\right]\right]\psi_{\ell,m}\\
&=(2p+1)^{1/2}\,\epsilon_{p,q}.
\end{aligned}
\tag{57}
$$

3.5 P_1 Approximation

The P_1 approximation is obtained by assuming that $\psi_{\ell,m}=0$ for $\ell>1$. In this case we get four equations which we will find the general form of using (57) by letting $N=1$:

$$
\begin{aligned}
&(2p+1)^{1/2}\left(\frac{n}{c}\frac{\partial}{\partial t}+\mu_t^p\right)\psi_{p,q}+\eta_p^q\left[\frac{\partial}{\partial z}-\frac{2}{n}\frac{\partial n}{\partial z}\right]\psi_{p-1,q}\\
&+\beta_p^q\left[\frac{1}{n}\left(\frac{\partial n}{\partial x}-i\frac{\partial n}{\partial y}\right)-\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\right]\psi_{p-1,q-1}\\
&-\beta_p^{-q}\left[\frac{1}{n}\left(\frac{\partial n}{\partial x}+i\frac{\partial n}{\partial y}\right)-\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\right]\psi_{p-1,q+1}+\xi_p^q\left[\frac{\partial}{\partial z}-\frac{2}{n}\frac{\partial n}{\partial z}\right]\psi_{p+1,q}\\
&-\alpha_p^{-q+1}\left[\frac{1}{n}\left(\frac{\partial n}{\partial x}-i\frac{\partial n}{\partial y}\right)-\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\right]\psi_{p+1,q-1}\\
&+\alpha_p^{q+1}\left[\frac{1}{n}\left(\frac{\partial n}{\partial x}+i\frac{\partial n}{\partial y}\right)-\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\right]\psi_{p+1,q+1}\\
&+\sum_{\ell=0}^{1}\sum_{m=-\ell}^{\ell}\left(\frac{2\ell+1}{n^2}\right)^{1/2}\left[\frac{\partial n}{\partial x}\left[|m|I_1(\ell,m;p,q)+\rho(\ell,m)I_2(\ell,m;p,q)\right.\right.\\
&-\left.(im)I_3(\ell,m;p,q)\right]+\frac{\partial n}{\partial y}\left[|m|I_4(\ell,m;p,q)+\rho(\ell,m)I_5(\ell,m;p,q)\right.\\
&+\left.\left.(im)I_6(\ell,m;p,q)\right]-\frac{\partial n}{\partial z}\left[|m|I_7(\ell,m;p,q)+\rho(\ell,m)I_8(\ell,m;p,q)\right]\right]\psi_{\ell,m}\\
&=(2p+1)^{1/2}\,\epsilon_{p,q}
\end{aligned}
$$

expanding the sums, taking note that $\rho(j,j)=0$ for all $j\in\mathbb{N}_0$, we arrive at:

$$
\begin{aligned}
&(2p+1)^{1/2}\left(\frac{n}{c}\frac{\partial}{\partial t}+\mu_t^p\right)\psi_{p,q}+\eta_p^q\left[\frac{\partial}{\partial z}-\frac{2}{n}\frac{\partial n}{\partial z}\right]\psi_{p-1,q}\\
&+\beta_p^q\left[\frac{1}{n}\left(\frac{\partial n}{\partial x}-i\frac{\partial n}{\partial y}\right)-\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\right]\psi_{p-1,q-1}\\
&-\beta_p^{-q}\left[\frac{1}{n}\left(\frac{\partial n}{\partial x}+i\frac{\partial n}{\partial y}\right)-\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\right]\psi_{p-1,q+1}+\xi_p^q\left[\frac{\partial}{\partial z}-\frac{2}{n}\frac{\partial n}{\partial z}\right]\psi_{p+1,q}\\
&-\alpha_p^{-q+1}\left[\frac{1}{n}\left(\frac{\partial n}{\partial x}-i\frac{\partial n}{\partial y}\right)-\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\right]\psi_{p+1,q-1}
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha_p^{q+1}\left[\frac{1}{n}\left(\frac{\partial n}{\partial x}+i\frac{\partial n}{\partial y}\right)-\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\right]\psi_{p+1,q+1}\\
&+\frac{\sqrt{3}}{n}\left[\frac{\partial n}{\partial x}\left[\rho(1,0)I_2(1,0;p,q)\right]+\frac{\partial n}{\partial y}\left[\rho(1,0)I_5(1,0;p,q)\right]\right.\\
&\left.-\frac{\partial n}{\partial z}\left[\rho(1,0)I_8(1,0;p,q)\right]\right]\psi_{1,0}\\
&+\frac{\sqrt{3}}{n}\left[\frac{\partial n}{\partial x}\left[I_1(1,1;p,q)-iI_3(1,1;p,q)\right]+\frac{\partial n}{\partial y}\left[I_4(1,1;p,q)+iI_6(1,1;p,q)\right]\right.\\
&\left.-\frac{\partial n}{\partial z}I_7(1,1;p,q)\right]\psi_{1,1}\\
&+\frac{\sqrt{3}}{n}\left[\frac{\partial n}{\partial x}\left[I_1(1,-1;p,q)+\rho(1,-1)I_2(1,-1;p,q)+iI_3(1,-1;p,q)\right]\right.\\
&+\frac{\partial n}{\partial y}\left[I_4(1,-1;p,q)+\rho(1,-1)I_5(1,-1;p,q)-iI_6(1,-1;p,q)\right]\\
&\left.-\frac{\partial n}{\partial z}\left[I_7(1,-1;p,q)+\rho(1,-1)I_8(1,-1;p,q)\right]\right]\psi_{1,-1}\\
&=(2p+1)^{1/2}\,\epsilon_{p,q}. \qquad (58)
\end{aligned}
$$

We will use (58) to find the four equations.

3.5.1 The First Equation

The first equation is obtained by taking $p=q=0$:

$$
\begin{aligned}
&\left(\frac{n}{c}\frac{\partial}{\partial t}+\mu_t^0\right)\psi_{0,0}+\xi_0^0\left[\frac{\partial}{\partial z}-\frac{2}{n}\frac{\partial n}{\partial z}\right]\psi_{1,0}\\
&-\alpha_0^1\left[\frac{1}{n}\left(\frac{\partial n}{\partial x}-i\frac{\partial n}{\partial y}\right)-\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\right]\psi_{1,-1}\\
&+\alpha_0^1\left[\frac{1}{n}\left(\frac{\partial n}{\partial x}+i\frac{\partial n}{\partial y}\right)-\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\right]\psi_{1,1}\\
&+\frac{\sqrt{3}}{n}\left[\frac{\partial n}{\partial x}\left[\rho(1,0)I_2(1,0;0,0)\right]+\frac{\partial n}{\partial y}\left[\rho(1,0)I_5(1,0;0,0)\right]\right.\\
&\left.-\frac{\partial n}{\partial z}\left[\rho(1,0)I_8(1,0;0,0)\right]\right]\psi_{1,0}\\
&+\frac{\sqrt{3}}{n}\left[\frac{\partial n}{\partial x}\left[I_1(1,1;0,0)-iI_3(1,1;0,0)\right]+\frac{\partial n}{\partial y}\left[I_4(1,1;0,0)+iI_6(1,1;0,0)\right]\right.\\
&\left.-\frac{\partial n}{\partial z}I_7(1,1;0,0)\right]\psi_{1,1}\\
&+\frac{\sqrt{3}}{n}\left[\frac{\partial n}{\partial x}\left[I_1(1,-1;0,0)+\rho(1,-1)I_2(1,-1;0,0)+iI_3(1,-1;0,0)\right]\right.\\
&+\frac{\partial n}{\partial y}\left[I_4(1,-1;0,0)+\rho(1,-1)I_5(1,-1;0,0)-iI_6(1,-1;0,0)\right]
\end{aligned}
$$

$$- \frac{\partial n}{\partial z}\left[I_7(1,-1;0,0) + \rho(1,-1) I_8(1,-1;0,0)\right]\Big]\,\psi_{1,-1}$$
$$= \epsilon_{0,0}.$$

Notice that $\alpha_0^1 = \sqrt{2}$, $\xi_0^0 = 1$, $\rho(1,0) = -\sqrt{2}$, $I_1(1,-1;0,0) = \frac{1}{2\sqrt{6}}$ and $I_1(1,1;0,0) = -\frac{1}{2\sqrt{6}}$ by equation (101), $I_2(1,-1;0,0) = 0$ because $J_0(2,0,0) = 0$, $I_2(1,0;0,0) = 0$ because $\delta_{q-1,m} = \delta_{q+1,m} = 0$, $I_3(1,-1;0,0) = -i\frac{\sqrt{3}}{2\sqrt{2}}$ by equation (102), $I_3(1,1;0,0) = -i\frac{\sqrt{3}}{2\sqrt{2}}$ by equation (102), $I_4(1,-1;0,0) = -i\frac{1}{2\sqrt{6}}$ by equation (105), $I_4(1,1;0,0) = -i\frac{1}{2\sqrt{6}}$ by equation (105), $I_5(1,0;0,0) = 0$ because $\delta_{q-1,m} = \delta_{q+1,m} = 0$, $I_5(1,-1;0,0) = 0$ because $J_0(2,0,0) = 0$, $I_6(1,-1;0,0) = \frac{\sqrt{3}}{2\sqrt{2}}$ by equation (106), $I_6(1,1;0,0) = -\frac{\sqrt{3}}{2\sqrt{2}}$ by equation (106), $I_7(1,-1;0,0) = 0$ because $\delta_{q,m} = 0$, $I_7(1,1;0,0) = 0$ because $\delta_{q,m} = 0$, $I_8(1,0;0,0) = \sqrt{\frac{2}{3}}$ by equation (107), $I_8(1,-1;0,0) = 0$ because $\delta_{q,m} = 0$, giving us the following representation of the equation:

$$\begin{aligned}
&\left(\frac{n}{c}\frac{\partial}{\partial t} + \mu_t^0\right)\psi_{0,0} + \left(\frac{\partial}{\partial z} - \frac{2}{n}\frac{\partial n}{\partial z}\right)\psi_{1,0} \\
&-\sqrt{2}\left[\frac{1}{n}\left(\frac{\partial n}{\partial x} - i\frac{\partial n}{\partial y}\right) - \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\right]\psi_{1,-1} \\
&+\sqrt{2}\left[\frac{1}{n}\left(\frac{\partial n}{\partial x} + i\frac{\partial n}{\partial y}\right) - \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right]\psi_{1,1} \\
&-\frac{\sqrt{3}}{n}\left[\frac{\partial n}{\partial z}\left(-\frac{2}{\sqrt{3}}\right)\right]\psi_{1,0} \\
&+\frac{\sqrt{3}}{n}\left[\frac{\partial n}{\partial x}\left(-\frac{1}{2\sqrt{6}} + i^2\frac{\sqrt{3}}{2\sqrt{2}}\right) + \frac{\partial n}{\partial y}\left(-i\frac{1}{2\sqrt{6}} - i\frac{\sqrt{3}}{2\sqrt{2}}\right)\right]\psi_{1,1} \\
&+\frac{\sqrt{3}}{n}\left[\frac{\partial n}{\partial x}\left(\frac{1}{2\sqrt{6}} - i^2\frac{\sqrt{3}}{2\sqrt{2}}\right) + \frac{\partial n}{\partial y}\left(-i\frac{1}{2\sqrt{6}} - i\frac{\sqrt{3}}{2\sqrt{2}}\right)\right]\psi_{1,-1} \\
&= \epsilon_{0,0}
\end{aligned}$$

with some simplification we get

$$\begin{aligned}
&\left(\frac{n}{c}\frac{\partial}{\partial t} + \mu_t^0\right)\psi_{0,0} + \left(\frac{\partial}{\partial z} - \frac{2}{n}\frac{\partial n}{\partial z}\right)\psi_{1,0} \\
&-\sqrt{2}\left[\frac{1}{n}\left(\frac{\partial n}{\partial x} - i\frac{\partial n}{\partial y}\right) - \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\right]\psi_{1,-1} \\
&+\sqrt{2}\left[\frac{1}{n}\left(\frac{\partial n}{\partial x} + i\frac{\partial n}{\partial y}\right) - \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right]\psi_{1,1} \\
&+\frac{1}{\sqrt{2}}\frac{2}{n}\left[\frac{\partial n}{\partial x} - i\frac{\partial n}{\partial y}\right]\psi_{1,-1} - \frac{1}{\sqrt{2}}\frac{2}{n}\left[\frac{\partial n}{\partial x} + i\frac{\partial n}{\partial y}\right]\psi_{1,1} \\
&+\frac{2}{n}\frac{\partial n}{\partial z}\psi_{1,0} = \epsilon_{0,0}.
\end{aligned}$$

which further simplifies to

$$\left(\frac{n}{c}\frac{\partial}{\partial t} + \mu_t^0\right)\psi_{0,0} + \frac{\partial}{\partial z}\psi_{1,0} + \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\psi_{1,-1} - \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\psi_{1,1} = \epsilon_{0,0}. \quad (59)$$

3.5.2 The Second Equation

We will find the second equation for the P_1 by taking $p = 1$, and $q = -1$. In this case, (58) becomes

$$\begin{aligned}
&\sqrt{3}\left(\frac{n}{c}\frac{\partial}{\partial t} + \mu_t^1\right)\psi_{1,-1} - \beta_1^1\left[\frac{1}{n}\left(\frac{\partial n}{\partial x} + i\frac{\partial n}{\partial y}\right) - \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right]\psi_{0,0} \\
&+\frac{\sqrt{3}}{n}\left[\frac{\partial n}{\partial x}\left[\rho(1,0)I_2(1,0;1,-1)\right] + \frac{\partial n}{\partial y}\left[\rho(1,0)I_5(1,0;1,-1)\right]\right. \\
&\left.-\frac{\partial n}{\partial z}\left[\rho(1,0)I_8(1,0;1,-1)\right]\right]\psi_{1,0} \\
&+\frac{\sqrt{3}}{n}\left[\frac{\partial n}{\partial x}\left[I_1(1,1;1,-1) - iI_3(1,1;1,-1)\right] + \frac{\partial n}{\partial y}\left[I_4(1,1;1,-1) + iI_6(1,1;1,-1)\right]\right. \\
&\left.-\frac{\partial n}{\partial z}I_7(1,1;1,-1)\right]\psi_{1,1} \\
&+\frac{\sqrt{3}}{n}\left[\frac{\partial n}{\partial x}\left[I_1(1,-1;1,-1) + \rho(1,-1)I_2(1,-1;1,-1) + iI_3(1,-1;1,-1)\right]\right. \\
&+\frac{\partial n}{\partial y}\left[I_4(1,-1;1,-1) + \rho(1,-1)I_5(1,-1;1,-1) - iI_6(1,-1;1,-1)\right] \\
&\left.-\frac{\partial n}{\partial z}\left[I_7(1,-1;1,-1) + \rho(1,-1)I_8(1,-1;1,-1)\right]\right]\psi_{1,-1} \\
&= \sqrt{3}\,\epsilon_{1,-1}.
\end{aligned}$$

Notice that $\beta_1^1 = \sqrt{\frac{2}{3}}$, $I_1(1,-1;1,-1) = 0$ because $\delta_{q-1,m} = \delta_{q+1,m} = 0$, $I_1(1,1;1,-1) = 0$ because $\delta_{q-1,m} = \delta_{q+1,m} = 0$, $I_2(1,-1;1,-1) = 0$ because $(\ell + m + \sigma_m + 1) = J_0(2,1,1) = 0$, $I_3(1,-1;1,-1) = 0$ because $\delta_{q+1,m} = \delta_{q-1,m} = 0$, $I_3(1,1;1,-1) = 0$ because $\delta_{q+1,m} = \delta_{q-1,m} = 0$, $I_4(1,-1;1,-1) = 0$ because $\delta_{q-1,m} = \delta_{q+1,m} = 0$, $I_4(1,1;1,-1) = 0$ because $\delta_{q-1,m} = \delta_{q+1,m} = 0$, $I_5(1,-1;1,-1) = 0$ because $(\ell + m + \sigma_m + 1) = J_0(2,1,1) = 0$, $I_6(1,-1;1,-1) = 0$ because $\delta_{q+1,m} = \delta_{q-1,m} = 0$, $I_6(1,1;1,-1) = 0$ because $\delta_{q+1,m} = \delta_{q-1,m} = 0$, $I_7(1,-1;1,-1) = 0$ because $\delta_{q,m} = 0$, $I_7(1,1;1,-1) = 0$ because $\delta_{\ell+1,p} = \delta_{\ell-1,p} = 0$, $I_8(1,-1;1,-1) = 0$ because $J_2(3,1,1) = J_1(3,1,1) = 0$ giving us our final representation of the equation:

$$\left(\frac{n}{c}\frac{\partial}{\partial t} + \mu_t^1\right)\psi_{1,-1} - \frac{\sqrt{2}}{3}\left[\frac{1}{n}\left(\frac{\partial n}{\partial x} + i\frac{\partial n}{\partial y}\right) - \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right]\psi_{0,0} = \epsilon_{1,-1}. \tag{60}$$

3.5.3 The Third Equation

We will find the third equation for the P_1 by taking $p = 1$, and $q = 0$. In this case, (58) becomes

$$\sqrt{3}\left(\frac{n}{c}\frac{\partial}{\partial t} + \mu_t^1\right)\psi_{1,0} + \eta_1^0\left(\frac{\partial}{\partial z} - \frac{2}{n}\frac{\partial n}{\partial z}\right)\psi_{0,0}$$

$$
\begin{aligned}
&+\frac{\sqrt{3}}{n}\left[\frac{\partial n}{\partial x}\left[\rho(1,0)I_2(1,0;1,0)\right]+\frac{\partial n}{\partial y}\left[\rho(1,0)I_5(1,0;1,0)\right]\right.\\
&\left.-\frac{\partial n}{\partial z}\left[\rho(1,0)I_8(1,0;1,0)\right]\right]\psi_{1,0}\\
&+\frac{\sqrt{3}}{n}\left[\frac{\partial n}{\partial x}\left[I_1(1,1;1,0)-iI_3(1,1;1,0)\right]+\frac{\partial n}{\partial y}\left[I_4(1,1;1,0)+iI_6(1,1;1,0)\right]\right.\\
&\left.-\frac{\partial n}{\partial z}I_7(1,1;1,0)\right]\psi_{1,1}\\
&+\frac{\sqrt{3}}{n}\left[\frac{\partial n}{\partial x}\left[I_1(1,-1;1,0)+\rho(1,-1)I_2(1,-1;1,0)+iI_3(1,-1;1,0)\right]\right.\\
&+\frac{\partial n}{\partial y}\left[I_4(1,-1;1,0)+\rho(1,-1)I_5(1,-1;1,0)-iI_6(1,-1;1,0)\right]\\
&\left.-\frac{\partial n}{\partial z}\left[I_7(1,-1;1,0)+\rho(1,-1)I_8(1,-1;1,0)\right]\right]\psi_{1,-1}\\
&=\sqrt{3}\,\epsilon_{1,0}.
\end{aligned}
$$

Observe that $\eta_1^0=\frac{1}{\sqrt{3}}$, $I_1(1,-1;1,0)=0$ by Proposition 1, $I_1(1,1;1,0)=0$ by Proposition 1, $I_2(1,-1;1,0)=0$ because $\delta_{q+1,m}=J_0(2,1,0)=(\ell+m+\sigma_m+1)=0$, $I_3(1,-1;1,0)=0$ by Proposition 1, $I_3(1,1;1,0)=0$ by Proposition 1, $I_4(1,-1;1,0)=0$ by Proposition 1, $I_4(1,1;1,0)=0$ by Proposition 1,, $I_5(1,-1;1,0)=0$ because $\delta_{q+1,m}=J_0(2,1,0)=(\ell+m+\sigma_m+1)=0$, $I_6(1,-1;1,0)=0$ by Proposition 1, $I_6(1,1;1,0)=0$ by Proposition 1, $I_7(1,-1;1,0)=0$ because $\delta_{m,q}=0$, $I_7(1,1;1,0)=0$ because $\delta_{m,q}=0$, $I_8(1,-1;1,0)=0$ because $\delta_{q,m}=0$ giving us our final representation of the equation:

$$
\left(\frac{n}{c}\frac{\partial}{\partial t}+\mu_t^1\right)\psi_{1,0}+\frac{1}{3}\left(\frac{\partial}{\partial z}-\frac{2}{n}\frac{\partial n}{\partial z}\right)\psi_{0,0}=\epsilon_{1,0}. \tag{61}
$$

3.5.4 The Fourth Equation

We will find the fourth equation for the P_1 by taking $p=1$, and $q=1$. In this case, (58) becomes

$$
\begin{aligned}
&\sqrt{3}\left(\frac{n}{c}\frac{\partial}{\partial t}+\mu_t^1\right)\psi_{1,1}+\beta_1^1\left[\frac{1}{n}\left(\frac{\partial n}{\partial x}-i\frac{\partial n}{\partial y}\right)-\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\right]\psi_{0,0}\\
&+\frac{\sqrt{3}}{n}\left[\frac{\partial n}{\partial x}\left[\rho(1,0)I_2(1,0;1,1)\right]+\frac{\partial n}{\partial y}\left[\rho(1,0)I_5(1,0;1,1)\right]\right.\\
&\left.-\frac{\partial n}{\partial z}\left[\rho(1,0)I_8(1,0;1,1)\right]\right]\psi_{1,0}\\
&+\frac{\sqrt{3}}{n}\left[\frac{\partial n}{\partial x}\left[I_1(1,1;1,1)-iI_3(1,1;1,1)\right]+\frac{\partial n}{\partial y}\left[I_4(1,1;1,1)+iI_6(1,1;1,1)\right]\right.\\
&\left.-\frac{\partial n}{\partial z}I_7(1,1;1,1)\right]\psi_{1,1}\\
&+\frac{\sqrt{3}}{n}\left[\frac{\partial n}{\partial x}\left[I_1(1,-1;1,1)+\rho(1,-1)I_2(1,-1;1,1)+iI_3(1,-1;1,1)\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\partial n}{\partial y}\left[I_4(1,-1;1,1)+\rho(1,-1)I_5(1,-1;1,1)-iI_6(1,-1;1,1)\right] \\
&\left.-\frac{\partial n}{\partial z}\left[I_7(1,-1;1,1)+\rho(1,-1)I_8(1,-1;1,1)\right]\right]\psi_{1,-1} \\
&=\sqrt{3}\,\epsilon_{1,1}.
\end{aligned}
$$

Now observe that $\beta_1^1=\sqrt{\frac{2}{3}}$, $I_1(1,-1;1,1)=0$ because $\delta_{q+1,m}=\delta_{q-1,m}=0$, $I_1(1,1;1,1)=0$ because $\delta_{q+1,m}=\delta_{q-1,m}=0$, $I_2(1,-1;1,1)=0$ because $\delta_{q+1,m}=\delta_{q-1,m}=0$, $I_3(1,-1;1,1)=0$ because $\delta_{q+1,m}=\delta_{q-1,m}=0$, $I_3(1,1;1,1)=0$ because $\delta_{q+1,m}=\delta_{q-1,m}=0$, $I_4(1,-1;1,1)=0$ because $\delta_{q+1,m}=\delta_{q-1,m}=0$, $I_4(1,1;1,1)=0$ because $\delta_{q+1,m}=\delta_{q-1,m}=0$, $I_5(1,-1;1,1)=0$ because $\delta_{q+1,m}=\delta_{q-1,m}=0$, $I_6(1,-1;1,1)=0$ because $\delta_{q+1,m}=\delta_{q-1,m}=0$, $I_6(1,1;1,1)=0$ because $\delta_{q+1,m}=\delta_{q-1,m}=0$, $I_7(1,-1;1,1)=0$ because $\delta_{m,q}=0$, $I_7(1,1;1,1)=0$ because $\delta_{\ell+1,p}=\delta_{\ell-1,p}=0$, $I_8(1,-1;1,1)=0$ because $\delta_{q,m}=0$ giving us our final representation of the equation:

$$
\left(\frac{n}{c}\frac{\partial}{\partial t}+\mu_t^1\right)\psi_{1,1}+\frac{\sqrt{2}}{3}\left[\frac{1}{n}\left(\frac{\partial n}{\partial x}-i\frac{\partial n}{\partial y}\right)-\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\right]\psi_{0,0}=\epsilon_{1,1}. \tag{62}
$$

When the refractive index n is taken to be the constant 1, then (59)-(62) reduce to

$$
\begin{aligned}
\left(\frac{1}{c}\frac{\partial}{\partial t}+\mu_t^0\right)\psi_{0,0}+\left(\frac{\partial}{\partial z}\right)\psi_{1,0}+\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\psi_{1,-1}-\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\psi_{1,1} &= \epsilon_{0,0} \\
\left(\frac{1}{c}\frac{\partial}{\partial t}+\mu_t^1\right)\psi_{1,-1}+\frac{\sqrt{2}}{6}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\psi_{0,0} &= \epsilon_{1,-1} \\
\left(\frac{1}{c}\frac{\partial}{\partial t}+\mu_t^1\right)\psi_{1,0}+\frac{1}{3}\left(\frac{\partial}{\partial z}\right)\psi_{0,0} &= \epsilon_{1,0} \\
\left(\frac{1}{c}\frac{\partial}{\partial t}+\mu_t^1\right)\psi_{1,1}-\frac{\sqrt{2}}{6}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\psi_{0,0} &= \epsilon_{1,1}
\end{aligned}
$$

which is consistent with Arridge [3].

3.6 P_0 or Diffusion Approximation

From (5)-(6) we get after applying previously stated assumptions that

$$
\mathbf{F}(\mathbf{r},t) = \int_{S^2} L(\mathbf{r},\boldsymbol{\Omega},t)\boldsymbol{\Omega}d\boldsymbol{\Omega} = \begin{pmatrix} \frac{1}{\sqrt{2}}[\psi_{1,-1}(\mathbf{r},t)-\psi_{1,1}(\mathbf{r},t)] \\ \frac{1}{i\sqrt{2}}[\psi_{1,-1}(\mathbf{r},t)+\psi_{1,1}(\mathbf{r},t)] \\ \psi_{1,0}(\mathbf{r},t) \end{pmatrix} \tag{63}
$$

$$
\Phi(\mathbf{r},t) = \frac{n}{c}\int_{S^2} L(\mathbf{r},\boldsymbol{\Omega},t)d\boldsymbol{\Omega} = \frac{n}{c}\psi_{0,0}(\mathbf{r},t). \tag{64}
$$

We define ϵ_0 and ϵ_1 by

$$
\epsilon_0(\mathbf{r},t) = \epsilon_{0,0}(\mathbf{r},t) \tag{65}
$$

$$\epsilon_1(\mathbf{r},t) = \int_{S^2} \epsilon(\mathbf{r},\mathbf{\Omega},t)\mathbf{\Omega}d\mathbf{\Omega} = \begin{pmatrix} \frac{1}{\sqrt{2}}[\epsilon_{1,-1}(\mathbf{r},t) - \epsilon_{1,1}(\mathbf{r},t)] \\ \frac{1}{i\sqrt{2}}[\epsilon_{1,-1}(\mathbf{r},t) + \epsilon_{1,1}(\mathbf{r},t)] \\ \epsilon_{1,0}(\mathbf{r},t) \end{pmatrix}. \tag{66}$$

The P_1 approximation is given by the following four equations:

$$\left(\frac{n}{c}\frac{\partial}{\partial t} + \mu_t^0\right)\psi_{0,0} + \frac{\partial}{\partial z}\psi_{1,0} + \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\psi_{1,-1} - \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\psi_{1,1} = \epsilon_{0,0}$$

$$\left(\frac{n}{c}\frac{\partial}{\partial t} + \mu_t^1\right)\psi_{1,-1} - \frac{\sqrt{2}}{3}\left[\frac{1}{n}\left(\frac{\partial n}{\partial x} + i\frac{\partial n}{\partial y}\right) - \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right]\psi_{0,0} = \epsilon_{1,-1}$$

$$\left(\frac{n}{c}\frac{\partial}{\partial t} + \mu_t^1\right)\psi_{1,0} + \frac{1}{3}\left(\frac{\partial}{\partial z} - \frac{2}{n}\frac{\partial n}{\partial z}\right)\psi_{0,0} = \epsilon_{1,0}$$

$$\left(\frac{n}{c}\frac{\partial}{\partial t} + \mu_t^1\right)\psi_{1,1} + \frac{\sqrt{2}}{3}\left[\frac{1}{n}\left(\frac{\partial n}{\partial x} - i\frac{\partial n}{\partial y}\right) - \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\right]\psi_{0,0} = \epsilon_{1,1}.$$

Using (63)-(66) we may write the P_1 approximation in the form

$$\left(\frac{\partial}{\partial t} + \frac{c}{n}\mu_t^0\right)\Phi(\mathbf{r},t) + \nabla\cdot\mathbf{F}(\mathbf{r},t) = \epsilon_0(\mathbf{r},t) \tag{67}$$

$$\left(\frac{n}{c}\frac{\partial}{\partial t} + \mu_t^1\right)\mathbf{F}(\mathbf{r},t) + \frac{1}{3}\frac{c}{n}\left(\nabla - \frac{2}{n}\nabla n\right)\Phi(\mathbf{r},t) = \epsilon_1(\mathbf{r},t). \tag{68}$$

The diffusion or P_0 approximation is obtained by assuming that

$$\frac{\partial \mathbf{F}}{\partial t}(\mathbf{r},t) = 0 \qquad \text{and} \qquad \epsilon_1(\mathbf{r},t) = 0.$$

We refer to Arridge [3], Khan et al. [14, 13], Marti-Lopez et al. [19, 18] for a detailed discussion about the assumptions and the validity of the diffusion approximation. Under these assumptions, (68) becomes

$$\mu_t^1\mathbf{F}(\mathbf{r},t) + \frac{1}{3}\frac{c}{n}\left(\nabla - \frac{2}{n}\nabla n\right)\Phi(\mathbf{r},t) = 0.$$

Solving for $\mathbf{F}(\mathbf{r},t)$ we arrive at

$$\mathbf{F}(\mathbf{r},t) = \frac{-1}{3\mu_t^1}\frac{c}{n}\left(\nabla - \frac{2}{n}\nabla n\right)\Phi(\mathbf{r},t).$$

Substituting this into (67) yields

$$\left(\frac{\partial}{\partial t} + \frac{c}{n}\mu_t^0\right)\Phi(\mathbf{r},t) + \nabla\cdot\left[\frac{-1}{3\mu_t^1}\frac{c}{n}\left(\nabla - \frac{2}{n}\nabla n\right)\Phi(\mathbf{r},t)\right] = \epsilon_0(\mathbf{r},t).$$

Now this is the same as the diffusion approximation derived by Khan [13], Tualle et al. [27] by assuming a linear approximation of Φ with respect to $\mathbf{\Omega}$. This result is consistent with conservation of energy as pointed out by Tualle et al. [27]. However the conservation of energy may be lost if we do not assume the P_1 approximation to derive equation (67). For example, if we integrate equation (43) with respect to $\mathbf{\Omega}$ directly, see equation (54) in [13].

3.7 A General Algorithm

Define the operator

$$\Lambda = \left[\frac{1}{n}\left(\frac{\partial n}{\partial x} + i\frac{\partial n}{\partial y}\right) - \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right]$$

and denote by Λ^* its complex conjugate (not its adjoint):

$$\Lambda^* = \left[\frac{1}{n}\left(\frac{\partial n}{\partial x} - i\frac{\partial n}{\partial y}\right) - \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\right].$$

Then we have after absorbing all the terms inside the sum, combining and rearranging yields,

$$\begin{aligned}
&\sum_{\ell=0}^{N}\sum_{m=-\ell}^{\ell}\left[(2p+1)^{1/2}\,\delta_{\ell,p}\delta_{m,q}\left(\frac{n}{c}\frac{\partial}{\partial t} + \mu_t^p\right)\right.\\
&+\left(\eta_p^q\delta_{\ell,p-1} + \xi_p^q\delta_{\ell,p+1}\right)\delta_{m,q}\left[\frac{\partial}{\partial z} - \frac{2}{n}\frac{\partial n}{\partial z}\right]\\
&-\left(\alpha_p^{-q+1}\delta_{\ell,p+1} - \beta_p^q\delta_{\ell,p-1}\right)\delta_{m,q-1}\Lambda^*\\
&+\left(\alpha_p^{q+1}\delta_{\ell,p+1} - \beta_p^{-q}\delta_{\ell,p-1}\right)\delta_{m,q+1}\Lambda\\
&+(2\ell+1)^{1/2}\left[|m|I_1(\ell,m;p,q) + \rho(\ell,m)I_2(\ell,m;p,q) - (im)I_3(\ell,m;p,q)\right]\frac{1}{n}\frac{\partial n}{\partial x}\\
&+(2\ell+1)^{1/2}\left[|m|I_4(\ell,m;p,q) + \rho(\ell,m)I_5(\ell,m;p,q) + (im)I_6(\ell,m;p,q)\right]\frac{1}{n}\frac{\partial n}{\partial y}\\
&\left.-(2\ell+1)^{1/2}\left[|m|I_7(\ell,m;p,q) + \rho(\ell,m)I_8(\ell,m;p,q)\right]\frac{1}{n}\frac{\partial n}{\partial z}\right]\psi_{\ell,m}\\
&= (2p+1)^{1/2}\,\epsilon_{p,q}.
\end{aligned}$$

Let

$$\begin{aligned}
A_{\ell,m}^{p,q} &= (2p+1)^{1/2}\,\delta_{\ell,p}\delta_{m,q}\\
B_{\ell,m}^{p,q} &= \left(\eta_p^q\delta_{\ell,p-1} + \xi_p^q\delta_{\ell,p+1}\right)\delta_{m,q}\\
C_{\ell,m}^{p,q} &= -\left(\alpha_p^{-q+1}\delta_{\ell,p+1} - \beta_p^q\delta_{\ell,p-1}\right)\delta_{m,q-1}\\
D_{\ell,m}^{p,q} &= \left(\alpha_p^{q+1}\delta_{\ell,p+1} - \beta_p^{-q}\delta_{\ell,p-1}\right)\delta_{m,q+1}\\
E_{\ell,m}^{p,q} &= (2\ell+1)^{1/2}\left[|m|I_1(\ell,m;p,q) + \rho(\ell,m)I_2(\ell,m;p,q) - (im)I_3(\ell,m;p,q)\right]\\
F_{\ell,m}^{p,q} &= (2\ell+1)^{1/2}\left[|m|I_4(\ell,m;p,q) + \rho(\ell,m)I_5(\ell,m;p,q) + (im)I_6(\ell,m;p,q)\right]\\
G_{\ell,m}^{p,q} &= -(2\ell+1)^{1/2}\left[|m|I_7(\ell,m;p,q) + \rho(\ell,m)I_8(\ell,m;p,q)\right]\\
H_{\ell,m}^{p,q} &= (2p+1)^{1/2}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\sum_{\ell=0}^{N}\sum_{m=-\ell}^{\ell}\left[A_{\ell,m}^{p,q}\left(\frac{n}{c}\frac{\partial}{\partial t} + \mu_t^p\right) + B_{\ell,m}^{p,q}\left[\frac{\partial}{\partial z} - \frac{2}{n}\frac{\partial n}{\partial z}\right] + C_{\ell,m}^{p,q}\Lambda^* + D_{\ell,m}^{p,q}\Lambda + E_{\ell,m}^{p,q}\frac{1}{n}\frac{\partial n}{\partial x}\right.\\
&\left.+F_{\ell,m}^{p,q}\frac{1}{n}\frac{\partial n}{\partial y} + G_{\ell,m}^{p,q}\frac{1}{n}\frac{\partial n}{\partial z}\right]\psi_{\ell,m} = H_{\ell,m}^{p,q}\epsilon_{p,q}.
\end{aligned}$$

Furthermore, we can calculate all of the above coefficients using Matlab. We can back substitute for Λ and Λ^* to get:

$$\sum_{\ell=0}^{N}\sum_{m=-\ell}^{\ell}\left[A_{\ell,m}^{p,q}\left(\frac{n}{c}\frac{\partial}{\partial t}+\mu_t^p\right)+B_{\ell,m}^{p,q}\left[\frac{\partial}{\partial z}-\frac{2}{n}\frac{\partial n}{\partial z}\right]\right.$$
$$+C_{\ell,m}^{p,q}\left[\frac{1}{n}\left(\frac{\partial n}{\partial x}-i\frac{\partial n}{\partial y}\right)-\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\right]+D_{\ell,m}^{p,q}\left[\frac{1}{n}\left(\frac{\partial n}{\partial x}+i\frac{\partial n}{\partial y}\right)-\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\right]$$
$$\left.+E_{\ell,m}^{p,q}\frac{1}{n}\frac{\partial n}{\partial x}+F_{\ell,m}^{p,q}\frac{1}{n}\frac{\partial n}{\partial y}+G_{\ell,m}^{p,q}\frac{1}{n}\frac{\partial n}{\partial z}\right]\psi_{\ell,m}=H_{\ell,m}^{p,q}\epsilon_{p,q}.$$

We have also written a Matlab program for computing the general coefficients for the P_N approximation.

4 SPHERICALLY SYMMETRIC SOLUTIONS

In this section we will derive the transport equation for a medium with spatially varying refractive index in spherical polar coordinates. Let $L(r,\tau_r,t)$ be the spherically symmetric photon density where

$$\tau_r=\frac{\mathbf{\Omega}\cdot\mathbf{r}}{r} \;=\; \mathbf{\Omega}_x\frac{x}{r}+\mathbf{\Omega}_y\frac{y}{r}+\mathbf{\Omega}_z\frac{z}{r} \tag{69}$$

and $r=\sqrt{x^2+y^2+z^2}$.

4.1 Calculating spherically symmetric $\mathbf{\Omega}\cdot\nabla$ term

$$\begin{aligned}\mathbf{\Omega}\cdot\nabla L &= \mathbf{\Omega}_x\frac{\partial L}{\partial x}+\mathbf{\Omega}_y\frac{\partial L}{\partial y}+\mathbf{\Omega}_z\frac{\partial L}{\partial z}\\ &= \mathbf{\Omega}_x\frac{\partial L}{\partial r}\frac{\partial r}{\partial x}+\mathbf{\Omega}_y\frac{\partial L}{\partial r}\frac{\partial r}{\partial y}+\mathbf{\Omega}_z\frac{\partial L}{\partial r}\frac{\partial r}{\partial z}\\ &+ \mathbf{\Omega}_x\frac{\partial L}{\partial \tau_r}\frac{\partial r}{\partial \tau_r}+\mathbf{\Omega}_y\frac{\partial L}{\partial \tau_r}\frac{\partial \tau_r}{\partial y}+\mathbf{\Omega}_z\frac{\partial L}{\partial \tau_r}\frac{\partial \tau_r}{\partial z}.\end{aligned} \tag{70}$$

If we compute and plug in the expressions for $\partial r/\partial x$, $\partial r/\partial y$, $\partial r/\partial z$, $\partial\tau_r/\partial x$, $\partial\tau_r/\partial y$, and $\partial\tau_r/\partial z$ into (70) and simplify we get

$$\mathbf{\Omega}\cdot\nabla L \;=\; \tau_r\frac{\partial L}{\partial r}+\frac{1-\tau_r^2}{r}\frac{\partial L}{\partial \tau_r}. \tag{71}$$

4.2 Calculating $(\nabla n\cdot\nabla_{\mathbf{\Omega}})/n$ term

If we recall

$$\nabla_{\mathbf{\Omega}}L \;=\; \left(\cos\vartheta\cos\varphi\frac{\partial L}{\partial\vartheta}-\frac{\sin\varphi}{\sin\vartheta}\frac{\partial L}{\partial\varphi}\right)\hat{x}+\left(\cos\vartheta\sin\varphi\frac{\partial L}{\partial\vartheta}+\frac{\cos\varphi}{\sin\vartheta}\frac{\partial L}{\partial\varphi}\right)\hat{y}-\sin\vartheta\frac{\partial L}{\partial\vartheta}\hat{z}. \tag{72}$$

Therefore to compute $\nabla_{\mathbf{\Omega}} L$ we need to find expressions for $\partial L/\partial\vartheta$ and $\partial L/\partial\varphi$ in terms of τ_r. Using chain rule we get,

$$\frac{\partial L}{\partial\vartheta} = \frac{\partial L}{\partial\tau_r}\frac{\partial\tau_r}{\partial\vartheta}$$

and similarly for $\partial L/\partial\varphi$. Now if we use the following definitions of τ_r, $\hat{\vartheta}$, and $\hat{\varphi}$:

$$\begin{aligned}
\tau_r &= \mathbf{\Omega}_x \sin\vartheta\cos\varphi + \mathbf{\Omega}_y \sin\vartheta\cos\varphi + \mathbf{\Omega}_z \cos\vartheta \\
\hat{r} &= \sin\vartheta\cos\varphi\hat{x} + \sin\vartheta\cos\varphi\hat{y} + \cos\vartheta\hat{z} \\
\hat{\vartheta} &= \cos\vartheta\cos\varphi\hat{x} + \cos\vartheta\sin\varphi\hat{y} - \sin\vartheta\hat{z} \\
\hat{\varphi} &= -\sin\varphi\hat{x} + \cos\varphi\hat{y}
\end{aligned}$$

we get

$$\begin{aligned}
\frac{\partial\tau_r}{\partial\vartheta} &= \mathbf{\Omega}\cdot\hat{\vartheta} \\
\frac{\partial\tau_r}{\partial\varphi} &= \sin\vartheta\mathbf{\Omega}\cdot\hat{\varphi}.
\end{aligned}$$

Now if we use plug these into (72) we get

$$\nabla_{\mathbf{\Omega}} L = \frac{\partial L}{\partial\tau_r}\left[\left(\mathbf{\Omega}\cdot\hat{\vartheta}\right)\hat{\vartheta} + (\mathbf{\Omega}\cdot\varphi)\,\hat{\varphi}\right].$$

If we use the definition of ∇n and use the spherical symmetric gradient operator as in section (4.1) and use definitions of $\hat{r}$, τ_r, and $\mathbf{\Omega}$ we get

$$\nabla n = \frac{\partial n}{\partial r}\hat{r} + \frac{\partial n}{\partial\tau_r}\left[\frac{\mathbf{\Omega}}{r} - \frac{1}{r}\tau_r\cdot\hat{r}\right]$$

and after multiplying ∇n with $\nabla_{\mathbf{\Omega}} L$ and simplifying by noting that $\hat{r}$, $\hat{\vartheta}$, $\hat{\varphi}$ form a local orthogonal system we arrive at

$$\frac{1}{n}\nabla n\cdot\nabla_{\mathbf{\Omega}} L = \frac{1}{rn}\frac{\partial n}{\partial\tau_r}\frac{\partial L}{\partial\tau_r}\left[(\mathbf{\Omega}\cdot\hat{\vartheta})^2 + (\mathbf{\Omega}\cdot\hat{\varphi})^2\right]$$

and if we further simplify the terms $\mathbf{\Omega}\cdot\hat{\vartheta}$ and $\mathbf{\Omega}\cdot\hat{\varphi}$, then

$$(\mathbf{\Omega}\cdot\vartheta)^2 + (\mathbf{\Omega}\cdot\varphi)^2 = 1 - \tau_r^2$$

and we get the following simplified equation

$$\frac{1}{n}\nabla n\cdot\nabla_{\mathbf{\Omega}} L = \frac{1}{rn}\frac{\partial n}{\partial\tau_r}\frac{\partial L}{\partial\tau_r}(1 - \tau_r^2).$$

4.3 Calculating spherically symmetric $-(2\boldsymbol{\Omega}\cdot\nabla n)/n$ term

Using the same arguments as in the calculation of $\boldsymbol{\Omega}\cdot\nabla$ term in section (4.1) we arrive at

$$\boldsymbol{\Omega}\cdot\nabla n = \tau_r\frac{\partial n}{\partial r}+\frac{1-\tau_r^2}{r}\frac{\partial n}{\partial \tau_r}. \tag{73}$$

Therefore we get,

$$-\frac{2}{n}\left(\boldsymbol{\Omega}\cdot\nabla n\right)L = -\frac{2\tau_r}{n}\frac{\partial n}{\partial r}L+-\frac{2\left(1-\tau_r^2\right)}{rn}\frac{\partial n}{\partial \tau_r}L. \tag{74}$$

4.4 Spherically symmetric transport equation

If we combine the results of sections 4.1 to 4.3, we arrive at the following spherically symmetric transport equation in terms of $L(r,\tau_r,t)$ for a spatially varying refractive index:

$$\begin{aligned}\left(\frac{n}{c}\frac{\partial}{\partial t}+\mu_{tr}\right)L+\tau_r\frac{\partial L}{\partial r}+\frac{1-\tau_r^2}{r}\frac{\partial L}{\partial \tau_r}+\frac{1}{n}\frac{1-\tau_r^2}{r}\frac{\partial n}{\partial \tau_r}\frac{\partial L}{\partial \tau_r}-\frac{2\tau_r}{n}\frac{\partial n}{\partial r}L\\ -\ \frac{2(1-\tau_r^2)}{r}\frac{1}{n}\frac{\partial n}{\partial \tau_r}L=\mu_s\int_{S^2}f(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')L(r,\tau_r',t)d\boldsymbol{\Omega}'+\epsilon(r,\tau_r,t).\end{aligned} \tag{75}$$

4.5 Spherically symmetric P_N approximation

We will now consider the simplified case where the refractive index $n(r)$ is independent of τ_r and t. This simplification leads to the following spherically symmetric transport equation of which we will find the spherical harmonics or P_N approximation:

$$\begin{aligned}\left(\frac{n}{c}\frac{\partial}{\partial t}+\mu_{tr}\right)L &+ \tau_r\frac{\partial L}{\partial r}+\frac{1-\tau_r^2}{r}\frac{\partial L}{\partial \tau_r}-\frac{2\tau_r}{n}\frac{\partial n}{\partial r}L\\ &= \mu_s\int_{S^2}f(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')L(r,\tau_r',t)d\boldsymbol{\Omega}'+\epsilon(r,\tau_r,t).\end{aligned} \tag{76}$$

Let

$$\begin{aligned}L(r,\tau_r,t) &= \sum_{\ell=0}^{\infty}\frac{2\ell+1}{4\pi}\psi_\ell(r,t)P_\ell(\tau_r)\\ \epsilon(r,\tau_r,t) &= \sum_{\ell=0}^{\infty}\frac{2\ell+1}{4\pi}\epsilon_\ell(r,t)P_\ell(\tau_r)\end{aligned}$$

and substitute L and ϵ into equation (76) as in the previous section. Then if we multiply by $P_m(\tau_r)$ and integrating over S^2 using the orthogonality and integral identities (see Appendix) we arrive at the following infinite system of partial differential equations

$$\begin{aligned}(2\ell+1)\left(\frac{n}{c}\frac{\partial}{\partial t}+\mu_a+(1-f_\ell)\mu_s\right)\psi_\ell &+ (\ell+1)\left(\frac{\partial}{\partial r}+\frac{\ell+2}{r}-\frac{2}{n}\frac{\partial n}{\partial r}\right)\psi_{\ell+1}\\ &+ \ell\left(\frac{\partial}{\partial r}+\frac{\ell-1}{r}-\frac{2}{n}\frac{\partial n}{\partial r}\right)\psi_{\ell-1}=(2\ell+1)\epsilon_\ell\end{aligned} \tag{77}$$

and if we make use the transformation $\psi_\ell = n^2\phi_\ell$, then the equation (77) becomes,

$$\begin{aligned}(2\ell+1)\left(\frac{n}{c}\frac{\partial}{\partial t}+\mu_a+(1-f_\ell)\mu_s\right)\phi_\ell \quad &+ \quad (\ell+1)\left(\frac{\partial}{\partial r}+\frac{\ell+2}{r}\right)\phi_{\ell+1} \\ &+ \quad \ell\left(\frac{\partial}{\partial r}+\frac{\ell-1}{r}\right)\phi_{\ell-1} = (2\ell+1)\epsilon_\ell \qquad (78)\end{aligned}$$

which is the same as the infinite set of partial differential equations corresponding to the spherically symmetric RTE with constant refractive index [3, 6]. If we convert equation (78) to the frequency domain, then the following choice of eigenfunction is desirable [6, 3]

$$\phi_\ell(r,\omega) \quad = \quad H_\ell(\lambda)k_\ell(-\lambda r)$$

where

$$k_\ell(y) = \sqrt{\frac{\pi}{2y}}K_{\ell+\frac{1}{2}}(y)$$

is the modified spherical Bessel function of the second kind. The eigenvalues $\lambda(\omega)$ are the roots of the secular equation

$$\begin{vmatrix} \beta_0(\omega) & \lambda & 0 & 0 & 0 & \dots & \dots & 0 \\ \lambda & 3\beta_1(\omega) & 2\lambda & 0 & 0 & \dots & \dots & 0 \\ 0 & 2\lambda & 5\beta_2(\omega) & 3\lambda & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \dots & \vdots \\ 0 & \dots & \dots & \dots & \dots & (N-1)\lambda & (2N-1)\beta_{N-1}(\omega) & N\lambda \\ 0 & \dots & \dots & \dots & \dots & 0 & N\lambda & (2N+1)\beta_N(\omega) \end{vmatrix} = 0 \qquad (79)$$

where

$$\beta_\ell(\omega) = \mu_a + (1-\Theta_\ell)\mu_s + \frac{in\omega}{c}$$

and the eigenfunctions $H_\ell(\lambda)$ satisfies the recurrence relations [6, 3]:

$$(2\ell+1)\beta_\ell(\omega)H_\ell(\lambda) + \lambda(\ell+1)H_{\ell+1}(\lambda) + \lambda\ell H_{\ell-1}(\lambda) \quad = \quad 0. \qquad (80)$$

The general solution to the system of PDEs (77) can be expressed as

$$\psi_\ell(r,\omega) = \sum_j a_j n^2(r)\lambda_j H_\ell(\lambda_j)k_\ell(-\lambda_j r) \qquad (81)$$

where our solution ψ_ℓ for the spatially varying case differs from spatially constant case ϕ_ℓ [6, 3] in that the eigenfunctions are changed by a multiple of the spatially varying refractive index $n^2(r)$ and the eigenvalues depend on the refractive index n since β_ℓ is a function of n.

In particular if we assume $H_0(\lambda) = 1$, $H_1(\lambda) = -\beta_1(\omega)/\lambda$ etc, and if we take $k_0(y) = \exp(-y)/y$, then we get the following approximate solutions for the flux:

$$\Phi(r,\omega) \quad = \quad \psi_\ell(r,\omega) = \sum_j a_j n^2(r)\frac{e^{-\lambda_j(\omega)r}}{r}. \qquad (82)$$

In the next section we will compute the eigenvalues for a few simple cases for comparison of the spatially varying versus spatially constant refractive index model.

5 COMPARISON OF CONSTANT VERSUS SPATIALLY VARYING REFRACTIVE INDEX MODEL

We computed the eigenvalues for the spatially varying case with the following parameters: the absorption coefficient $\mu_a = 0.025mm^{-1}$, the scattering coefficient $\mu_s = 20mm^{-1}$, and the frequency of the wave $\omega = 1GHz$. In Figures 1 and 2, we plot the real and imaginary parts of the eigenvalues respectively for a medium with refractive index $n = 1.4$. We compared the $n = 1.4$ case with the $n = 1.0$, the constant refractive index case, calculated by Arridge [3] by keeping all the other parameters the same. We find that the imaginary part of the eigenvalues change significantly which makes sense as changing refractive index will effectively change the imaginary part of the β_ℓ term in equation (80). The most notable change in the real part of the eigenvalues occurs in the lowest eigenvalue which has the most significant effect on the solution as it is the dominant term in the eigenfunction expansion (82).

In Figure 3, we plot the real part of the the lowest eigenvalues for the refractive indices $n = 1.4(\Diamond)$, $n = 1.2(*)$, and $n = 1.0(\circ)$ for comparison. It is evident from Figure 3 that the first eigenvalues corresponding to the refractive indices $n = 1.4$, $n = 1.2$, and $n = 1.0$ are substantially different. In Figure 4, we also plot the imaginary part of the eigenvalues for comparison. We observe from Figure 4 that the eigenvalues for higher refractive index tend to spread more than the constant refractive index case.

Furthermore, we recall that the eigenfunctions for spatially varying refractive index are different than those for the constant refractive index by a factor of $n^2(r)$, see equation (82). Therefore the eigenfunction expansion solution of the non-constant refractive index case is different from the constant refractive index case.

6 CONCLUSIONS

In this report, we derived the radiative transport equation and its P_N approximation for a medium with a spatially varying refractive index. We found the analytical solution of the coupled system of partial differential equations corresponding to the P_N approximation in a spherically symmetric geometry. We computed the eigenfunctions and eigenvalues of the system. We showed that the P_N model with spatially varying refractive index for photon transport is substantially different than the spatially constant model. Therefore the new model with spatially varying refractive index may be used for potential biomedical imaging applications giving us more insight into the existing optical imaging research.

APPENDIX: SPHERICAL HARMONICS

A The Associated Legendre Functions

Recall that the associated Legendre functions are defined in terms of the Legendre polynomials by

$$P_n^m(x) = (-1)^m(1 - x^2)^{m/2}\frac{\partial^m P_n(x)}{\partial x^m} \tag{83}$$

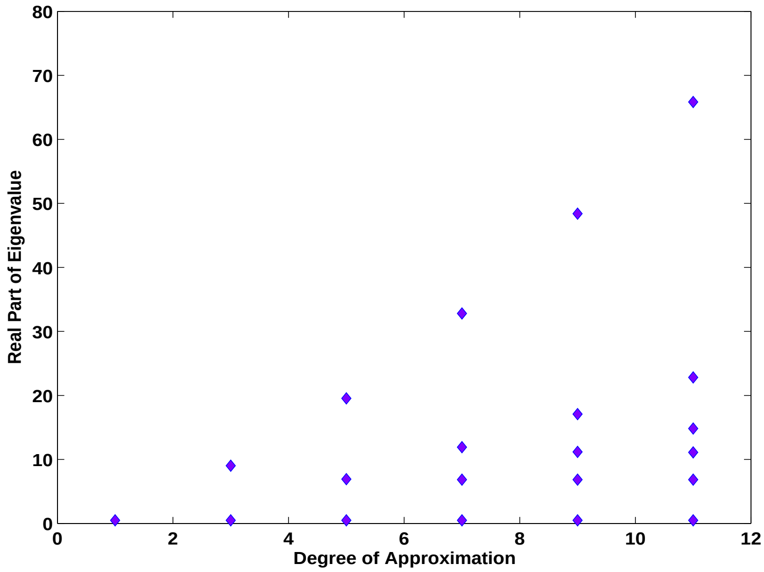

Figure 1: Real parts of eigenvalues when refractive index is $n = 1.4$ with absorption coefficient $\mu_a = 0.025mm^{-1}$, scattering coefficient $\mu_s = 20mm^{-1}$, and the frequency of the wave $\omega = 1GHz$.

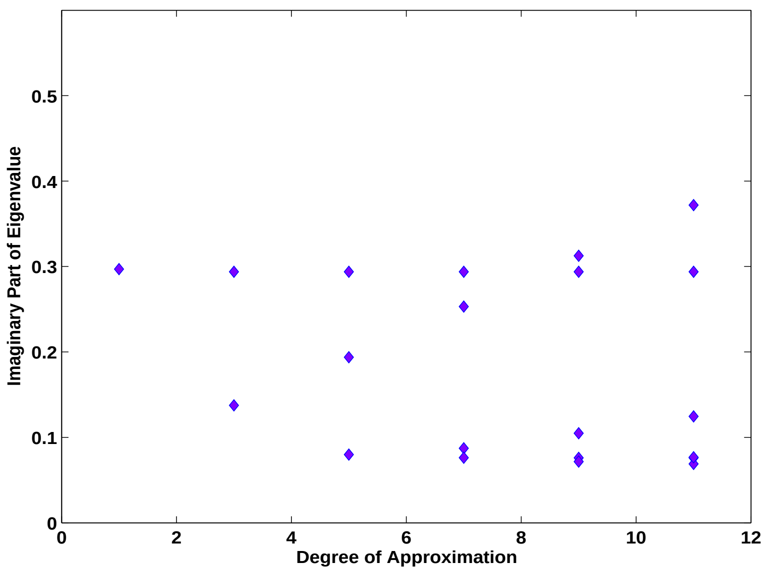

Figure 2: Imaginary parts of eigenvalues when refractive index is $n = 1.4$ with absorption coefficient $\mu_a = 0.025mm^{-1}$, the scattering coefficient $\mu_s = 20mm^{-1}$, and the frequency of the wave $\omega = 1GHz$.

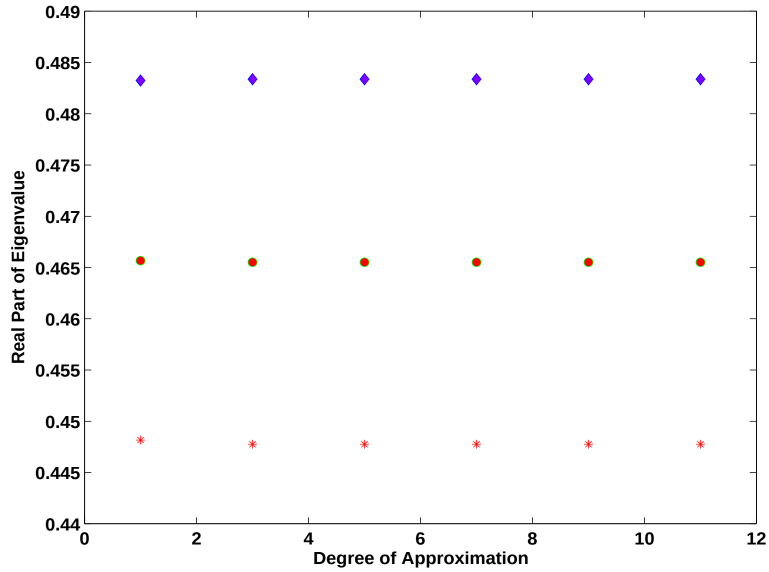

Figure 3: Real parts of the lowest eigenvalues when refractive index is $n = 1.4(\diamondsuit)$, $n = 1.2(\circ)$, and $n = 1.0(*)$ with absorption coefficient $\mu_a = 0.025mm^{-1}$, the scattering coefficient $\mu_s = 20mm^{-1}$, and the frequency of the wave $\omega = 1GHz$.

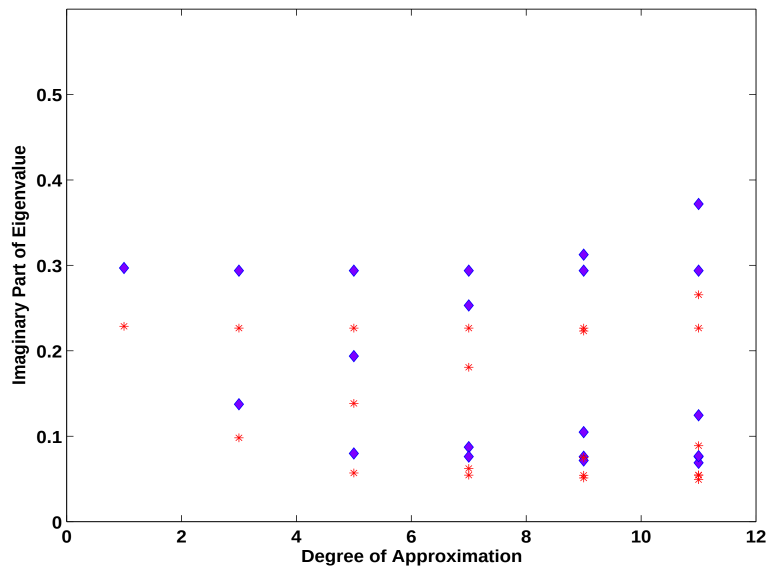

Figure 4: Imaginary parts of eigenvalues when refractive index is $n = 1.4(\diamondsuit)$ and $n = 1.0(*)$ with absorption coefficient $\mu_a = 0.025mm^{-1}$, the scattering coefficient $\mu_s = 20mm^{-1}$, the frequency of the wave $\omega = 1GHz$.

where P_n is the Legendre polynomial of degree $n,m \in \mathbb{N}_0$, $m \leq n$. So we get that

$$\begin{aligned}
\frac{\partial}{\partial \vartheta} P_n^m(\cos\vartheta) &= \frac{\partial}{\partial \vartheta}(-1)^m (1-\cos^2\vartheta)^{m/2} \frac{\partial^m P_n(\cos\vartheta)}{\partial x^m} \\
&= \frac{\partial}{\partial \vartheta}(-1)^m sin^m \vartheta \frac{\partial^m P_n(\cos\vartheta)}{\partial x^m} \\
&= (-1)^m \Big[m \sin^{m-1}\vartheta \cos\vartheta \frac{\partial^m P_n(\cos\vartheta)}{\partial x^m} \\
&\qquad + \sin^m \vartheta \frac{\partial^{m+1} P_n(\cos\vartheta)}{\partial x^{m+1}} (-\sin\vartheta) \Big] \\
&= m \frac{\cos\vartheta}{\sin\vartheta} (-1)^m (1-\cos^2\vartheta)^{m/2} \frac{\partial^m P_n(\cos\vartheta)}{\partial x^m} \\
&\qquad + (-1)^{m+1} (1-\cos^2\vartheta)^{m+1/2} \frac{\partial^{m+1} P_n(\cos\vartheta)}{\partial x^{m+1}}
\end{aligned}$$

Again employing (83) we arrive at

$$\frac{\partial}{\partial \vartheta} P_n^m(\cos\vartheta) = m(\cot\vartheta) P_n^m(\cos\vartheta) + P_n^{m+1}(\cos\vartheta). \tag{84}$$

B Spherical Harmonic Functions

Now consider the spherical harmonic functions which may be represented in terms of the associated Legendre functions by

$$Y_{n,m}(\vartheta, \varphi) = \left(\frac{(2n+1)}{4\pi} \frac{(n-|m|)!}{(n+|m|)!} \right)^{1/2} (-1)^{\frac{1}{2}(m-|m|)} P_n^{|m|}(\cos\vartheta) e^{im\varphi} \tag{85}$$

where $n \in \mathbb{N}_0$, and $m \in \mathbb{Z}$ with $-n \leq m \leq n$.

The spherical harmonic functions satisfy the orthogonality relation [3]

$$\int_{S^2} Y_{n,m}(\mathbf{\Omega}) Y^*_{\ell,k}(\mathbf{\Omega}) d\mathbf{\Omega} = \int_0^{2\pi} \int_0^{\pi} Y_{n,m}(\vartheta,\varphi) Y^*_{\ell,k}(\vartheta,\varphi) \sin\vartheta d\vartheta d\varphi \quad = \quad \delta_{n,\ell}\, \delta_{m,k} \tag{86}$$

and the addition theorem is given by [3]

$$P_\ell(\mathbf{\Omega}' \cdot \mathbf{\Omega}) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y^*_{\ell,m}(\mathbf{\Omega}') Y_{l,m}(\mathbf{\Omega}). \tag{87}$$

B.1 Differentiating $Y_{n,m}(\vartheta, \varphi)$

We use (85) to calculate $\frac{\partial}{\partial \vartheta} Y_{n,m}(\vartheta, \varphi)$:

$$\frac{\partial}{\partial \vartheta} \left(\frac{(2n+1)}{4\pi} \frac{(n-|m|)!}{(n+|m|)!} \right)^{1/2} (-1)^{\frac{1}{2}(m+|m|)} P_n^{|m|}(\cos\vartheta) e^{im\varphi}$$

$$
\begin{aligned}
&= \left(\frac{(2n+1)}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}\right)^{1/2}(-1)^{\frac{1}{2}(m+|m|)}\left(\frac{\partial}{\partial\vartheta}P_n^{|m|}(\cos\vartheta)\right)e^{im\varphi} \\
&= \left(\frac{(2n+1)}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}\right)^{1/2}(-1)^{\frac{1}{2}(m+|m|)}\left[|m|(\cot\vartheta)P_n^{|m|}(\cos\vartheta)+P_n^{|m|+1}(\cos\vartheta)\right]e^{im\varphi}
\end{aligned}
$$

where the last line is obtained from employing (84).

$$
\begin{aligned}
&= |m|(\cot\vartheta)\left(\frac{(2n+1)}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}\right)^{1/2}(-1)^{\frac{1}{2}(m+|m|)}P_n^{|m|}(\cos\vartheta)e^{im\varphi} \\
&\quad+\left(\frac{(2n+1)}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}\right)^{1/2}(-1)^{\frac{1}{2}(m+|m|)}P_n^{|m+\sigma_m|}(\cos\vartheta)e^{im\varphi}
\end{aligned}
\tag{88}
$$

where

$$
\sigma_m = \begin{cases} 1 & m \geq 0 \\ -1 & m < 0. \end{cases}
$$

Notice that

$$
\begin{aligned}
(-1)^{\frac{1}{2}(m+|m|)}(-\sigma_m) &= (-1)^{\frac{1}{2}(m+|m|)}(-1)^{\frac{1}{2}(\sigma_m+1)} \\
&= (-1)^{\frac{1}{2}(m+\sigma_m+|m|+1)} \\
&= (-1)^{\frac{1}{2}(m+\sigma_m+|m+\sigma_m|)}.
\end{aligned}
$$

This enables us to write (88) as

$$
\begin{aligned}
&\frac{\partial}{\partial\vartheta}Y_{n,m}(\vartheta,\varphi) = \\
&\quad |m|(\cot\vartheta)\left(\frac{(2n+1)}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}\right)^{1/2}(-1)^{\frac{1}{2}(m+|m|)}P_n^{|m|}(\cos\vartheta)e^{im\varphi} \\
&\quad +(-\sigma_m)\left(\frac{(2n+1)}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}\right)^{1/2}(-1)^{\frac{1}{2}(m+\sigma_m+|m+\sigma_m|)}P_n^{|m+\sigma_m|}(\cos\vartheta)e^{im\varphi}.
\end{aligned}
$$

So (85) gives us

$$
\begin{aligned}
\frac{\partial}{\partial\vartheta}Y_{n,m}(\vartheta,\varphi) &= |m|(\cot\vartheta)Y_{n,m}(\vartheta,\varphi) \\
&\quad +(-\sigma_m)\left[(n-|m|)(n+|m|+1|)\right]^{1/2}e^{-i\sigma_m\varphi}Y_{n,m+\sigma_m}(\vartheta,\varphi)
\end{aligned}
$$

which we will write simply as

$$
\frac{\partial}{\partial\vartheta}Y_{n,m}(\vartheta,\varphi) = |m|(\cot\vartheta)Y_{n,m}(\vartheta,\varphi)+\rho(n,m)e^{-i\sigma_m\varphi}Y_{n,m+\sigma_m}(\vartheta,\varphi) \tag{89}
$$

where

$$
\rho(n,m) := (-\sigma_m)\left[(n-|m|)(n+|m|+1|)\right]^{1/2}. \tag{90}
$$

Now observe that

$$\begin{aligned}\frac{\partial}{\partial\varphi}Y_{n,m}(\vartheta,\varphi) &= \frac{\partial}{\partial\varphi}\left(\frac{(2n+1)}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}\right)^{1/2}(-1)^{\frac{1}{2}(m+|m|)}P_n^{|m|}(\cos\vartheta)e^{im\varphi}\\ &= \left(\frac{(2n+1)}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}\right)^{1/2}(-1)^{\frac{1}{2}(m+|m|)}P_n^{|m|}(\cos\vartheta)\frac{\partial}{\partial\varphi}e^{im\varphi}\\ &= \left(\frac{(2n+1)}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}\right)^{1/2}(-1)^{\frac{1}{2}(m+|m|)}P_n^{|m|}(\cos\vartheta)ime^{im\varphi}.\end{aligned}$$

So we arrive at

$$\frac{\partial}{\partial\varphi}Y_{n,m}(\vartheta,\varphi) = imY_{n,m}(\vartheta,\varphi). \tag{91}$$

We have from [3] the following recurrence relations for the spherical harmonic functions:

$$\cos\vartheta Y_{n,m} = \left(\frac{(n+m)(n-m)}{(2n+1)(2n-1)}\right)^{1/2}Y_{n-1,m} + \left(\frac{(n+m+1)(n-m+1)}{(2n+1)(2n+3)}\right)^{1/2}Y_{n+1,m} \tag{92}$$

$$\sin\vartheta e^{i\varphi}Y_{n,m} = \left(\frac{(n-m)(n-m-1)}{(2n+1)(2n-1)}\right)^{1/2}Y_{n-1,m+1} - \left(\frac{(n+m+1)(n+m+2)}{(2n+1)(2n+3)}\right)^{1/2}Y_{n+1,m+1} \tag{93}$$

$$\sin\vartheta e^{-i\varphi}Y_{n,m} = -\left(\frac{(n+m)(n+m-1)}{(2n+1)(2n-1)}\right)^{1/2}Y_{n-1,m-1} + \left(\frac{(n-m+1)(n-m+2)}{(2n+1)(2n+3)}\right)^{1/2}Y_{n+1,m-1}. \tag{94}$$

C Type I integrals

The following proposition is a special case of other much more general results and may make a nice example. It will be useful to us later to be able to compute the integral

$$\int_{-1}^{1}\frac{P_n^m(x)P_l^k(x)}{\sqrt{1-x^2}}\,dx \tag{95}$$

for $n, m, \ell, k \in \mathbb{N}$, such that $m < n$ and $k < \ell$.

Proposition 1 *Let $n, m, \ell, k \in \mathbb{N}$, $m < n$, $k < l$ be such that $(n+\ell-m-k)$ is odd. Then*

$$\int_{-1}^{1}\frac{P_n^m(x)P_l^k(x)}{\sqrt{1-x^2}}\,dx = 0$$

Proof: We use a simple symmetry argument. Suppose $(n+\ell-m-k)$ is odd. Then $(n-m)+(\ell-k)$ is odd, and hence exactly one of $(n-m)$ and $(\ell-k)$ is odd. Without loss of generality we will assume that $(n-m)$ is odd and $(\ell-k)$ is even. Then $P_n^m(x)$ is an odd function and $P_\ell^k(x)$ is an even function. Observing that $\frac{1}{\sqrt{1-x^2}}$ is an even function, we conclude that the function

$$f(x) = \frac{P_n^m(x)P_l^k(x)}{\sqrt{1-x^2}}$$

is odd. Thus by symmetry integrating $f(x)$ over the interval $(-1, 1)$ yields zero as a result. □

In particular, we would like to be able to compute (95) when k and m differ by 1. When this occurs, Proposition 1 requires that $(n + \ell)$ be odd for (95) to be nonzero. We may then write ℓ as $\ell = n + 2j + 1$ for some $j \in \mathbb{Z}$. So we now consider the integrals

$$\int_{-1}^{1} \frac{P_n^m(x)P_{n+2j+1}^{m+1}(x)}{\sqrt{1-x^2}}\, dx$$

and

$$\int_{-1}^{1} \frac{P_n^{k+1}(x)P_{n+2j+1}^{k}(x)}{\sqrt{1-x^2}}\, dx$$

The following proposition is taken from [25]:

Proposition 2 *Let $n, m, k \in \mathbb{N}_0$ and $j \in \mathbb{Z}$. Then*

$$\int_{-1}^{1} \frac{P_n^m(x)P_{n+2j+1}^{m+1}(x)}{\sqrt{1-x^2}}\, dx \;=\; \begin{cases} 0 & \text{if } j < 0 \\ -\frac{2(n+m)!}{(n-m)!} & \text{if } j \geq 0 \end{cases}$$

and

$$\int_{-1}^{1} \frac{P_n^{k+1}(x)P_{n+2j+1}^{k}(x)}{\sqrt{1-x^2}}\, dx \;=\; \begin{cases} -\frac{2(n+k+2j+1)!}{(n-k+2j+1)!} & \text{if } j < 0 \\ 0 & \text{if } j \geq 0. \end{cases}$$

D Type II integrals

It will also be useful to us later to be able to compute a second type of integral of the form

$$\int_{-1}^{1} \frac{P_n^m(x)P_\ell^k(x)}{x}\, dx$$

for $n, m, l, k \in \mathbb{N}_0$, such that $m \leq n$ and $k \leq l$.

Proposition 3 *Let $n, m, \ell, k \in \mathbb{N}_0$ be not all zero, $m \leq n$, $k \leq \ell$ be such that $(n+\ell-m-k)$ is even. Then*

$$\int_{-1}^{1} \frac{P_n^m(x)P_\ell^k(x)}{x}\, dx \;=\; 0$$

in the Cauchy principle value sense.

Proof: Again we use a simple symmetry argument similar to that of Proposition 1. Suppose $(n+\ell-m-k)$ is even. Then $(n-m)+(\ell-k)$ is even, and hence $(n-m)$ and $(\ell-k)$ are either both even or both odd. Then $P_n^m(x)$ and $P_\ell^k(x)$ are either both even or both odd functions. Observing that $\frac{1}{\sqrt{x}}$ is an odd function, we conclude that the function

$$f(x) = \frac{P_n^m(x)P_\ell^k(x)}{x}$$

is odd. Thus by symmetry integrating $f(x)$ over the interval $(-1,1)$ yields zero as a result. $\square$

The following proposition gives a closed expression for most cases of this integral in terms of the parameters n, m, ℓ, and k.

Proposition 4 *Let $n, m \in \mathbb{N}$, $\ell, k \in \mathbb{N}_0$ be such that $k \leq \ell$, and $0 < n-m$ is odd. Also let $N_1 = \lfloor \frac{n-m}{2} \rfloor$, $N_2 = \lfloor \frac{\ell-k}{2} \rfloor$. Then we have the following representation:*

$$\int_{-1}^{1} \frac{P_n^m(x)P_\ell^k(x)}{x}\,dx \;=\; \sum_{q=0}^{N_1}\left[\frac{(2n-4q-1)}{(-1)^q}\left(\prod_{s=0}^{2q}\left[n+(-1)^{s+1}m-s\right]^{(-1)^{s+1}}\right)J_{n-2q-1}(m,l,k)\right] \tag{96}$$

where

$$J_n(m,l,k) \;:=\; \begin{cases} \sum_{p_1=0}^{N_1}\sum_{p_2=0}^{N_2} C_{n,m}^{p_1} C_{\ell,k}^{p_2} \frac{\Gamma\left(\frac{n+\ell-m-k-2p_1-2p_2+1}{2}\right)\Gamma\left(\frac{m+k+2p_1+2p_2+2}{2}\right)}{\Gamma\left(\frac{n+\ell+3}{2}\right)} & \text{if n+l-m-k is even} \\ 0 & \text{if n+l-m-k is odd} \end{cases}$$

and

$$C_{n,m}^{p} := \frac{(-1)^p(n+m)!}{2^{m+2p}(m+p)!\,p!\,(n-m-2p)!}.$$

Proof: Throughout this proof we will make use of the identity

$$J_n \;=\; \int_{-1}^{1} P_n^m(x)P_\ell^k(x)\,dx$$

taken from [29].

We proceed by induction on n. Notice that $n, m \in \mathbb{N}$ and $0 < n-m$ together imply that $n \geq 2$. So we will first verify the case $n = 2$ in which case we must have $m = 1$. A manipulation of the recurrence relation in [1] gives us

$$P_n^m(x) = \frac{(2n-1)}{(n-m)}xP_{n-1}^m(x) - \frac{(n+m-1)}{(n-m)}P_{n-2}^m$$

Multiplying by $\frac{P_\ell^k(x)}{x}$ and integrating gives the relation

$$\int_{-1}^{1} \frac{P_n^m(x)P_\ell^k(x)}{x}\,dx \;=\; \frac{(2n-1)}{(n-m)}\int_{-1}^{1} P_{n-1}^m(x)P_\ell^k(x)\,dx - \frac{(n+m-1)}{(n-m)}\int_{-1}^{1} \frac{P_{n-2}^m(x)P_\ell^k(x)}{x}\,dx \tag{97}$$

Making the substitutions $n=2$ and $m=1$ we get:

$$\begin{aligned}\int_{-1}^{1} \frac{P_2^1(x)P_\ell^k(x)}{x}\,dx &= \frac{3}{(2-1)}\int_{-1}^{1} P_1^1(x)P_\ell^k(x)\,dx - \frac{(1+1)}{(2-1)}\int_{-1}^{1} \frac{P_0^1(x)P_\ell^k(x)}{x}\,dx \\ &= 3J_1 - 2\int_{-1}^{1} \frac{P_0^1(x)P_\ell^k(x)}{x}\,dx\end{aligned}$$

Taking note that $P_0^1(x)=0$, we find that the left-hand side of (96) is

$$\int_{-1}^{1} \frac{P_2^1(x)P_\ell^k(x)}{x}\,dx \;=\; 3J_1$$

Now we check the right-hand side, noticing that $N_1=0$:

$$\begin{aligned}&\sum_{q=0}^{0}\left[\frac{(4-4q-1)}{(-1)^q}\left(\prod_{s=0}^{2q}[2+(-1)^{s+1}-s]^{(-1)^{s+1}}\right)J_{2-2q-1}\right] \\ &\qquad = 3\left(\prod_{s=0}^{0}[2+(-1)^{s+1}-s]^{(-1)^{s+1}}\right)J_1 = 3J_1\end{aligned}$$

So our hypothesis holds in the case $n=2$.

Now we verify the case when $n=3$. We will have to consider only the possibility that $m=2$ as when $m=1,3$ we have that $n-m$ is even. Making the substitutions $n=3$ and $m=2$ into (97) yields:

$$\begin{aligned}\int_{-1}^{1} \frac{P_3^2(x)P_\ell^k(x)}{x}\,dx &= \frac{(6-1)}{(3-2)}\int_{-1}^{1} P_{3-1}^2(x)P_\ell^k(x)\,dx - \frac{(3+2-1)}{(3-2)}\int_{-1}^{1} \frac{P_{3-2}^2(x)P_\ell^k(x)}{x}\,dx \\ &= 5\int_{-1}^{1} P_2^2(x)P_\ell^k(x)\,dx - 4\int_{-1}^{1} \frac{P_1^2(x)P_\ell^k(x)}{x}\,dx \\ &= 5J_2 + 0 \;=\; 5J_2\end{aligned}$$

Again we check the right-hand side, noticing that $N_1=0$:

$$\begin{aligned}&\sum_{q=0}^{0}\left[\frac{(6-4q-1)}{(-1)^q}\left(\prod_{s=0}^{2q}\left[3+(-2)^{s+1}-s\right]^{(-1)^{s+1}}\right)J_{3-2q-1}\right] \\ &\qquad = 5\left(\prod_{s=0}^{0}\left[3+(-2)^{s+1}-s\right]^{(-1)^{s+1}}\right)J_2 \;=\; 5J_2\end{aligned}$$

So our hypothesis holds in the case $n = 3$.

Suppose that our hypothesis holds for the n^{th} case and consider the $n+2$ case. Then again using (97) we get

$$\int_{-1}^{1} \frac{P_{n+2}^m(x)P_\ell^k(x)}{x}\,dx = \frac{(2n+3)}{(n-m+2)}\int_{-1}^{1} P_{n+1}^m(x)P_\ell^k(x)\,dx - \frac{(n+m+1)}{(n-m+2)}\int_{-1}^{1} \frac{P_n^m(x)P_\ell^k(x)}{x}\,dx.$$

Recognizing the first integral as J_{n+1} and the second as the n^{th} case, we get

$$\int_{-1}^{1} \frac{P_{n+1}^m(x)P_\ell^k(x)}{x}\,dx =$$

$$\frac{(2n+3)}{(n-m+2)}J_n - \frac{(n+m+1)}{(n-m+2)}\sum_{q=0}^{N_1}\left[\frac{(2n-4q-1)}{(-1)^q}\left(\prod_{s=0}^{2q}\left[n+(-m)^{s+1}-s\right]^{(-1)^{s+1}}\right)J_{n-2q-1}\right] =$$

$$\frac{(2n+3)}{(n-m+2)}J_n + \sum_{q=0}^{\lfloor (n-m)/2\rfloor}\left[\frac{(2n-4q-1)}{(-1)^{q+1}}\left(\prod_{s=2}^{2(q+1)}\left[n+(-m)^{s+1}+2-s\right]^{(-1)^{s+1}}\right)J_{n-2q-1}\right] =$$

$$\sum_{q=0}^{\lfloor (n+2-m)/2\rfloor}\left[\frac{(2(n+2)-4q-1)}{(-1)^q}\left(\prod_{s=0}^{2q}\left[n+(-m)^{s+1}+2-s\right]^{(-1)^{s+1}}\right)J_{n-2q-1}\right].$$

Thus our hypothesis holds in the $n+2$ case. Then by the first principle of mathematical induction, our hypothesis is proven for all $n \geq 2$. □

E Integrals to evaluate

Here we will compute the integrals associated with the derivation of the P_N or spherical harmonics expansion. The following eight integrals are used to derive our main result in this report but note that most of these are just straightforward computation and the two most complicated integrals were proved in the previous sections as proposition.

E.1 Calculating I_1

Consider the integral

$$I_1 = \int_0^{2\pi}\int_0^{\pi}(\cos^2\vartheta\cos\varphi)Y_{\ell,m}(\vartheta,\varphi)Y_{p,q}^*(\vartheta,\varphi)d\vartheta d\varphi. \tag{98}$$

By employing (92) we obtain

$$I_1 = \int_0^{2\pi}\int_0^{\pi}(\cos\vartheta\cos\varphi)\left[\left(\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}\right)^{1/2}Y_{\ell-1,m} + \left(\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2}Y_{\ell+1,m}\right]Y_{p,q}^*d\vartheta d\varphi,$$

$$
\begin{aligned}
I_1 &= \left(\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}\right)^{1/2} \int_0^{2\pi}\int_0^{\pi} (\cos\vartheta\cos\varphi) Y_{\ell-1,m} Y^*_{p,q} d\vartheta d\varphi \\
&\quad + \left(\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2} \int_0^{2\pi}\int_0^{\pi} (\cos\vartheta\cos\varphi) Y_{\ell+1,m} Y^*_{p,q} d\vartheta d\varphi.
\end{aligned}
$$

Again we use (92):

$$
\begin{aligned}
I_1 &= \left(\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}\right)^{1/2} \int_0^{2\pi}\int_0^{\pi} \cos\varphi \left[\left(\frac{(\ell+m-1)(\ell-m-1)}{(2\ell-3)(2\ell-1)}\right)^{1/2} Y_{\ell-2,m}\right. \\
&\quad \left. + \left(\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}\right)^{1/2} Y_{\ell,m}\right] Y^*_{p,q} d\vartheta d\varphi \\
&\quad + \left(\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2} \int_0^{2\pi}\int_0^{\pi} \cos\varphi \left[\left(\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2} Y_{\ell,m}\right. \\
&\quad \left. + \left(\frac{(\ell+m+2)(\ell-m+2)}{(2\ell+5)(2\ell+3)}\right)^{1/2} Y_{\ell+2,m}\right] Y^*_{p,q} d\vartheta d\varphi,
\end{aligned}
$$

$$
\begin{aligned}
I_1 &= \left(\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}\right)^{1/2} \left(\frac{(\ell+m-1)(\ell-m-1)}{(2\ell-3)(2\ell-1)}\right)^{1/2} \int_0^{2\pi}\int_0^{\pi} \cos\varphi Y_{\ell-2,m} Y^*_{p,q} d\vartheta d\varphi \\
&\quad + \left(\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)} + \frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right) \int_0^{2\pi}\int_0^{\pi} \cos\varphi Y_{\ell,m} Y^*_{p,q} d\vartheta d\varphi \\
&\quad + \left(\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2} \left(\frac{(\ell+m+2)(\ell-m+2)}{(2\ell+5)(2\ell+3)}\right)^{1/2} \int_0^{2\pi}\int_0^{\pi} \cos\varphi Y_{\ell+2,m} Y^*_{p,q} d\vartheta d\varphi.
\end{aligned}
$$

Let $I_1^i(\ell,m)$ denote the i^{th} integral in this expression of I_1 and take note that

$$
\begin{aligned}
I_1^1(\ell,m) &= I_1^2(\ell-2,m) \\
I_1^3(\ell,m) &= I_1^2(\ell+2,m).
\end{aligned}
$$

We will find I_1^2 and use it to compute the remaining integrals:

$$
I_1^2 = \int_0^{2\pi}\int_0^{\pi} \cos\varphi Y_{\ell,m} Y^*_{p,q} d\vartheta d\varphi. \tag{99}
$$

We expand cosine in terms of exponential functions

$$
\begin{aligned}
I_1^2 &= \int_0^{2\pi}\int_0^{\pi} \left(\frac{e^{i\varphi}+e^{-i\varphi}}{2}\right) Y_{\ell,m} Y^*_{p,q} d\vartheta d\varphi \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi} e^{i\varphi} Y_{\ell,m} Y^*_{p,q} d\vartheta d\varphi + \frac{1}{2}\int_0^{2\pi}\int_0^{\pi} e^{-i\varphi} Y_{\ell,m} Y^*_{p,q} d\vartheta d\varphi.
\end{aligned}
$$

We now employ (85) to obtain

$$
\begin{aligned}
I_1^2 &= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi} e^{i\varphi} [\nu_\ell^m P_\ell^{|m|}(\cos\vartheta) e^{im\varphi}][\nu_p^q P_p^{|q|}(\cos\vartheta) e^{-iq\varphi}] d\vartheta d\varphi \\
&\quad + \frac{1}{2}\int_0^{2\pi}\int_0^{\pi} e^{-i\varphi} [\nu_\ell^m P_\ell^{|m|}(\cos\vartheta) e^{im\varphi}][\nu_p^q P_p^{|q|}(\cos\vartheta) e^{-iq\varphi}] d\vartheta d\varphi
\end{aligned}
$$

where

$$\nu_\ell^m := \left(\frac{(2\ell+1)}{4\pi} \frac{(\ell-|m|)!}{(\ell+|m|)!} \right)^{1/2} (-1)^{\frac{1}{2}(m-|m|)}, \tag{100}$$

$$\begin{aligned} I_1^2 &= \frac{\nu_\ell^m \nu_p^q}{2} \int_0^{2\pi} \int_0^{\pi} P_\ell^{|m|}(\cos\vartheta) P_p^{|q|}(\cos\vartheta) e^{i(m-q+1)\varphi} d\vartheta d\varphi \\ &\quad + \frac{\nu_\ell^m \nu_p^q}{2} \int_0^{2\pi} \int_0^{\pi} P_\ell^{|m|}(\cos\vartheta) P_p^{|q|}(\cos\vartheta) e^{i(m-q-1)\varphi} d\vartheta d\varphi. \end{aligned}$$

We employ Fubini's Theorem to seperate the integrals and arrive at

$$\begin{aligned} I_1^2 &= \frac{\nu_\ell^m \nu_p^q}{2} \left(\int_0^{\pi} P_\ell^{|m|}(\cos\vartheta) P_p^{|q|}(\cos\vartheta) d\vartheta \right) \left(\int_0^{2\pi} e^{i(m-q+1)\varphi} d\varphi \right) \\ &\quad + \frac{\nu_\ell^m \nu_p^q}{2} \left(\int_0^{\pi} P_\ell^{|m|}(\cos\vartheta) P_p^{|q|}(\cos\vartheta) d\vartheta \right) \left(\int_0^{2\pi} e^{i(m-q-1)\varphi} d\varphi \right). \end{aligned}$$

Recalling the orthogonality of the exponential functions, we write

$$\begin{aligned} I_1^2 &= \frac{\nu_\ell^m \nu_p^q}{2} \left(\int_0^{\pi} P_\ell^{|m|}(\cos\vartheta) P_p^{|q|}(\cos\vartheta) d\vartheta \right) 2\pi \delta_{\{q-1,m\}} \\ &\quad + \frac{\nu_\ell^m \nu_p^q}{2} \left(\int_0^{\pi} P_\ell^{|m|}(\cos\vartheta) P_p^{|q|}(\cos\vartheta) d\vartheta \right) 2\pi \delta_{\{q+1,m\}} \\ &= \pi \nu_\ell^m \nu_p^q \left(\delta_{\{q-1,m\}} + \delta_{\{q+1,m\}} \right) \left(\int_0^{\pi} P_\ell^{|m|}(\cos\vartheta) P_p^{|q|}(\cos\vartheta) d\vartheta \right). \end{aligned}$$

Making the change of variable $x = \cos\vartheta$ we obtain

$$I_1^2 = \pi \nu_\ell^m \nu_p^q \left(\delta_{\{q-1,m\}} + \delta_{\{q+1,m\}} \right) \left(\int_{-1}^{1} \frac{P_\ell^{|m|}(x) P_p^{|q|}(x)}{\sqrt{1-x^2}} dx \right).$$

This remaining integral may be computed using Propositions 1 and 2. Therefore we get,

$$\begin{aligned} I_1 &= \left(\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)} \right)^{1/2} \left(\frac{(\ell+m-1)(\ell-m-1)}{(2\ell-3)(2\ell-1)} \right)^{1/2} I_1^2(\ell-2,m) \\ &\quad + \left(\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)} + \frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)} \right) I_1^2(\ell,m) \\ &\quad + \left(\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)} \right)^{1/2} \left(\frac{(\ell+m+2)(\ell-m+2)}{(2\ell+5)(2\ell+3)} \right)^{1/2} I_1^2(\ell+2,m). \end{aligned} \tag{101}$$

E.2 Calculating I_2

Now consider the integral

$$I_2(\ell,m) = \int_0^{2\pi} \int_0^{\pi} (\cos\vartheta \cos\varphi \sin\vartheta) e^{-i\sigma_m \varphi} Y_{\ell,m+\sigma_m} Y_{p,q}^* d\vartheta d\varphi.$$

We will again make use of (92) arriving at:

$$\begin{aligned}
I_2 &= \int_0^{2\pi}\int_0^{\pi}(\cos\varphi\sin\vartheta)e^{-i\sigma_m\varphi}\left[\left(\frac{(\ell+m+\sigma_m)(\ell-m-\sigma_m)}{(2\ell+1)(2\ell-1)}\right)^{1/2}Y_{\ell-1,m+\sigma_m}\right. \\
&\left.+\left(\frac{(\ell+m+\sigma_m+1)(\ell-m-\sigma_m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2}Y_{\ell+1,m+\sigma_m}\right]Y^*_{p,q}d\vartheta d\varphi,
\end{aligned}$$

$$\begin{aligned}
I_2 &= \left(\frac{(\ell+m+\sigma_m)(\ell-m-\sigma_m)}{(2\ell+1)(2\ell-1)}\right)^{1/2}\int_0^{2\pi}\int_0^{\pi}(\cos\varphi\sin\vartheta)e^{-i\sigma_m\varphi}Y_{\ell-1,m+\sigma_m}Y^*_{p,q}d\vartheta d\varphi \\
&+\left(\frac{(\ell+m+\sigma_m+1)(\ell-m-\sigma_m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2}\int_0^{2\pi}\int_0^{\pi}(\cos\varphi\sin\vartheta)e^{-i\sigma_m\varphi}Y_{\ell+1,m+\sigma_m}Y^*_{p,q}d\vartheta d\varphi.
\end{aligned}$$

Expanding cosine as before we find that

$$\begin{aligned}
I_2 &= \left(\frac{(\ell+m+\sigma_m)(\ell-m-\sigma_m)}{(2\ell+1)(2\ell-1)}\right)^{1/2}\int_0^{2\pi}\int_0^{\pi}\left(\frac{e^{i\varphi}+e^{-i\varphi}}{2}\right)\sin\vartheta e^{-i\sigma_m\varphi}Y_{\ell-1,m+\sigma_m}Y^*_{p,q}d\vartheta d\varphi \\
&+ \left(\frac{(\ell+m+\sigma_m+1)(\ell-m-\sigma_m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2}\int_0^{2\pi}\int_0^{\pi}\left(\frac{e^{i\varphi}+e^{-i\varphi}}{2}\right)\sin\vartheta e^{-i\sigma_m\varphi}Y_{\ell+1,m+\sigma_m}Y^*_{p,q}d\vartheta d\varphi,
\end{aligned}$$

$$\begin{aligned}
I_2 &= \left(\frac{(\ell+m+\sigma_m)(\ell-m-\sigma_m)}{4(2\ell+1)(2\ell-1)}\right)^{1/2}\int_0^{2\pi}\int_0^{\pi}\sin\vartheta e^{i(1-\sigma_m)\varphi}Y_{\ell-1,m+\sigma_m}Y^*_{p,q}d\vartheta d\varphi \\
&+\left(\frac{(\ell+m+\sigma_m)(\ell-m-\sigma_m)}{4(2\ell+1)(2\ell-1)}\right)^{1/2}\int_0^{2\pi}\int_0^{\pi}\sin\vartheta e^{-i(1+\sigma_m)\varphi}Y_{\ell-1,m+\sigma_m}Y^*_{p,q}d\vartheta d\varphi \\
&+\left(\frac{(\ell+m+\sigma_m+1)(\ell-m-\sigma_m+1)}{4(2\ell+1)(2\ell+3)}\right)^{1/2}\int_0^{2\pi}\int_0^{\pi}\sin\vartheta e^{i(1-\sigma_m)\varphi}Y_{\ell+1,m+\sigma_m}Y^*_{p,q}d\vartheta d\varphi \\
&+\left(\frac{(\ell+m+\sigma_m+1)(\ell-m-\sigma_m+1)}{4(2\ell+1)(2\ell+3)}\right)^{1/2}\int_0^{2\pi}\int_0^{\pi}\sin\vartheta e^{-i(1+\sigma_m)\varphi}Y_{\ell+1,m+\sigma_m}Y^*_{p,q}d\vartheta d\varphi.
\end{aligned}$$

Now we use the Fubini's theorem to separate the θ and the φ integral and using equation 97 for the θ variable, we get

$$\begin{aligned}
I_2 &= \pi\nu_{l-1}^{m+\sigma_m}\nu_p^q\left(\frac{(\ell+m+\sigma_m)(\ell-m-\sigma_m)}{(2\ell+1)(2\ell-1)}\right)^{1/2}\delta_{q-1,m}J_{l-1}(|m+\sigma_m|,p,|q|) \\
&+\pi\nu_{l-1}^{m+\sigma_m}\nu_p^q\left(\frac{(\ell+m+\sigma_m)(\ell-m-\sigma_m)}{(2\ell+1)(2\ell-1)}\right)^{1/2}\delta_{q+1,m}J_{l-1}(|m+\sigma_m|,p,|q|) \\
&+\pi\nu_{l+1}^{m+\sigma_m}\nu_p^q\left(\frac{(\ell+m+\sigma_m+1)(\ell-m-\sigma_m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2}\delta_{q-1,m}J_{l+1}(|m+\sigma_m|,p,|q|) \\
&+\pi\nu_{l+1}^{m+\sigma_m}\nu_p^q\left(\frac{(\ell+m+\sigma_m+1)(\ell-m-\sigma_m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2}\delta_{q+1,m}J_{l+1}(|m+\sigma_m|,p,|q|).
\end{aligned}$$

E.3 Calculating I_3

Now consider the integral

$$
\begin{aligned}
I_3(\ell,m) &= \int_0^{2\pi}\int_0^{\pi} \sin\varphi Y_{\ell,m}Y_{p,q}^{*} d\vartheta d\varphi \\
&= \int_0^{2\pi}\int_0^{\pi} \left(\frac{e^{i\varphi}-e^{-i\varphi}}{2i}\right) Y_{\ell,m}Y_{p,q}^{*} d\vartheta d\varphi \\
&= \frac{1}{2i}\int_0^{2\pi}\int_0^{\pi} e^{i\varphi}Y_{\ell,m}Y_{p,q}^{*} d\vartheta d\varphi - \frac{1}{2i}\int_0^{2\pi}\int_0^{\pi} e^{-i\varphi}Y_{\ell,m}Y_{p,q}^{*} d\vartheta d\varphi \\
&= i\pi\nu_\ell^m \nu_p^q \left(\delta_{q+1,m}-\delta_{q-1,m}\right)\left(\int_{-1}^{1}\frac{P_\ell^{|m|}(x)P_p^{|q|}(x)}{\sqrt{1-x^2}}dx\right) \qquad (102)
\end{aligned}
$$

This remaining integral may be computed using Propositions 1 and 2.

E.4 Calculating I_4

Now consider the integral

$$
I_4 = \int_0^{2\pi}\int_0^{\pi} (\cos^2\vartheta\sin\varphi) Y_{\ell,m}(\vartheta,\varphi)Y_{p,q}^{*}(\vartheta,\varphi) d\vartheta d\varphi. \qquad (103)
$$

By employing (92) we obtain

$$
\begin{aligned}
I_4 &= \int_0^{2\pi}\int_0^{\pi} (\cos\vartheta\sin\varphi)\left[\left(\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}\right)^{1/2} Y_{\ell-1,m}\right. \\
&\qquad \left. + \left(\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2} Y_{\ell+1,m}\right] Y_{p,q}^{*} d\vartheta d\varphi,
\end{aligned}
$$

$$
\begin{aligned}
I_4 &= \left(\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}\right)^{1/2}\int_0^{2\pi}\int_0^{\pi}(\cos\vartheta\sin\varphi)Y_{\ell-1,m}Y_{p,q}^{*} d\vartheta d\varphi \\
&\quad + \left(\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2}\int_0^{2\pi}\int_0^{\pi}(\cos\vartheta\sin\varphi)Y_{\ell+1,m}Y_{p,q}^{*} d\vartheta d\varphi.
\end{aligned}
$$

Again we use (92):

$$
\begin{aligned}
I_4 &= \left(\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}\right)^{1/2}\int_0^{2\pi}\int_0^{\pi}\sin\varphi\left[\left(\frac{(\ell+m-1)(\ell-m-1)}{(2\ell-3)(2\ell-1)}\right)^{1/2} Y_{\ell-2,m}\right. \\
&\quad \left. + \left(\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}\right)^{1/2} Y_{\ell,m}\right] Y_{p,q}^{*} d\vartheta d\varphi \\
&\quad + \left(\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2}\int_0^{2\pi}\int_0^{\pi}\sin\varphi\left[\left(\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2} Y_{\ell,m}\right. \\
&\quad \left. + \left(\frac{(\ell+m+2)(\ell-m+2)}{(2\ell+5)(2\ell+3)}\right)^{1/2} Y_{\ell+2,m}\right] Y_{p,q}^{*} d\vartheta d\varphi,
\end{aligned}
$$

$$
\begin{aligned}
I_4 &= \left(\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}\right)^{1/2}\left(\frac{(\ell+m-1)(\ell-m-1)}{(2\ell-3)(2\ell-1)}\right)^{1/2}\int_0^{2\pi}\int_0^{\pi}\sin\varphi Y_{\ell-2,m}Y^*_{p,q}d\vartheta d\varphi \\
&+\left(\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}+\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right)\int_0^{2\pi}\int_0^{\pi}\sin\varphi Y_{\ell,m}Y^*_{p,q}d\vartheta d\varphi \\
&+\left(\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2}\left(\frac{(\ell+m+2)(\ell-m+2)}{(2\ell+5)(2\ell+3)}\right)^{1/2}\int_0^{2\pi}\int_0^{\pi}\sin\varphi Y_{\ell+2,m}Y^*_{p,q}d\vartheta d\varphi.
\end{aligned}
$$

Let $I_4^i(\ell,m)$ denote the i^{th} integral in this expression of I_4 and take note that

$$
\begin{aligned}
I_4^1(\ell,m) &= I_4^2(\ell-2,m) \\
I_4^3(\ell,m) &= I_4^2(\ell+2,m).
\end{aligned}
$$

We will find I_4^2 and use it to compute the remaining integrals:

$$
I_4^2 = \int_0^{2\pi}\int_0^{\pi}\sin\varphi Y_{\ell,m}Y^*_{p,q}d\vartheta d\varphi. \tag{104}
$$

We expand sine in terms of exponential functions

$$
\begin{aligned}
I_4^2 &= \int_0^{2\pi}\int_0^{\pi}\left(\frac{e^{i\varphi}-e^{-i\varphi}}{2i}\right)Y_{\ell,m}Y^*_{p,q}d\vartheta d\varphi \\
&= \frac{1}{2i}\int_0^{2\pi}\int_0^{\pi}e^{i\varphi}Y_{\ell,m}Y^*_{p,q}d\vartheta d\varphi-\frac{1}{2i}\int_0^{2\pi}\int_0^{\pi}e^{-i\varphi}Y_{\ell,m}Y^*_{p,q}d\vartheta d\varphi.
\end{aligned}
$$

We now employ (85) to obtain

$$
\begin{aligned}
I_4^2 &= \frac{1}{2i}\int_0^{2\pi}\int_0^{\pi}e^{i\varphi}[\nu_\ell^m P_\ell^{|m|}(\cos\vartheta)e^{im\varphi}][\nu_p^q P_p^{|q|}(\cos\vartheta)e^{-iq\varphi}]d\vartheta d\varphi \\
&-\frac{1}{2i}\int_0^{2\pi}\int_0^{\pi}e^{-i\varphi}[\nu_\ell^m P_\ell^{|m|}(\cos\vartheta)e^{im\varphi}][\nu_p^q P_p^{|q|}(\cos\vartheta)e^{-iq\varphi}]d\vartheta d\varphi
\end{aligned}
$$

$$
\begin{aligned}
I_4^2 &= \frac{\nu_\ell^m\nu_p^q}{2i}\int_0^{2\pi}\int_0^{\pi}P_\ell^{|m|}(\cos\vartheta)P_p^{|q|}(\cos\vartheta)e^{i(m-q+1)\varphi}d\vartheta d\varphi \\
&-\frac{\nu_\ell^m\nu_p^q}{2i}\int_0^{2\pi}\int_0^{\pi}P_\ell^{|m|}(\cos\vartheta)P_p^{|q|}(\cos\vartheta)e^{i(m-q-1)\varphi}d\vartheta d\varphi.
\end{aligned}
$$

We employ Fubini's Theorem to seperate the integrals and arrive at

$$
\begin{aligned}
I_4^2 &= \frac{\nu_\ell^m\nu_p^q}{2i}\left(\int_0^{\pi}P_\ell^{|m|}(\cos\vartheta)P_p^{|q|}(\cos\vartheta)d\vartheta\right)\left(\int_0^{2\pi}e^{i(m-q+1)\varphi}d\varphi\right) \\
&-\frac{\nu_\ell^m\nu_p^q}{2i}\left(\int_0^{\pi}P_\ell^{|m|}(\cos\vartheta)P_p^{|q|}(\cos\vartheta)d\vartheta\right)\left(\int_0^{2\pi}e^{i(m-q-1)\varphi}d\varphi\right).
\end{aligned}
$$

Recalling the orthogonality of the exponential functions, we write

$$\begin{aligned} I_4^2 &= \frac{\nu_\ell^m \nu_p^q}{2i}\left(\int_0^\pi P_\ell^{|m|}(\cos\vartheta)P_p^{|q|}(\cos\vartheta)d\vartheta\right)2\pi\delta_{\{q-1,m\}} \\ &\quad -\frac{\nu_\ell^m \nu_p^q}{2i}\left(\int_0^\pi P_\ell^{|m|}(\cos\vartheta)P_p^{|q|}(\cos\vartheta)d\vartheta\right)2\pi\delta_{\{q+1,m\}} \\ &= i\pi\nu_\ell^m\nu_p^q\left(\delta_{\{q+1,m\}}-\delta_{\{q-1,m\}}\right)\left(\int_0^\pi P_\ell^{|m|}(\cos\vartheta)P_p^{|q|}(\cos\vartheta)d\vartheta\right). \end{aligned}$$

Making the change of variable $x = \cos\vartheta$ we obtain

$$I_4^2 = i\pi\nu_\ell^m\nu_p^q\left(\delta_{\{q+1,m\}}-\delta_{\{q-1,m\}}\right)\left(\int_{-1}^1 \frac{P_\ell^{|m|}(x)P_p^{|q|}(x)}{\sqrt{1-x^2}}dx\right)$$

This remaining integral may be computed using Propositions 1 and 2. Therefore we get,

$$\begin{aligned} I_4 &= \left(\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}\right)^{1/2}\left(\frac{(\ell+m-1)(\ell-m-1)}{(2\ell-3)(2\ell-1)}\right)^{1/2} I_4^2(\ell-2,m) \\ &\quad+\left(\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}+\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right)I_4^2(\ell,m) \\ &\quad+\left(\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2}\left(\frac{(\ell+m+2)(\ell-m+2)}{(2\ell+5)(2\ell+3)}\right)^{1/2} I_4^2(\ell+2,m). \end{aligned} \tag{105}$$

E.5 Calculating I_5

Now consider the integral Now consider the integral Now consider the integral

$$I_5(\ell,m) = \int_0^{2\pi}\int_0^\pi (\cos\vartheta\sin\varphi\sin\vartheta)e^{-i\sigma_m\phi}Y_{\ell,m+\sigma_m}Y_{p,q}^* d\vartheta d\varphi.$$

We will again make use of (92) arriving at:

$$\begin{aligned} I_5 &= \int_0^{2\pi}\int_0^\pi (\sin\varphi\sin\vartheta)e^{-i\sigma_m\phi}\left[\left(\frac{(\ell+m+\sigma_m)(\ell-m-\sigma_m)}{(2\ell+1)(2\ell-1)}\right)^{1/2}Y_{\ell-1,m+\sigma_m}\right. \\ &\quad\left.+\left(\frac{(\ell+m+\sigma_m+1)(\ell-m-\sigma_m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2}Y_{\ell+1,m+\sigma_m}\right]Y_{p,q}^* d\vartheta d\varphi, \end{aligned}$$

$$\begin{aligned} I_5 &= \left(\frac{(\ell+m+\sigma_m)(\ell-m-\sigma_m)}{(2\ell+1)(2\ell-1)}\right)^{1/2}\int_0^{2\pi}\int_0^\pi(\sin\varphi\sin\vartheta)e^{-i\sigma_m\phi}Y_{\ell-1,m+\sigma_m}Y_{p,q}^* d\vartheta d\varphi \\ &\quad+\left(\frac{(\ell+m+\sigma_m+1)(\ell-m-\sigma_m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2}\int_0^{2\pi}\int_0^\pi(\sin\varphi\sin\vartheta)e^{-i\sigma_m\phi}Y_{\ell+1,m+\sigma_m}Y_{p,q}^* d\vartheta d\varphi. \end{aligned}$$

Expanding cosine as before we find that

$$
\begin{aligned}
I_5 &= \left(\frac{(\ell+m+\sigma_m)(\ell-m-\sigma_m)}{(2\ell+1)(2\ell-1)}\right)^{1/2}\int_0^{2\pi}\int_0^{\pi}\left(\frac{e^{i\varphi}-e^{-i\varphi}}{2i}\right)\sin\vartheta e^{-i\sigma_m\phi}Y_{\ell-1,m+\sigma_m}Y^*_{p,q}d\vartheta d\varphi \\
&+ \left(\frac{(\ell+m+\sigma_m+1)(\ell-m-\sigma_m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2}\int_0^{2\pi}\int_0^{\pi}\left(\frac{e^{i\varphi}-e^{-i\varphi}}{2i}\right)\sin\vartheta e^{-i\sigma_m\phi}Y_{\ell+1,m+\sigma_m}Y^*_{p,q}d\vartheta d\varphi,
\end{aligned}
$$

$$
\begin{aligned}
I_5 &= -i\left(\frac{(\ell+m+\sigma_m)(\ell-m-\sigma_m)}{4(2\ell+1)(2\ell-1)}\right)^{1/2}\int_0^{2\pi}\int_0^{\pi}\sin\vartheta e^{i(1-\sigma_m)\varphi}Y_{\ell-1,m+\sigma_m}Y^*_{p,q}d\vartheta d\varphi \\
&\quad +i\left(\frac{(\ell+m+\sigma_m)(\ell-m-\sigma_m)}{4(2\ell+1)(2\ell-1)}\right)^{1/2}\int_0^{2\pi}\int_0^{\pi}\sin\vartheta e^{-i(1+\sigma_m)\varphi}Y_{\ell-1,m+\sigma_m}Y^*_{p,q}d\vartheta d\varphi \\
&\quad -i\left(\frac{(\ell+m+\sigma_m+1)(\ell-m-\sigma_m+1)}{4(2\ell+1)(2\ell+3)}\right)^{1/2}\int_0^{2\pi}\int_0^{\pi}\sin\vartheta e^{i(1-\sigma_m)\varphi}Y_{\ell+1,m+\sigma_m}Y^*_{p,q}d\vartheta d\varphi \\
&\quad +i\left(\frac{(\ell+m+\sigma_m+1)(\ell-m-\sigma_m+1)}{4(2\ell+1)(2\ell+3)}\right)^{1/2}\int_0^{2\pi}\int_0^{\pi}\sin\vartheta e^{-i(1+\sigma_m)\varphi}Y_{\ell+1,m+\sigma_m}Y^*_{p,q}d\vartheta d\varphi.
\end{aligned}
$$

Now we use the Fubini's theorem to separate the θ and the φ integral and using equation 97 for the θ variable, we get

$$
\begin{aligned}
I_5 &= -i\pi\nu^{m+\sigma_m}_{l-1}\nu^q_p\left(\frac{(\ell+m+\sigma_m)(\ell-m-\sigma_m)}{(2\ell+1)(2\ell-1)}\right)^{1/2}\delta_{q-1,m}J_{l-1}(|m+\sigma_m|,p,|q|) \\
&\quad +i\pi\nu^{m+\sigma_m}_{l-1}\nu^q_p\left(\frac{(\ell+m+\sigma_m)(\ell-m-\sigma_m)}{(2\ell+1)(2\ell-1)}\right)^{1/2}\delta_{q+1,m}J_{l-1}(|m+\sigma_m|,p,|q|) \\
&\quad -i\pi\nu^{m+\sigma_m}_{l+1}\nu^q_p\left(\frac{(\ell+m+\sigma_m+1)(\ell-m-\sigma_m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2}\delta_{q-1,m}J_{l+1}(|m+\sigma_m|,p,|q|) \\
&\quad +i\pi\nu^{m+\sigma_m}_{l+1}\nu^q_p\left(\frac{(\ell+m+\sigma_m+1)(\ell-m-\sigma_m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2}\delta_{q+1,m}J_{l+1}(|m+\sigma_m|,p,|q|).
\end{aligned}
$$

E.6 Calculating I_6

Consider

$$
\begin{aligned}
I_6(\ell,m) &= \int_0^{2\pi}\int_0^{\pi}\cos\varphi Y_{\ell,m}Y^*_{p,q}d\vartheta d\varphi \\
&= \int_0^{2\pi}\int_0^{\pi}\left(\frac{e^{i\phi}+e^{-i\phi}}{2}\right)Y_{\ell,m}Y^*_{p,q}d\vartheta d\varphi \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}e^{i\phi}Y_{\ell,m}Y^*_{p,q}d\vartheta d\varphi+\frac{1}{2}\int_0^{2\pi}\int_0^{\pi}e^{-i\phi}Y_{\ell,m}Y^*_{p,q}d\vartheta d\varphi \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}e^{i\phi}[\nu^m_\ell P^{|m|}_\ell(\cos\vartheta)e^{im\varphi}][\nu^q_p P^{|q|}_p(\cos\vartheta)e^{-iq\varphi}]d\vartheta d\varphi
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{2}\int_0^{2\pi}\int_0^{\pi} e^{-i\phi}[\nu_\ell^m P_\ell^{|m|}(\cos\vartheta)e^{im\varphi}][\nu_p^q P_p^{|q|}(\cos\vartheta)e^{-iq\varphi}]d\vartheta d\varphi \\
= & \frac{\nu_\ell^m\nu_p^q}{2}\left(\int_0^{\pi} P_\ell^{|m|}(\cos\vartheta)P_p^{|q|}(\cos\vartheta)d\vartheta\right)\left(\int_0^{2\pi} e^{i(m-q+1)\varphi}d\varphi\right) \\
& +\frac{\nu_\ell^m\nu_p^q}{2}\left(\int_0^{\pi} P_\ell^{|m|}(\cos\vartheta)P_p^{|q|}(\cos\vartheta)d\vartheta\right)\left(\int_0^{2\pi} e^{i(m-q-1)\varphi}d\varphi\right) \\
= & \pi\nu_\ell^m\nu_p^q\left(\delta_{m,q-1}+\delta_{m,q+1}\right)\left(\int_0^{\pi} P_\ell^{|m|}(\cos\vartheta)P_p^{|q|}(\cos\vartheta)d\vartheta\right) \\
= & \pi\nu_\ell^m\nu_p^q\left(\delta_{m,q+1}+\delta_{m,q-1}\right)\left(\int_{-1}^{1}\frac{P_\ell^{|m|}(x)P_p^{|q|}(x)}{\sqrt{1-x^2}}dx.\right) \qquad (106)
\end{aligned}
$$

This remaining integral may be computed using Propositions 1 and 2.

E.7 Calculating I_7

Consider

$$
\begin{aligned}
I_7(\ell,m) &= \int_0^{2\pi}\int_0^{\pi}(\cot\vartheta\sin^2\vartheta)Y_{\ell,m}Y_{p,q}^*d\vartheta d\varphi \\
&= \int_0^{2\pi}\int_0^{\pi}(\cos\vartheta\sin\vartheta)Y_{\ell,m}Y_{p,q}^*d\vartheta d\varphi.
\end{aligned}
$$

Now we use the recurrence relation (92) to get

$$
\begin{aligned}
I_7 = & \int_0^{2\pi}\int_0^{\pi}\Bigg[\left(\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}\right)^{1/2}Y_{\ell-1,m} \\
& +\left(\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2}Y_{\ell+1,m}\Bigg]Y_{p,q}^*\sin\vartheta d\vartheta d\varphi \\
= & \left(\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}\right)^{1/2}\int_0^{2\pi}\int_0^{\pi}Y_{\ell-1,m}Y_{p,q}^*\sin\vartheta d\vartheta d\varphi \\
& +\left(\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2}\int_0^{2\pi}\int_0^{\pi}Y_{\ell+1,m}Y_{p,q}^*\sin\vartheta d\vartheta d\varphi.
\end{aligned}
$$

The by the orthogonality relation (86) we arrive at

$$
\begin{aligned}
I_7(\ell,m) = & \left(\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}\right)^{1/2}\delta_{\ell-1,p}\,\delta_{m,q} \\
& +\left(\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right)^{1/2}\delta_{\ell+1,p}\,\delta_{m,q}.
\end{aligned}
$$

E.8 Calculating I_8

Finally consider

$$I_8(\ell,m) \;=\; \int_0^{2\pi}\int_0^{\pi} (\sin^2\vartheta) e^{-i\sigma_m\varphi} Y_{\ell,m+\sigma_m} Y^*_{p,q} d\vartheta d\varphi.$$

Now using the Fubini's theorem and the definition of $Y_{\ell,m}$ (see 85 Appendix) we can separate the θ and ϑ integral and we get

$$\begin{aligned} I_8(\ell,m) &= \nu_p^q \nu_\ell^{m+\sigma_m} \left(\int_0^{\pi} P_\ell^{|m+\sigma_m|}(\cos\vartheta) P_p^{|q|}(\cos\vartheta)\sin^2\vartheta d\vartheta\right)\left(\int_0^{2\pi} e^{i(m-q)\varphi} d\varphi\right) \\ &= 2\pi\nu_p^q\nu_\ell^{m+\sigma_m}\delta_{m,q}\left(\int_{-1}^{1} P_\ell^{|m+\sigma_m|}(x) P_p^{|q|}(x)\sqrt{1-x^2}dx\right) \end{aligned}$$

where we made a change of variable with $x = \cos\vartheta$ for integration. Now if we use the recurrence relations for the associated Legendre functions (see equation 8.5.5. in [1]):

$$\sqrt{1-x^2} P_\ell^{|m+\sigma_m|}(x) \;=\; \frac{P_{\ell+1}^{|m+\sigma_m|+1}(x) - P_{\ell-1}^{|m+\sigma_m|+1}(x)}{-(2\ell+1)}$$

we get the following for I_8:

$$\begin{aligned} &I_8(\ell,m) = \\ &\frac{2\pi\nu_p^q\nu_\ell^{m+\sigma_m}\delta_{m,q}}{2\ell+1}\left(J_{\ell-1}(|m+\sigma_m|+1,p,|q|) - J_{\ell+1}(|m+\sigma_m|+1,p,|q|)\right). \end{aligned} \quad (107)$$

References

[1] M. Abramowitz and A. Stegun. *Handbook of Mathematical Functions.* Dover, New York, 1965.

[2] L. A. Apresyan and Yu. A. Kravtsov. *Radiation Transfer: Statistical and Wave Aspects.* Gordon and Breach Publishers, Netherlands, 1996.

[3] S.R. Arridge. Optical tomography in medical imaging: Topical review. *Inverse Problems*, 15:R41–R93, 1999.

[4] A.B. Bakushinsky, T. Khan, and A. Smirnova. Inverse problem in optical tomog- raphy and its numerical investigation by iteratively regularized methods. *Journal of Inverse and Ill-Posed Problems*, In Press, 2005.

[5] G. Bekefi. *Radiation Process in Plasmas.* John Wiley & Sons, New York, 1966.

[6] B. Davison. *Neutron Transport Theory.* Oxford University Press, Oxford, 1957.

[7] H Ferwerda. The radiative transfer equation for scattering media with spatially varying refractive index. *J. Opt. A: Pure Appl. Opt.*, 1:L1–L2, 1999.

[8] H. Goldstein. *Classical Mechanics 2nd Edition.* Addison-Wesley Publishing Company, New York, 1980.

[9] E. G. Harris. Radiative transfer in dispersive media. *Physical Review*, 138(2B):B479–B485, 1965.

[10] H. Jiang. The diffusion approximation for turbid media with a spatially varying refractive index. *Proceedings of Advances in Optical Imaging and Photon Migration, Optical Society of America*, pages 366–368, 2000.

[11] H. Jiang, K.D. Paulsen, and U.L. Osterberg. Optical image reconstruction using frequency frequency-domain data: simulations and experiments. *J. Opt. Soc. Am. A*, 13.

[12] F. John. *Partial Differential Equations 4th Edition.* Springer, New York, 1982.

[13] T. Khan. On derivation of the radiative transfer equation and its diffusion approximation for scattering media with spatially varying refractive indices. *Clemson University Mathematical Sciences Technical Report*, TR2003-07-TK:1–6, July 2003.

[14] T. Khan and H. Jiang. A new diffusion approximation to the radiative transfer equation for scattering media with spatially varying refractive indices. *J. Opt. A: Pure Appl. Opt.*, 5:137–141, 2003.

[15] T. Khan and M. Pinsky. On some integrals over a unit sphere. *Clemson University Mathematical Sciences Technical Report*, TR2003-05-KP:1–9, May 2003.

[16] T. Khan and A. Smirnova. 1d inverse problem in duffusion based optical tomography using iteratively regularized gauss-newton algorithm. *Applied Mathematics and Computation*, 161(1), 2005.

[17] Yu. A. Kravtsov and Yu. I. Orlov. *Geometrical Optics of Inhomogenous Media.* Springer-Verlag, Berlin, 1990.

[18] L. Marti-Lopez, J. Bouza-Dominguez, and J. Hebden. Interpretation of the failure of the time-dependent diffusion equation near a point source. *Optics Communications*, In Press, 2004.

[19] L. Marti-Lopez, J. Bouza-Dominguez, J. Hebden, S. Arridge, and R. Martinez-Celorio. Validity conditions for the radiative transfer equation. *J. Opt. Soc. Am. A*, 20(11):2046–2056, 2003.

[20] J.C. Minano. Refractive-index distribution in two-dimensional geometry for a given one-parameter manifold of rays. *J. Opt. Soc. Am. A*, 11(2):1821–1825, 1985.

[21] Barrett O'Neill. *Elementary Differential Geometry.* Academic Press, New York, 1967.

[22] M. Pinsky. Stochastic riemannian geometry. *Probabilistic Analysis and Related Topics Edited By A.T. Bharucha-Reid*, 1:199–236, 1978.

[23] MC Roggermann, BM Welsh, PJ Gardner, RL Johnson, and BL Pedersen. Sensing three-dimensional index-of-refraction variations by means of optical wavefront sensor measurements and tomographic reconstruction. *Optical Engineering*, 34(5):1374–1384, 1995.

[24] L. Ryzhik, G. Papanicolaou, and J.B. Keller. Transport equations for elastic and other waves in random media. *Wave Motion*, 24:327–370, 1996.

[25] S. N. Samaddar. Some integrals involving asssociated legendre functions. *Mathematics of Computaion*, 28(125):257–263, 1974.

[26] GJ Tearney, ME Brezinski, JF Southern, BE Bouma, MR Hee, and JG Fujimoto. Determination of the refractive index of highly scattering human tissue by optical coherence tomography. *Optics Letters*, 20(21):2258–2260, 1995.

[27] J. Tualle and E. Tenet. Derivation of the radiative transfer equation for scattering media with spatially varying refractive index. *Optics Communications*, 228:33–38, 2003.

[28] J. Wloka. *Partial Differential Equations*. Cambridge University Press, 1982.

[29] B. R. Wong. On the overlap integral of associated legendre polynomials. *J. Phys. A: Math Gen.*, 31:1101–1103, 1998.

[30] Y. Xu, X. Gu, T. Khan, and H. Jiang. Absorption and scattering image of heterogenous scattering media can be simultaneously reconstructed using dc data. *Applied Optics*, 41(25), 2002.