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Clemson University, South Carolina, USA

Technical Report TR2005_10_EW

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October 2005

This report is availabe online at

http://www.math.clemson.edu/reports/TR2005_10_EW.pdf

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Cone characterizations of approximate solutions in real-vector optimization

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Abstract

Borrowing concepts from linear algebra and convex analysis, it has been shown how the feasible set for a general vector optimization problem can be mapped under a linear transformation so that Pareto points in the image correspond to nondominated solutions for the original problem. The focus of this paper is to establish corresponding results for approximate nondominated points, based on a new characterization of these solutions using the concept of translated cones. The problem of optimizing over this set of approximate solutions is addressed and possible applications are given in the references.

Keywords: Approximate solutions, epsilon-efficient solutions, epsilon-minimal elements, translated cones

1 Introduction

The general goal in any optimization or decision making process is to identify a single or all best solutions within a set of given feasible points or alternatives. For decision problems with multiple criteria, however, the notion of a best solution needs special attention due to the lack of a canonical order of vectors and usually depends on the underlying preferences of the optimizing decision maker.

*This research has been supported by the Automotive Research Center, a U.S. Army TACOM Center of Excellence for Modeling and Simulation of Ground Vehicles at the University of Michigan, and by the National Science Foundation, grant number DMS-0425768.

The preference order traditionally adopted was introduced into the economic theory by Edgeworth (1881) although usually credited to and thus more commonly associated with the name of Pareto (1896). A main characteristic of this concept is that in contrast to problems with only one criterion, in general there does not exist a unique best solution, but a solution set of so-called efficient or nondominated points.

Based on the observed equivalence between other partial orderings and certain convex cones, Yu (1974) and Bergstresser et al. (1976) generalized the Pareto concept to more general domination structures and sets. Following up these works, Lin (1976) provided a comparison of the defined optimality concepts, and Chew (1979) proposed a reformulation for general vector spaces. The papers by Bergstresser and Yu (1977) and Takeda and Nishida (1980) also illustrate domination structures for multicriteria games and fuzzy multicriteria decision making, respectively. Many of these earlier results were collected in the monograph by Yu (1985), while further investigation was continued by Weidner (1985, 1987, 1990). More recently, Chen and Yang (2002) examined variable domination structures in the context of variational inequalities, while Weidner (2003) also studied domination structures with respect to tradeoff directions.

Miettinen and Mäkelä (2000, 2001) provided a detailed characterization of optimality concepts based on the original concept of cones. Realizing that this concept is widely used for theoretical investigations but hardly employed in real life applications, Hunt and Wiecek (2003) promoted the use of cones also for practical decision making. A tradeoff-based cone construction for modeling the decision maker's preferences was developed in Hunt (2004), while Wu (2004) further examined the relevance of convex cones for a solution concept in fuzzy multiobjective programming.

Nevertheless, although providing a theoretical framework for the definition of various optimality concepts, the use of cones does not in general provide us with applicable tools to actually identify the associated solutions. Hence, a second major research area in vector optimization is the development of practical methods to solve these problems, thereby usually adopting the traditional concept of Pareto efficiency. In this context, Wierzbicki (1986) collected and examined various approaches with respect to their capability to identify the complete set of optimal solutions. Since this set, however, might consist of an infinite number of points, finding an exact description often turns out to be practically impossible or at least computationally too expensive, and consequently many research efforts focused on approximation concepts and procedures, see Vályi (1985), Lemaire (1992) or Ruzika and Wiecek (2005), among others.

The notion of approximate solutions adopted in this text follows from the concept of epsilon-efficiency originally introduced into multiple objective programming by Loridan (1984). Two years later, White (1986) introduced six alternative definitions of epsilon-efficient solutions and established the relationships between those. Following either Loridan or White, related definitions or examinations of these concepts were given by Helbig and Pateva (1994), Tanaka (1996), Yokoyama (1996, 1999) and Li and Wang (1998), while Németh (1989), Loridan et al. (1999) and Rong and Wu (2000) also considered epsilon-efficiency for more general vector optimization problems.

Another significant portion of the literature deals with necessary and sufficient conditions for epsilon-efficient solutions, among those the works by Yokoyama (1992, 1994), Liu (1996), Deng (1997, 1998) and Dutta and Vetrivel (2001). Further results were obtained by Kazmi (2001) who also derived conditions for the existence of epsilon-minima. Recently, Engau and Wiecek (2005a,b) also described practical decision making situation in which suboptimal solutions are of relevance other than merely for approximation purposes and developed a methodology for the generation of epsilon-efficient solutions in multiple objective programming.

The purpose of this paper is to define and investigate approximate solutions for real vector optimization problems in the framework of cones, based on the concept of epsilon-efficiency. However, thereby it turns out that the underlying domination structure cannot be described by a cone anymore, following the classical definition as given by Rockafellar (1970), but is described by a cone that is translated from the origin. Therefore we adopt the more general notion presented in Luenberger (1969) which allows for a cone with an arbitrary vertex and corresponds to what Rockafellar calls a skew orthant or generalized m-dimensional simplex with one ordinary vertex and $m-1$ directions (or vertices at infinity). Other than the two monographs above, Nachbin (1996) introduced affine cones to describe mappings between convex vector spaces, and Bauschke (2003) mentioned translated cones in the context of duality results for Bregman projections onto linear constraints, both of which, however, do not relate to the translation of a cone in the context of this paper.

We formulate the notion of translated cones as needed for our purposes and investigate some preliminary properties and possible representations of translated cones as polyhedral sets. Borrowing concepts from linear algebra and convex analysis, we review results from Yu (1985), Weidner (1990), Hunt and Wiecek (2003) and the more general treatment in Cambini et al. (2003) which show how the feasible set for a general vector optimization problem can be mapped under a linear transformation so that Pareto points in the image correspond to nondominated solutions for the original problem. We then generalize these results in the proposed new context by deriving corresponding relationships for approximate solutions and polyhedral translated cones.

The organization of the remaining text is now as follows. After this introduction, Section 2 provides some preliminaries by defining (weakly) minimal and epsilon-minimal elements as the adopted concepts of approximate solutions and formalizing the notion of translated cones as cones that are shifted from the origin along some translation vector. Alternative representations of translated cones are derived in Section 3 with the main focus on possible descriptions by a system of linear inequalities. In Section 4, these representations are used to characterize epsilon-minimal elements with respect to polyhedral cones as minimal solutions with respect to translated polyhedral cones, and of particular interest for practical applications, it is shown how the set of epsilon-minimal solutions with respect to a polyhedral cones can be transformed into a set of minimal elements with respect to a Pareto cone. The problem to identify maximal elements among the set of epsilon-minimal elements is addressed in Section 5, and some final remarks in Section 6 and the list of references conclude the paper.

2 Preliminaries

Let Z be a real linear space. The sum of two subsets $X, Y \subseteq Z$ is defined as the Minkowski sum $X + Y := \{x + y : x \in X, y \in Y\}$, and for $z \in Z$ a single element, $Y + z$ is written instead of $Y + \{z\}$.

Definition 2.1. A *cone* $C \subseteq Z$ is a set that is closed under nonnegative scalar multiplication, $\lambda C \subseteq C$ for all $\lambda \geq 0$. The notation C° is used to denote $C \setminus \{0\}$. An *ordering cone* is a pointed convex cone, that is a convex cone which does not contain any nontrivial subspaces,

$$C + C \subseteq C \text{ and } C^\circ \cap -C = \emptyset.$$

Given an ordering cone $C \subseteq Z$, a partial order on Z can be defined by $z^1 \leqq_C z^2$ if and only if $z^2 - z^1 \in C$, where z^1 and z^2 are any two elements in Z. Furthermore we write $z^1 \leq_C z^2$ if and only if $z^2 - z^1 \in C^\circ$ and $z^1 \geqq_C z^2$ and $z^1 \geq_C z^2$ if and only if $z^2 \leqq_C z^1$ and $z^2 \leq_C z^1$, respectively.

For $Z = \mathbb{R}^m$ a Euclidean vector space, the abbreviated notation $z^1 \leqq z^2$ is used if and only if $z_i^1 \leq z_i^2$ for all $i = 1, \ldots, m$, $z^1 \leq z^2$ if and only if $z^1 \leq z^2$ and $z^1 \neq z^2$, $z^1 < z^2$ if and only if $z_i^1 < z_i^2$ for all $i = 1, \ldots, m$. With the relations $\geqq$, $\geq$ and $>$ defined accordingly, then the nonnegative, the nonnegative nonzero and the positive orthants are denoted by $\mathbb{R}^m_{\geqq} := \{z \in \mathbb{R}^m : z \geqq 0\}$, $\mathbb{R}^m_{\geq} := \{z \in \mathbb{R}^m : z \geq 0\}$ and $\mathbb{R}^m_{>} := \{z \in \mathbb{R}^m : z > 0\}$.

An important class of Euclidean ordering cones is given by pointed polyhedral cones.

Definition 2.2. A *polyhedral cone* is a cone $C \subseteq \mathbb{R}^m$ for which there exists a matrix $A \in \mathbb{R}^{p\times m}$ such that $C = \{z \in \mathbb{R}^m : Az \geqq 0\}$. The *kernel* of a polyhedral cone is defined as the kernel (or nullspace) of the associated matrix

$$\operatorname{Ker} C = \operatorname{Ker} A = \{z \in \mathbb{R}^m : Az = 0\}.$$

It is an easy task to verify that a polyhedral cone is always convex, but in general not pointed. Given a pointed polyhedral cone C, the relations $\leqq_C$ and $\leq_C$ are defined as before, and in addition $z^1 <_C z^2$ is written if and only if $z^2 - z^1 \in \operatorname{int} C$, where $\operatorname{int} C$ denotes the interior of C using the standard Euclidean topology. Then in particular for $A = I \in \mathbb{R}^{m\times m}$ the identity matrix, the three induced polyhedral cones coincide with the three orthants defined above.

While our main interest lies in the characterization of approximate solutions in real-vector optimization for the particular application to multiobjective programming, the initial concepts and results do not depend on the Euclidean nature of the underlying space and thus are presented for an arbitrary real linear space Z.

2.1 Minimal and epsilon-minimal elements

Let Z be a real linear space, $Y \subseteq Z$ be a given set and $C \subseteq Z$ be a given cone.

Definition 2.3. An element $z \in Z$ is called a *minimal element* of the set Y with respect to the cone C if $z \in Y$ and if there does not exist a point $y \in Y$ with $y \leq_C z$, or equivalently

$$Y \cap (z - C^\circ) = \emptyset.$$

The set of all minimal elements of Y with respect to the cone C is denoted by $\mathrm{MIN}(Y, C)$.

Alternative terminology refers to the set $\mathrm{MIN}(Y, C)$ as the set of *efficient* or *nondominated solutions* and calls C the *domination cone* or the *cone of dominated directions*. Hereby, the particular assumptions of convexity and pointedness of an ordering cone guarantee that the sum of two dominated directions $d^1, d^2 \in C$ is again a dominated direction, $d^1 + d^2 \in C$, and that if both d and $-d \in C$ are dominated directions, then $d = 0$.

By replacing the cone $C \subseteq Z$ with an arbitrary set $D \subseteq Z$, we extend Definition 2.3 and furthermore introduce the notion of weakly minimal elements.

Definition 2.4. The set $\mathrm{MIN}(Y, D)$ of all minimal elements of Y with respect to the set D is defined as the set of all points $z \in Y$ for which

$$Y \cap (z - D^{\circ}) = \emptyset.$$

Furthermore, if the space Z carries a topology allowing to define the interior of a set, then the set of *weakly* minimal elements of Y with respect to D is defined as

$$\mathrm{WMIN}(Y, D) := \mathrm{MIN}(Y, \mathrm{int}\, D).$$

The following lemma gives two equivalent characterizations of (weakly) minimal elements, among which we choose the most convenient without further explanation for all subsequent proofs throughout the rest of the paper.

Lemma 2.1. *Let $z \in \mathrm{MIN}(Y, D)$ be a minimal element of Y with respect to the set D. Then the following are equivalent:*

(i) $Y \cap (z - D^{\circ}) = \emptyset$;

(ii) there does not exist $y \in Y, y \neq z$ such that $z - y \in D$;

(iii) there does not exist $y \in Y, d \in D, d \neq 0$ such that $z = y + d$.

Proof. Rewrite (i) $Y \cap (z - D^{\circ}) = \emptyset$

$$\begin{aligned}
&\iff \nexists\, y \in Y: && y \in z - D \setminus \{0\} \\
&\iff \nexists\, y \in Y, y \neq z: && y \in z - D \Leftrightarrow z - y \in D && \text{(ii)} \\
&\iff \nexists\, y \in Y, y \neq z, d \in D: && y = z - d \\
&\iff \nexists\, y \in Y, d \in D, d \neq 0: && z = y + d && \text{(iii).}
\end{aligned}$$

□

For a cone C, conditions for the existence of minimal elements are established by Hartley (1978), Corley (1980), Borwein (1983) and Sawaragi et al. (1985), among others. For a comparison of various existence results, the reader is referred to the recent survey provided by Sonntag and Zalinescu (2000). A major role in these results is the pointedness condition

$C^\circ \cap -C = \emptyset$ of the cone C which can be ensured to hold for a set D if this set is specifically chosen as the translation of the pointed cone C from the origin along one of its nonzero elements $\varepsilon \in C^\circ$. Note that $\varepsilon \in C^\circ$ can also be written as $\varepsilon \geq_C 0$.

Definition 2.5. For given $\varepsilon \in C^\circ$, an element $z \in Z$ is called an *ε-minimal element* of the set Y with respect to the cone C if $z \in Y$ and if there does not exist a point $y \in Y$ with $y \leq_C z - \varepsilon$, or equivalently

$$Y \cap (z - \varepsilon - C) = \emptyset.$$

The set of all ε-minimal elements of Y with respect to the cone C is denoted by $\mathrm{MIN}(Y, C, \varepsilon)$. Furthermore, if the space Z carries a topology allowing to define the interior of a set, then the set of *weakly* ε-minimal elements of Y with respect to C is defined as

$$\mathrm{WMIN}(Y, C, \varepsilon) = \mathrm{MIN}(Y, \operatorname{int} C, \varepsilon).$$

Remark 2.1. Note that opposed to Definitions 2.3 and 2.4 for minimal elements, Definition 2.5 does not exclude zero from the cone C and therefore does not define 0-minimal in generalization of minimal elements, for which it would be necessary to require that $Y \cap (z - \varepsilon - C^\circ) = \emptyset$.

The reason for the slightly varying definition of minimal and ε-minimal elements is to guarantee the following identity which provides a convenient characterization of (weakly) ε-minimal elements with respect to a cone C as (weakly) minimal elements with respect to a set D.

Proposition 2.1. *Let C be a pointed cone and $\varepsilon \in C^\circ$. Define $D = C_\varepsilon := C + \varepsilon$. Then*

$$\mathrm{MIN}(Y, D) = \mathrm{MIN}(Y, C, \varepsilon) \text{ and } \mathrm{WMIN}(Y, D) = \mathrm{WMIN}(Y, C, \varepsilon).$$

Proof. From the definition of minimal elements with respect to some set D we have that

$$\begin{aligned}\mathrm{MIN}(Y, D) &= \{z \in Y : Y \cap (z - D^\circ) = \emptyset\} \\ &= \{z \in Y : Y \cap (z - (C + \varepsilon) \setminus \{0\}) = \emptyset\} \\ &= \{z \in Y : Y \cap (z - (C \setminus \{-\varepsilon\} + \varepsilon)) = \emptyset\} \\ &= \{z \in Y : Y \cap (z - C - \varepsilon) = \emptyset\} = \mathrm{MIN}(Y, C, \varepsilon)\end{aligned}$$

where the last equality follows from $-\varepsilon \notin C$ since C is pointed and $\varepsilon \in C^\circ$. Then the second statement follows immediately from Definitions 2.4 and 2.5 and the observation that $\operatorname{int}(C+\varepsilon) = \operatorname{int} C + \varepsilon$. □

Note that the set $D = C_\varepsilon$ corresponds to the translation of the cone C along the vector ε and therefore, in principle, is not a cone anymore. In the following section, we investigate some preliminary properties of sets that are cones translated from the origin.

2.2 Translated cones

As before, let Z be an arbitrary real linear space.

Definition 2.6. Let $D \subseteq Z$ be a given set. If there exists a cone $C \subseteq Z$ and a vector $\varepsilon \in Z$ such that $D = C + \varepsilon$, then D is said to be a *translated cone* with *translation vector* ε, written $D = C_\varepsilon$. The translated cone $D = C_\varepsilon$ is called *convex* or *pointed* if and only if the cone C is convex or pointed, respectively.

The following proposition justifies the previous "only if" statement be demonstrating that if such a cone C exists, then it must be unique.

Proposition 2.2. *Let $C^1, C^2 \subseteq Z$ be two cones and $\varepsilon_1, \varepsilon_2 \in Z$ be two translation vectors. If the translated cones $C^1_{\varepsilon^1}$ and $C^2_{\varepsilon^2}$ are equal, $C^1_{\varepsilon^1} = C^2_{\varepsilon^2}$, then so are the cones,*

$$C^1 = C^2.$$

Proof. Suppose $C^1_{\varepsilon^1} = C^2_{\varepsilon^2}$ and let $d \in C^1$. Then $d + \varepsilon^1 \in C^1_{\varepsilon^1} = C^2_{\varepsilon^2}$ and thus $d + \varepsilon^1 - \varepsilon^2 \in C^2$. Since C^2 is a cone, we also have that $\frac{1}{2}(d + \varepsilon^1 - \varepsilon^2) \in C^2$, thus $\varepsilon^2 + \frac{1}{2}(d + \varepsilon^1 - \varepsilon^2) \in C^2_{\varepsilon^2} = C^1_{\varepsilon^1}$ and $\varepsilon^2 + \frac{1}{2}(d + \varepsilon^1 - \varepsilon^2) - \varepsilon^1 = \frac{1}{2}(d + \varepsilon^2 - \varepsilon^1) \in C^1$. The fact that C^1 is a cone now gives that also $d + \varepsilon^2 - \varepsilon^1 \in C^1$, and we conclude $d + \varepsilon^2 - \varepsilon^1 + \varepsilon^1 = d + \varepsilon^2 \in C^1_{\varepsilon^1} = C^2_{\varepsilon^2}$ and finally $d + \varepsilon^2 - \varepsilon^2 = d \in C^2$. Hence $C^1 \subseteq C^2$, and by interchanging the roles of C^1 and C^2 we obtain the result. □

Note that the translation vector, in general, is not unique.

Example 2.1. Let $Z = \mathbb{R}^2$ and consider the cone $C = \{(d_1, d_2)^T \in \mathbb{R}^2 : d_1 \geq 0\}$. Then $C = C_\varepsilon$ for all translation vectors $\varepsilon \in \{(\varepsilon_1, \varepsilon_2)^T \in \mathbb{R}^2 : \varepsilon_1 = 0\}$.

However, uniqueness of the translation vector can be guaranteed in the case of a pointed (translated) cone.

Proposition 2.3. *Let $D \subseteq Z$ be a pointed translated cone. Then there exists a unique cone $C \subseteq \mathbb{R}^m$ and a unique translation vector $\varepsilon \in Z$ such that $D = C_\varepsilon$.*

Proof. Apply Proposition 2.2 to obtain the unique cone C, which by definition is pointed, $C^\circ \cap -C = \emptyset$. Since D is a translated cone, there exists $\varepsilon \in Z$ such that $D = C_\varepsilon$, and in order to show uniqueness, let $C_{\varepsilon^1} = C_{\varepsilon^2}$, where $\varepsilon^1, \varepsilon^2 \in Z$ are two translation vectors. Then we have to show that $\varepsilon^1 = \varepsilon^2$. First, knowing that $\varepsilon^1 \in C_{\varepsilon^1} = C_{\varepsilon^2} \ni \varepsilon^2$, there must exist vectors $d^1, d^2 \in C$ such that $\varepsilon^1 = d^2 + \varepsilon^2$ and $\varepsilon^2 = d^1 + \varepsilon^1$. Hence, we obtain $d^1 = -d^2$ which then implies $d^1 = d^2 = 0$ as C is pointed, thus yielding $\varepsilon^1 = \varepsilon^2$. $\square$

Definition 2.7. Given a translated cone $D = C_\varepsilon$ for which the translation vector ε is unique, then ε is also said to be the *vertex* of the translated cone D.

Now let $C \subseteq Z$ be a given cone and $\varepsilon^1, \varepsilon^2 \in Z$ be two translation vectors. We close this section by addressing the relationships between $D^1 = C_{\varepsilon^1}$, $D^2 = C_{\varepsilon^2}$ and the associated sets $\text{MIN}(Y, D^1)$ and $\text{MIN}(Y, D^2)$ of minimal elements, based on the relationship between ε^1 and ε^2, thereby following Sawaragi et al. (1985) who established the following containment result for the set of minimal elements with respect to two different cones.

Proposition 2.4. *Let $Y \subseteq Z$ be a set and $C^1, C^2 \subseteq Z$ be two cones with $C^2 \subseteq C^1$. Then*

$$\text{MIN}(Y, C^1) \subseteq \text{MIN}(Y, C^2).$$

Remark 2.2. Note that analogously to Proposition 2.4, it follows that $\text{MIN}(Y, D^1) \subseteq \text{MIN}(Y, D^2)$ whenever $D^2 \subseteq D^1$, so that in particular $\text{MIN}(Y, C^1_\varepsilon) \subseteq \text{MIN}(Y, C^2_\varepsilon)$ for any vector $\varepsilon \in Z$.

Now given only one cone $C \subseteq Z$ but two translation vectors $\varepsilon^1, \varepsilon^2 \in Z$, we formulate a corresponding condition on ε^1 and ε^2 which guarantees that $\text{MIN}(Y, C, \varepsilon^1) \subseteq \text{MIN}(Y, C, \varepsilon^2)$.

Proposition 2.5. *Let $D^1 = C_{\varepsilon^1}$ and $D^2 = C_{\varepsilon^2}$ be two convex translated cones with $\varepsilon^1 \leqq_C \varepsilon^2$. Then*

$$\text{MIN}(Y, D^1) \subseteq \text{MIN}(Y, D^2).$$

Proof. Since $\varepsilon^1 \leqq_C \varepsilon^2$, we have that $\varepsilon^2 - \varepsilon^1 \in C$, or $\varepsilon^2 \in C + \varepsilon^1$. It follows that $C + \varepsilon^2 \subseteq C + C + \varepsilon^1 \subseteq C + \varepsilon^1$ since C is a convex cone, and so $D^2 \subseteq D^1$. Now Remark 2.2 implies that $\text{MIN}(Y, D^1) \subseteq \text{MIN}(Y, D^2)$. $\square$

In particular, we obtain that $\text{MIN}(Y, C) \subseteq \text{MIN}(Y, C, \varepsilon)$ for every $\varepsilon \in C^\circ$.

3 Inequality representations of polyhedral translated cones

For the subsequent two sections, we let $Z = \mathbb{R}^m$ be a Euclidean space and first focus on the derivation of linear inequality representations of translated polyhedral cones in terms of polyhedral sets. The first definition serves to clarify the adopted notation.

Definition 3.1. Let $A \in \mathbb{R}^{p \times m}$ be a matrix and $b \in \mathbb{R}^p$ be a vector. Then

$$D(A, b) := \{d \in \mathbb{R}^m : Ad \geqq b\}$$

defines a *polyhedral set*. For $b = 0$, the *polyhedral cone* implied by A is denoted by

$$D(A) := D(A, 0) = \{d \in \mathbb{R}^m : Ad \geqq 0\}.$$

Remark 3.1. Note here that the representation $D(A, b)$ of a polyhedral set is not unique and that, in general, a polyhedral set $D = D(A, b)$ may be empty. However, if $b \in -\mathbb{R}^p_{\geqq}$, then $0 \in D$ and hence $D \neq \emptyset$. In particular, a polyhedral cone is always nonempty.

As mentioned before, an ordering cone is a cone that is pointed and convex. It is an easy task to verify that a polyhedral cone is always convex, while conditions for a pointed polyhedral cone are characterized in the following proposition.

Proposition 3.1. *Let $C = \{d \in \mathbb{R}^m : Ad \geqq 0\}$ be a polyhedral cone with matrix $A \in \mathbb{R}^{p \times m}$. Then the following are equivalent:*

(i) C is pointed;

(ii) $C^\circ = \{d \in \mathbb{R}^m : Ad \geq 0\}$;

(iii) A has full column rank, $\operatorname{rank} A = m$.

3.1 Representing translated polyhedral cones as polyhedral sets

The first theorem states that every translated polyhedral cone can be represented as a polyhedral set.

Theorem 3.1. *Let $C_\varepsilon \subseteq \mathbb{R}^m$ be a translated polyhedral cone with $C = D(A) \subseteq \mathbb{R}^m$ for some matrix $A \in \mathbb{R}^{p\times m}$, and $\varepsilon \in \mathbb{R}^m$ be a translation vector. Set $b = A\varepsilon \in \mathbb{R}^p$ and let $D = D(A, b) \subseteq \mathbb{R}^m$ be the polyhedral set implied by A and b. Then*

$$D = C_\varepsilon.$$

Proof. We have to show that $D = C_\varepsilon$, where $D = D(A, b)$ with $b = A\varepsilon$ and $C = D(A)$. For the first inclusion, let $d \in D$. Then $Ad \geqq b = A\varepsilon$ and thus

$$Ad - A\varepsilon = A(d - \varepsilon) \geqq 0 \Rightarrow d - \varepsilon \in C \Rightarrow d \in C + \varepsilon = C_\varepsilon,$$

yielding $D \subseteq C_\varepsilon$. For the reversed inclusion, let $d = c + \varepsilon$ where $c \in C$. Then $Ac \geqq 0$, or

$$Ad = A(c + \varepsilon) = Ac + A\varepsilon \geqq 0 + b = b,$$

which implies $d \in D$ and therefore gives $C_\varepsilon \subseteq D$ to conclude the proof. □

Given a translated cone D which can be represented as polyhedral set $D = D(A, b)$, we also call D a *polyhedral translated cone*.

Remark 3.2. By definition, a polyhedral translated cone D is pointed if and only if the polyhedral cone C is pointed, in which case Proposition 2.3 established that the translation vector ε is the unique vertex of $D = C_\varepsilon$.

Now the pointedness conditions from Proposition 3.1 can be generalized for polyhedral translated cones.

Proposition 3.2. *Let $D = C_\varepsilon \subseteq \mathbb{R}^m$ be a polyhedral translated cone with $C = D(A) \subseteq \mathbb{R}^m$ and translation vector $\varepsilon \in \mathbb{R}^m$. Then the following are equivalent:*

(i) D is pointed with vertex ε;

(ii) $D \setminus \{\varepsilon\} = \{d \in \mathbb{R}^m : Ad \geq b\}$, where $b = A\varepsilon$;

(iii) A has full column rank, $\operatorname{rank} A = m$.

Proof. By Remark 3.2, the translated cone $D = C_\varepsilon$ is pointed if and only if C is pointed, and hence (i) is equivalent to the pointedness of C. Next, note that (ii) means that there does not exist $d \in D, d \neq \varepsilon$ such that $Ad - b = Ad - A\varepsilon = A(d - \varepsilon) = 0$ or equivalently, by writing $c = d - \varepsilon \in D - \varepsilon = C$, that there does not exist $c \in C, c \neq 0$ such that $Ac = 0$. Therefore, (ii) is equivalent to saying that $C^\circ = C \setminus \{0\} = \{d \in \mathbb{R}^m : Ad \geq 0\}$, and now the proof follows from Proposition 3.1. □

3.2 Polyhedral sets describing polyhedral translated cones

In this section we investigate a statement similar to Theorem 3.1 and derive conditions for a given polyhedral set to describe a polyhedral translated cone. For this purpose we first extend the concept of the kernel of a polyhedral cone to polyhedral sets.

Definition 3.2. Given a polyhedral set $D = D(A, b) \subseteq \mathbb{R}^m$, the *kernel* of D with respect to A and b is defined as

$$\operatorname{Ker} D(A, b) := \{d \in \mathbb{R}^m : Ad = b\}.$$

If A and b are clear from the context, we simply write $\operatorname{Ker} D$ instead of $\operatorname{Ker} D(A, b)$. However, note that the kernel in general depends on the specific choice of $A \in \mathbb{R}^{p \times m}$ and $b \in \mathbb{R}^p$.

Example 3.1. Let $Z = \mathbb{R}$, $A = (1, 1)^T$ and $b = (1, 0)^T$. Then

$$D^1 = D(A, b) = \{d \in \mathbb{R} : d \geq 1, d \geq 0\} = \{d \in \mathbb{R} : d \geq 1\}$$

with $\operatorname{Ker} D^1 = \emptyset$. However, if we let $D^2 = D(1, 1)$, then $D^1 = D^2$ but $\operatorname{Ker} D^2 = \{1\}$.

The following results collect some properties of the kernel which eventually lead to a second characterization of polyhedral translated cones.

Proposition 3.3. *The kernel of a polyhedral set $D = D(A) \subseteq \mathbb{R}^m$ is again a polyhedral set. In particular, the kernel of a polyhedral cone $C = D(A) \subseteq \mathbb{R}^m$ is a polyhedral cone.*

Proof. Note that

$$\operatorname{Ker} D = \{d \in \mathbb{R}^m : Ad = b\} = \{d \in \mathbb{R}^m : Ad \geqq b, -Ad \geqq -b\};$$
$$\operatorname{Ker} C = \{d \in \mathbb{R}^m : Ad = 0\} = \{d \in \mathbb{R}^m : Ad \geqq 0, -Ad \geqq 0\},$$

and hence $\operatorname{Ker} D = D((A, -A)^T, (b, -b)^T)$ and $\operatorname{Ker} C = D((A, -A)^T)$. □

Proposition 3.3 also follows from basic linear algebra by observing that the kernel of a polyhedral cone $D(A)$ is the solution set to the homogeneous system of equations $Ad = 0$.

Analogously, the kernel of a polyhedral set $D(A, b)$ is given by the general solution to the inhomogeneous system of equations $Ad = b$.

Now the following property is well known.

Proposition 3.4. *Let* $\operatorname{Ker} C$ *and* $\operatorname{Ker} D$ *be the kernel of the polyhedral cone* $C = D(A)$ *and the polyhedral set* $D = D(A, b)$*, respectively. If* $\varepsilon \in \operatorname{Ker} D$ *is any kernel element of the polyhedral set, then*

$$\operatorname{Ker} D = \operatorname{Ker} C + \varepsilon.$$

Similar to Proposition 3.1 pointedness of polyhedral translated cones can be characterized in terms of the kernel of the associated polyhedral sets.

Proposition 3.5. *Let* $D = D(A, b) \subseteq \mathbb{R}^m$ *be a polyhedral set. Then*

(i) $D \setminus \operatorname{Ker} D = \{d \in \mathbb{R}^m : Ad \geq b\}$*;*

(ii) $\operatorname{int} D = \{d \in \mathbb{R}^m : Ad > b\}$.

In particular, if $C = D(A) \subseteq \mathbb{R}^m$ *is a polyhedral cone, then*

(iii) $0 \in \operatorname{Ker} C$*;*

(iv) $\operatorname{Ker} C = \{0\}$ *if and only if* C *is pointed.*

Proof. (i), (ii) and (iii) follow immediately from the representation of a polyhedral set and the definition of the kernel. Then (iv) follows from (iii), (i) and Proposition 3.1. □

Based on Theorem 3.1, we can formulate the analogous statements for polyhedral translated cones.

Proposition 3.6. *Let* $C_\varepsilon \subseteq \mathbb{R}^m$ *be a polyhedral translated cone with* $C = D(A)$ *and translation vector* $\varepsilon \in \mathbb{R}^m$*. Then there exists a representation* $C_\varepsilon = D(A, b)$ *such that*

(i) $\varepsilon \in \operatorname{Ker} D(A, b)$*;*

(ii) $\operatorname{Ker} D(A, b) = \{\varepsilon\}$ *if and only if* D *is pointed.*

Proof. Refer to Theorem 3.1 in which it is shown that the polyhedral translated cone C_ε can be represented as $C_\varepsilon = D(A, b)$ with $b = A\varepsilon$, and thus in particular $\varepsilon \in \operatorname{Ker} D(A, b)$ which establishes (i). Analogously to Proposition 3.5(iv), then (ii) follows from (i), Proposition 3.5(i) and Proposition 3.2. □

Corollary 3.1. *Let $C_\varepsilon \subseteq \mathbb{R}^m$ be a polyhedral translated cone with $C = D(A)$ and translation vector $\varepsilon \in \operatorname{Ker} D$. Then there exists a representation $C_\varepsilon = D(A, b)$ such that the following are equivalent:*

(i) $\operatorname{Ker} C = \{0\}$*;*

(ii) $\operatorname{Ker} D(A, b) = \{\varepsilon\}$*;*

(iii) C and $D(A, b)$ are pointed.

Proof. By definition, a translated cone C_ε is pointed if and only if C is pointed. Now combine Proposition 3.5(iv) with Proposition 3.6(ii) to obtain the result. □

The preceding results now enable the statement of a second characterization of polyhedral translated cones by giving both a necessary and sufficient condition for a polyhedral set $D \subseteq \mathbb{R}^m$ to describe a polyhedral translated cone $C_\varepsilon \subseteq \mathbb{R}^m$.

Theorem 3.2. *Let $D \subseteq \mathbb{R}^m$ be a polyhedral set.*

(i) If D has a representation $D = D(A, b)$ such that the kernel of D with respect to A and b is nonempty, $\operatorname{Ker} D(A, b) \neq \emptyset$*, then D describes a polyhedral translated cone $D = C_\varepsilon$. Moreover, $C = D(A)$ and the translation vector $\varepsilon \in \operatorname{Ker} D(A, b)$.*

(ii) If the kernel in (i) is a singleton, $\operatorname{Ker} D(A, b) = \{\varepsilon\}$*, then the polyhedral translated cone $D = C_\varepsilon$ is pointed.*

Proof. For (i) we have to show that if $D = D(A, b)$ with $\operatorname{Ker} D \neq \emptyset$, then $D = C_\varepsilon$, where $C = D(A)$ and $\varepsilon \in \operatorname{Ker} D(A, b)$. Thus, let $\varepsilon \in \operatorname{Ker} D(A, b)$ and $D = D(A, b)$. Then $A\varepsilon = b$ and $d \in D$ if and only if $Ad \geqq b = A\varepsilon$, or $Ad - A\varepsilon = A(d - \varepsilon) \geqq 0$. This implies that $D - \varepsilon = C$ and therefore $D = C_\varepsilon$, and then (ii) follows from Corollary 3.1. □

Remark 3.3. Theorem 3.2 and Theorem 3.1 imply that the condition of a nonempty kernel, $\operatorname{Ker} D \neq \emptyset$, is in fact a necessary and sufficient condition for D to describe a polyhedral translated cone.

The condition that $\operatorname{Ker} D$ is a singleton is merely sufficient, but in general not necessary for C_ε to be pointed. If the kernel of an arbitrary representation is empty, $\operatorname{Ker} D = \emptyset$, then C_ε can still be pointed or not pointed. If this kernel contains more than one element, then C_ε is always not pointed.

Example 3.2. As in Example 3.1, let $Z = \mathbb{R}$, $A^1 = (1,1)^T$ and $b^1 = (1,0)^T$. Then $D^1 = \{d \in \mathbb{R} : d \geq 1, d \geq 0\} = \{d \in \mathbb{R} : d \geq 1\}$ is pointed with $\operatorname{Ker} D^1 = \emptyset$. On the other hand, consider $D^2 = D(0,-1) = \{d \in \mathbb{R} : 0d \geq -1\}$, then $\operatorname{Ker} D^2 = \emptyset$ while $D^2 = \mathbb{R}$ is not pointed.

The concluding corollary restates Theorem 3.2 in terms of the rank of the matrix A and follows again from well established results in linear algebra and the theory of matrices.

Corollary 3.2. *Let $D = D(A,b) \subseteq \mathbb{R}^m$ be a polyhedral set with matrix $A \in \mathbb{R}^{p \times m}$ and vector $b \in \mathbb{R}^m$.*

(i) If $\operatorname{rank} A = \operatorname{rank}(A,b)$ or equivalently, $b \in \operatorname{Lin} A$, where $\operatorname{Lin} A$ denotes the linear hull of A, then D describes a polyhedral translated cone.

(ii) If $\operatorname{rank} A = \operatorname{rank}(A,b) = m$, then D describes a pointed polyhedral translated cone.

4 Cone characterizations of epsilon-minimal elements

While the previous representation results do not depend on any particular properties of the underlying cones, for a reasonable definition of minimal elements it essentially suffices to require some ordering properties such as convexity or pointedness. Nevertheless, most practical applications restrict consideration to the special case of a Pareto cone and hence motivate efforts to find relationships between minimal elements with respect to general ordering cones and Pareto efficient solutions. Yu (1985), Weidner (1990) and Hunt and Wiecek (2003) established results for polyhedral cones which show that in this case the set of minimal elements can be transformed into a set of minimal elements with respect to a Pareto cone, while Cambini et al. (2003) also studied the effects of other linear transformations on the partial orders induced by a given ordering cone. After examining some of these established results, we derive the corresponding relationships for minimal elements with respect to arbitrary polyhedral sets and, as a special case, for ε-minimal elements with respect to polyhedral (translated) cones.

4.1 Minimal elements and polyhedral cones

The first theorem is adopted from the above mentioned references and describes the relationship between the images of minimal elements of Y and the minimal elements among the image of the initial set Y under the linear mapping induced by the cone matrix A.

Theorem 4.1. *Let $C = D(A) \subseteq \mathbb{R}^m$ be a polyhedral cone. Then*

$$A[\mathrm{MIN}(Y, C)] \subseteq \mathrm{MIN}(A[Y], \mathbb{R}^p_{\geqq}),$$

where $A[\mathrm{MIN}(Y, C)] := \{u \in \mathbb{R}^p : u = Ay \text{ for some } y \in \mathrm{MIN}(Y, C)\}$. If $\mathrm{Ker}\, A = \{0\}$, then

$$\mathrm{MIN}(A[Y], \mathbb{R}^p_{\geqq}) \subseteq A[\mathrm{MIN}(Y, C)].$$

Recall that for a polyhedral cone $C = D(A)$, the kernel of the matrix A also defines the kernel of the cone C, $\mathrm{Ker}\, C = \mathrm{Ker}\, A$. Hence, Proposition 3.5 implies the following corollary to Theorem 4.1.

Corollary 4.1. *Let $C = D(A) \subseteq \mathbb{R}^m$ be a pointed polyhedral cone. Then*

$$A[\mathrm{MIN}(Y, C)] = \mathrm{MIN}(A[Y], \mathbb{R}^p_{\geqq}).$$

Weidner (1990) established equality in Theorem 4.1 under the weaker assumption that

$$\mathrm{Ker}\, A \cap (Y - Y) = \{0\}.$$

We shortly discuss this condition and derive conditions for its equivalence with the pointedness condition $\mathrm{Ker}\, A = \{0\}$ employed in Theorem 4.1.

Proposition 4.1. *Let $A \in \mathbb{R}^{p \times m}$ be a matrix and $Y \subseteq \mathbb{R}^m$ be a set with $0 \in \mathrm{int}(Y - Y)$. Then*

$$\mathrm{Ker}\, A \cap (Y - Y) = \{0\} \Longleftrightarrow \mathrm{Ker}\, A = \{0\}.$$

Proof. Let $0 \in \mathrm{int}(Y - Y)$. Then for all $d \in \mathbb{R}^m$, there exists $\lambda > 0$ such that $\mu d \in (Y - Y)$ for all $0 \leq \mu \leq \lambda$. Furthermore, recall from Proposition 3.3 that the kernel of a polyhedral cone is a polyhedral cone. Then let $\mathrm{Ker}\, A \cap (Y - Y) = \{0\}$ and suppose by contradiction that there exists $d \in \mathrm{Ker}\, A, d \neq 0$. From the above, we know that $\mu d \in (Y - Y)$ for some (sufficiently small) $\mu > 0$ and hence $\mu d \neq 0$. Since $\mathrm{Ker}\, A$ is a cone, we also have that $\mu d \in \mathrm{Ker}\, A$ and thus $\mu d \in \mathrm{Ker}\, A \cap (Y - Y)$, giving the contradiction. The opposite direction is immediate. □

The next proposition provides a sufficient criterion for $Y \subseteq \mathbb{R}^m$ which guarantees that the condition $0 \in \mathrm{int}(Y - Y)$ in Proposition 4.1 holds true. We also present an example which demonstrates that this criterion, in general, does not necessarily need to be satisfied.

Proposition 4.2. *If the set* $Y \subseteq \mathbb{R}^m$ *has a nonempty interior,* $\operatorname{int} Y \neq \emptyset$*, then*

$$0 \in \operatorname{int}(Y - Y).$$

Proof. Let $y \in \operatorname{int} Y$. Then, for all $d \in \mathbb{R}^m$, there exists $\lambda > 0$ such that $y + \mu d \in Y$ for all $0 \leq \mu \leq \lambda$. It follows that for all $d \in \mathbb{R}^m, (y + \mu d - y) = \mu d \in (Y - Y)$ for all $\lambda \geq \mu \geq 0$ and hence $0 \in \operatorname{int}(Y - Y)$. □

From the above, it can be concluded that Weidner's condition is equivalent with the pointedness of the ordering cone for continuous problems in which the set Y is connected and has nonempty interior, but might provide a true generalization for discrete vector optimization problems.

In addition, it turns out that the statement of Proposition 4.2 and thus the equivalence in Proposition 4.1 may also hold true for a set Y whose interior is empty, as illustrated by the following example.

Example 4.1. Consider the following set for which the above condition $\operatorname{int} Y \neq \emptyset$ is violated, while the statement of Proposition 4.2 and thus the equivalence in Proposition 4.1 still holds. Given the set

$$Y := \{(y_1, 0)^T \in \mathbb{R}^2 : -1 \leq y_1 \leq 1\} \cup \{(0, y_2)^T \in \mathbb{R}^2 : 0 \leq y_2 \leq 2\},$$

we have that $\operatorname{int} Y = \emptyset$, but $Y - Y = \{(y_1, y_2)^T \in \mathbb{R}^2 : -2 \leq y_1 \leq 2, -2 \leq y_2 \leq 2\}$ and hence

$$0 \in \operatorname{int}(Y - Y) = \{(y_1, y_2)^T \in \mathbb{R}^2 : -2 < y_1 < 2, -2 < y_2 < 2\}.$$

Based on the above observations we refrain from further investigation of Weidner's condition and continue to use the initial condition $\operatorname{Ker} A = \{0\}$. Now returning to the discussion of Theorem 4.1, note that this theorem is formulated for minimal elements. The generalization to weakly minimal elements is also possible, but has not been addressed in the literature.

Theorem 4.2. *Let* $C = D(A) \subseteq \mathbb{R}^m$ *be a polyhedral cone. Then*

$$A[\mathrm{WMIN}(Y, C)] = \mathrm{WMIN}(A[Y], \mathbb{R}^p_{\geqq}).$$

Proof. Let $\hat{u} \in A[\mathrm{WMIN}(Y, C)]$ with $\hat{u} = A\hat{y}$, $\hat{y} \in \mathrm{WMIN}(Y, C)$, i.e., there do not exist $y \in Y, d \in \operatorname{int} C, d \neq 0$ such that $\hat{y} = y + d$. Now suppose by contradiction that $\hat{u} \notin$

$\mathrm{WMIN}(A[Y], \mathbb{R}^p_{\geqq})$, then $\hat{u} > u$ for some $u \in A[Y], u = Ay$, where $y \in Y, y \neq \hat{y}$. It follows that

$$\hat{u} - u = A\hat{y} - Ay = A(\hat{y} - y) > 0,$$

and setting $d = \hat{y} - y$ gives $\hat{y} = y + d$, $d \in \operatorname{int} C, d \neq 0$ in contradiction to the above. For the opposite direction, let $\hat{u} \in \mathrm{WMIN}(A[Y], \mathbb{R}^p_{\geqq})$ with $\hat{u} = A\hat{y}, \hat{y} \in Y$. Then there does not exist $y \in Y$ such that $A\hat{y} > Ay$ and hence $A(\hat{y} - y) > 0$. Now suppose by contradiction that $\hat{u} \notin A[\mathrm{WMIN}(Y, C)]$, or $\hat{y} \notin \mathrm{WMIN}(Y, C)$. Then there exist $y \in Y, d \in \operatorname{int} C$ such that $\hat{y} = y + d$ and hence

$$A(\hat{y} - y) = Ad > 0.$$

This yields the contradiction. □

Note that this result can also be derived from Corollary 4.1. First, recall from Definition 2.4 that $\mathrm{WMIN}(Y, C) = \mathrm{MIN}(Y, \operatorname{int} C)$ and then observe that $\operatorname{int} C$ is always pointed for $C \subset \mathbb{R}^m$ a polyhedral cone, and clearly $\mathrm{MIN}(Y, C) = \mathrm{WMIN}(Y, C) = \emptyset$ for $C = \mathbb{R}^m$ and Y not a singleton.

4.2 Epsilon-minimal elements and polyhedral translated cones

In this section, we derive various generalizations of Theorems 4.1 and 4.2. In its original formulation restricted to minimal elements with respect to polyhedral cones, we derive the corresponding results for minimal elements with respect to arbitrary polyhedral sets and, as a special case, for ε-minimal elements with respect to polyhedral translated cones.

Theorem 4.3. *Let $D = D(A, b) \subseteq \mathbb{R}^m$ be a polyhedral set. Then*

$$A[\mathrm{MIN}(Y, D)] \subseteq \mathrm{MIN}(A[Y], \mathbb{R}^p_{\geqq b}),$$

where $\mathbb{R}^p_{\geqq b} := \mathbb{R}^p_{\geqq} + b = \{w \in \mathbb{R}^p : w \geqq b\}$. If $\operatorname{Ker} A = \{0\}$, then

$$\mathrm{MIN}(A[Y], \mathbb{R}^p_{\geqq b}) \subseteq A[\mathrm{MIN}(Y, D)].$$

Proof. Let $\hat{u} \in A[\mathrm{MIN}(Y, D)]$ with $\hat{u} = A\hat{y}$, $\hat{y} \in \mathrm{MIN}(Y, D)$, i.e., there do not exist $y \in Y, d \in D, d \neq 0$ such that $\hat{y} = y + d$. Now suppose by contradiction that $\hat{u} \notin \mathrm{MIN}(A[Y], \mathbb{R}^p_{\geqq b})$, then $\hat{u} \geqq u + b$ for some $u = Ay \in A[Y], u \neq \hat{u}$ and thus $y \neq \hat{y}$. It follows that

$$\hat{u} - u = A\hat{y} - Ay = A(\hat{y} - y) \geqq b,$$

and setting $d = \hat{y} - y$ gives $\hat{y} = y + d$, $d \in D$, $d \neq 0$ in contradiction to the above. For the opposite direction, let $\hat{u} \in \mathrm{MIN}(A[Y], \mathbb{R}^p_{\geqq b})$ with $\hat{u} = A\hat{y}, \hat{y} \in Y$. Then there does not exist $u = Ay \in A[Y], Ay \neq A\hat{y}$ such that $A\hat{y} \geqq Ay + b$, or $A(\hat{y} - y) \geqq b$. Now suppose by contradiction that $\hat{u} \notin A[\mathrm{MIN}(Y, D)]$, or $\hat{y} \notin \mathrm{MIN}(Y, D)$. Then there exist $y \in Y, d \in D, d \neq 0$ such that $\hat{y} = y + d$ and hence

$$A(\hat{y} - y) = Ad \geqq b$$

with $A\hat{y} \neq Ay$ if $\operatorname{Ker} A = \{0\}$. This yields the contradiction. □

Proposition 4.3 provides an alternative condition to guarantee equality in Theorem 4.3.

Proposition 4.3. *Let $D = D(A, b) \subseteq \mathbb{R}^m$ be a polyhedral set with $b \notin -\mathbb{R}^p_{\geqq}$. Then*

$$A[\mathrm{MIN}(Y, D)] = \mathrm{MIN}(A[Y], \mathbb{R}^p_{\geqq b}).$$

Proof. The inclusion $A[\mathrm{MIN}(Y, D)] \subseteq \mathrm{MIN}(A[Y], \mathbb{R}^p_{\geqq b})$ follows as in Theorem 4.3. For the reversed inclusion the final contradiction follows from

$$A(\hat{y} - y) = Ad \geqq b$$

since $b \notin -\mathbb{R}^p_{\geqq}$ implies that $Ad \neq 0$ and hence $A\hat{y} \neq Ay$. This completes the proof. □

Following exactly as before, we can prove the corresponding result for weakly minimal elements.

Theorem 4.4. *Let $D = D(A, b) \subseteq \mathbb{R}^m$ be a polyhedral set. Then*

$$A[\mathrm{WMIN}(Y, D)] = \mathrm{WMIN}(A[Y], \mathbb{R}^p_{\geqq b}).$$

From here, we now can use Proposition 2.1 to relate the notion of ε-minimal elements with respect to a polyhedral cone C to minimal elements with respect to a polyhedral translated cone D with translation vector ε. Based thereon, we then derive results corresponding to Theorems 4.1 and 4.2 for ε-minimal elements. In order to avoid pathological results and maintain notational clarity, we assume that $\varepsilon \neq 0$ throughout the remaining part of this section.

Lemma 4.1. *Given a matrix $A \in \mathbb{R}^{p\times m}$ and a vector $\varepsilon \in \mathbb{R}^m$, set $b = A\varepsilon$ and let $C = D(A)$ be the polyhedral cone and $D = D(A,b)$ be the polyhedral set implied by A and b. Then*

$$\mathrm{MIN}(Y, C, \varepsilon) = \mathrm{MIN}(Y, D) \text{ and } \mathrm{WMIN}(Y, C, \varepsilon) = \mathrm{WMIN}(Y, D).$$

Proof. From Theorem 3.1, first observe that under the given assumptions $D = C_\varepsilon$, i.e., the polyhedral set D describes the polyhedral translated cone C_ε. Then apply Proposition 2.1 to conclude with the result. □

Theorem 4.5. *Let $C = D(A) \subseteq \mathbb{R}^m$ be a polyhedral cone with matrix $A \in \mathbb{R}^{p\times m}$ and $\varepsilon \in \mathbb{R}^m$ a vector. Then*

$$A[\mathrm{MIN}(Y, C, \varepsilon)] \subseteq \mathrm{MIN}(A[Y], \mathbb{R}^p_{\geqq b}),$$

where $b = A\varepsilon$. If $\mathrm{Ker}\, A = \{0\}$*, then*

$$\mathrm{MIN}(A[Y], \mathbb{R}^p_{\geqq b}) \subseteq A[\mathrm{MIN}(Y, C, \varepsilon)].$$

In any case

$$A[\mathrm{WMIN}(Y, C, \varepsilon)] = \mathrm{WMIN}(A[Y], \mathbb{R}^p_{\geqq b}).$$

Proof. From Lemma 4.1 we have that $\mathrm{MIN}(Y, C, \varepsilon) = \mathrm{MIN}(Y, D)$ and $\mathrm{WMIN}(Y, C, \varepsilon) = \mathrm{WMIN}(Y, D)$, where $b = A\varepsilon$ and $D = D(A, b)$ is the translated polyhedral cone implied by A and $b = A\varepsilon$. Thus we obtain that $A[\mathrm{MIN}(Y, C, \varepsilon)] = A[\mathrm{MIN}(Y, D)]$ and $A[\mathrm{WMIN}(Y, C, \varepsilon)] = A[\mathrm{WMIN}(Y, D)]$, and now Theorems 4.3 and 4.4 give the result. □

Finally, we collect two further corollaries to Theorem 4.5.

Corollary 4.2. *Let $C = D(A) \subseteq \mathbb{R}^m$ be a pointed polyhedral cone. Then*

$$A[\mathrm{MIN}(Y, C, \varepsilon)] = \mathrm{MIN}(A[Y], \mathbb{R}^p_{\geqq b}).$$

Proof. Use Theorem 4.5 and Corollary 3.1. □

Alternatively, we obtain the same result from Proposition 4.3.

Corollary 4.3. *If $b = A\varepsilon \notin -\mathbb{R}^p_{\geqq}$, then*

$$A[\mathrm{MIN}(Y, C, \varepsilon)] = \mathrm{MIN}(A[Y], \mathbb{R}^p_{\geqq b}).$$

5 Maximizing over the set of epsilon-minimal elements

Over the years, many authors have shown interest in the problem of optimizing over the efficient set of a multiobjective programming problem, see Benson and Sayin (1994), Tu (2000) and Jorge (2005), among others. Here we consider the particular problem of maximizing over the set of ε-minimal elements while retaining to the same underlying ordering cone C. Motivation was provided by Engau and Wiecek (2005a,b) who described applications of multiobjective programming problems which require the generation of suboptimal solutions in practical decision making situations.

Now let Z be a topological real linear space, $Y \subseteq Z$ be a given set and $C \subseteq Z$ be a given ordering cone. Let $\varepsilon \in C^\circ$ be given and as before denote by $\mathrm{MIN}(Y, C, \varepsilon)$ and $\mathrm{WMIN}(Y, C, \varepsilon)$ the sets of all (weakly) ε-minimal elements of Y with respect to C.

Analogously to the definition of minimal elements, we first define the set of (weakly) maximal elements for a set $X \subseteq Z$.

Definition 5.1. An element $z \in Z$ is called a *maximal element* of the set X with respect to the cone C if $z \in X$ and if there does not exist a point $x \in X$ with $x \geq_C z$, or equivalently

$$X \cap (z + C^\circ) = \emptyset$$

The set of all maximal elements of X with respect to the cone C is denoted by $\mathrm{MAX}(X, C)$. The set of *weakly* maximal elements $\mathrm{WMAX}(X, C)$ is defined as the set of all elements $z \in X$ for which there does not exist a point $x \in X$ with $x >_C z$,

$$X \cap (z + \mathrm{int}\, C^\circ) = \emptyset,$$

or $\mathrm{WMAX}(X, C) = \mathrm{MAX}(X, \mathrm{int}\, C)$, equivalently.

Remark 5.1. Using Definition 5.1 together with Definitions 2.3 and 2.4, it is easily verified that $\mathrm{MAX}(X, C) = \mathrm{MIN}(X, -C)$ and $\mathrm{WMAX}(X, C) = \mathrm{WMIN}(X, -C)$.

Now the problem of interest is to identify the set of (weakly) maximal elements within the set of (weakly) ε-minimal elements, that is to find

$$\mathrm{WMAX}(\mathrm{WMIN}(Y, C, \varepsilon), C).$$

Although the underlying set of weakly ε-minimal elements is in general unknown and thus an actual optimization over this set in practice not possible, we can provide an alternative characterization of (at least) a subset of the above set of interest.

Definition 5.2. Let $Y \subseteq Z$ be a set, $C \subseteq Z$ be a cone and $\varepsilon \in C^\circ$ be a vector. Then

$$\mathrm{MIN}_\varepsilon(Y, C) := (\mathrm{MIN}(Y, C) + \varepsilon) \cap Y \text{ and } \mathrm{WMIN}_\varepsilon(Y, C) := (\mathrm{WMIN}(Y, C) + \varepsilon) \cap Y.$$

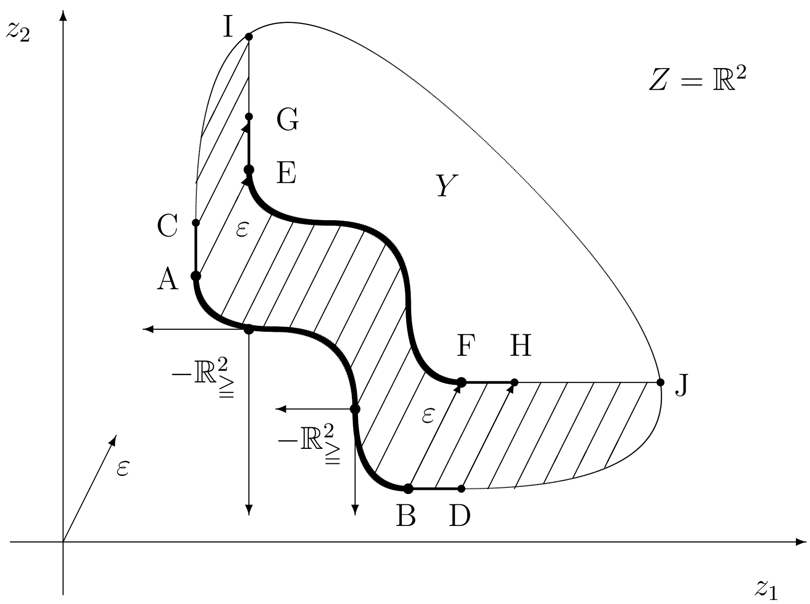

Figure 1: Relationships between different domination concepts

Figure 1 illustrates the relationships between the different sets defined for the depicted set Y as subset of $Z = \mathbb{R}^2$. The set $\mathrm{MIN}(Y, \mathbb{R}^2_{\geqq})$ of minimal elements with respect to the Pareto cone $\mathbb{R}^2_{\geqq}$ is given by the curve connecting points A and B, while the set $\mathrm{WMIN}(Y, \mathbb{R}^2_{\geqq})$ of all weakly minimal elements corresponds to the extended curve from C to D. Translating these sets by the specified translation vector ε yields the sets $\mathrm{MIN}_\varepsilon(Y, \mathbb{R}^2_{\geqq})$ and $\mathrm{WMIN}_\varepsilon(Y, \mathbb{R}^2_{\geqq})$ as the curves connecting points E and F, and G and H, respectively. Finally, the set $\mathrm{MIN}(Y, \mathbb{R}^2_{\geqq}, \varepsilon)$ of ε-minimal elements is the shaded area enclosed by all marked points without the curve connecting I, G, E, F, H and J, while the set $\mathrm{WMIN}(Y, \mathbb{R}^2_{\geqq}, \varepsilon)$ of weakly ε-minimal elements includes the curve from I to J. Maximization over this set of weakly minimal elements with respect to the weak Pareto cone $\mathbb{R}^2_{>}$ then yields the set $\mathrm{WMAX}(\mathrm{WMIN}(Y, \mathbb{R}^2_{\geqq}, \varepsilon), \mathbb{R}^2_{\geqq})$ and consists of all points along the curve connecting I and J.

The above discussion of Figure 1 suggests that the set $\mathrm{WMAX}(\mathrm{WMIN}(Y, C, \varepsilon), C)$ contains the set of those weakly ε-minimal elements that are not minimal and, moreover, that this set has the sets $\mathrm{MIN}_\varepsilon(Y,C)$ and $\mathrm{WMIN}_\varepsilon(Y,C)$ as (possibly proper) subsets. The verification of this intuitive conjecture, however, requires some further preparation.

Lemma 5.1. *Let $\hat{y} \in \mathrm{WMIN}(Y,C)$ be a weakly minimal element and let $\tilde{y} \in Y$.*

(i) If $\tilde{y} <_C \hat{y} + \varepsilon$, then $\tilde{y}$ is an ε-minimal element, $\tilde{y} \in \mathrm{MIN}(Y,C,\varepsilon)$.

(ii) If $\tilde{y} \leqq_C \hat{y} + \varepsilon$, then $\tilde{y}$ is a weakly ε-minimal element, $\tilde{y} \in \mathrm{WMIN}(Y,C,\varepsilon)$.

Proof. We know $\hat{y} \in \mathrm{WMIN}(Y,C)$, so there does not exist $y \in Y$ such that $y <_C \hat{y}$, or $y <_C (\hat{y} + \varepsilon) - \varepsilon$. Then we have for (i), if $\tilde{y} <_C \hat{y} + \varepsilon$, that there does not exist $y \in Y$ such that $y \leqq_C \tilde{y} - \varepsilon$, and hence $\tilde{y}$ is an ε-minimal element. Similarly for (ii), if $\tilde{y} \leqq_C \hat{y} + \varepsilon$, we obtain that there does not exist $y \in Y$ such that $y <_C \tilde{y} - \varepsilon$, and hence $\tilde{y}$ is a weakly ε-minimal element. □

Now the main result consists of two parts.

Theorem 5.1. *Let $Y \subseteq Z$ a set, $C \subseteq Z$ a cone and $\varepsilon \in C^\circ$ be given. Then*

$$\mathrm{MIN}_\varepsilon(Y,C) \subseteq \mathrm{WMIN}_\varepsilon(Y,C) \subseteq \mathrm{WMIN}(Y,C,\varepsilon) \setminus \mathrm{MIN}(Y,C,\varepsilon).$$

Proof. The first inclusion follows directly from Definition 5.2. For the second, we obtain that $\mathrm{WMIN}_\varepsilon(Y,C) \subseteq \mathrm{WMIN}(Y,C,\varepsilon)$ from Lemma 5.1(ii), and then $\mathrm{WMIN}_\varepsilon(Y,C) \cap \mathrm{MIN}(Y,C,\varepsilon) = \emptyset$ follows since $\varepsilon \in C^\circ$ is a dominated direction. □

While Theorem 5.1 follows under quite general conditions, the second part requires some further assumptions, namely convexity and pointedness of the underlying cone C.

Theorem 5.2. *Let $Y \subseteq Z$ a set, $C \subseteq Z$ an ordering cone and $\varepsilon \in C^\circ$ be given. Then*

$$\mathrm{WMIN}(Y,C,\varepsilon) \setminus \mathrm{MIN}(Y,C,\varepsilon) \subseteq \mathrm{WMAX}(\mathrm{WMIN}(Y,C,\varepsilon),C).$$

Proof. Let $\hat{y} \in \mathrm{WMIN}(Y,C,\varepsilon) \setminus \mathrm{MIN}(Y,C,\varepsilon)$, then there do not exist $y \in Y, d \in \operatorname{int} C$ ($d \neq -\varepsilon$ as $\varepsilon \in C^\circ$ where C is pointed) such that $\hat{y} = y + d + \varepsilon$, but $\hat{y} = \bar{y} + \bar{d} + \varepsilon$ for some $\bar{y} \in Y$ and $\bar{d} \in C$. We have to show that $\hat{y} \in \mathrm{WMAX}(\mathrm{WMIN}(Y,C,\varepsilon),C)$, or that there do not exist $y \in \mathrm{WMIN}(Y,C,\varepsilon), d \in \operatorname{int} C, d \neq 0$ such that $\hat{y} = y - d$. Therefore, suppose by contradiction that $\hat{y} = y - d$ with $y \in Y$, $d \in \operatorname{int} C$, $d \neq 0$ and show that $y \notin \mathrm{WMIN}(Y,C,\varepsilon)$. From the above, we have $\hat{y} = \bar{y} + \bar{d} + \varepsilon$ with $\bar{y} \in Y$ and $\bar{d} \in C$, and thus we obtain $y = \bar{y} + \bar{d} + d + \varepsilon$ with $\bar{y} \in Y$ and $\bar{\bar{d}} = \bar{d} + d \in \operatorname{int} C$ as C is convex, with $\bar{\bar{d}} \neq -\varepsilon$ again since $\varepsilon \in C^\circ$ with C pointed. Hence, $y \notin \mathrm{WMIN}(Y,C,\varepsilon)$, and the proof is complete.
□

Combining Theorem 5.1 and Theorem 5.2 now implies the concluding result.

Corollary 5.1. *Let $Y \subseteq Z$ a set, $C \subseteq Z$ an ordering cone and $\varepsilon \in C^\circ$ be given. Then*

$$\mathrm{MIN}_\varepsilon(Y, C) \subseteq \mathrm{WMIN}_\varepsilon(Y, C) \subseteq \mathrm{WMIN}(Y, C, \varepsilon)\backslash\mathrm{MIN}(Y, C, \varepsilon) \subseteq \mathrm{WMAX}(\mathrm{WMIN}(Y, C, \varepsilon), C).$$

An application of this result is given in Engau and Wiecek (2005a).

6 Conclusions

In this paper, we study approximate solutions in real-vector optimization and propose the formal framework of cones for their characterization.

It is well established that minimal elements among a set of vectors can be defined with respect to some partial order induced upon introduction of an underlying ordering cone. In direct generalization, approximate solutions are defined as epsilon-minimal elements and give rise to a number of interesting and previously unaddressed research questions.

At first, realizing that the cones describing epsilon-minimal elements must be shifted from the origin, preliminary results on translated cones are reported and deal primarily with the uniqueness of their representation. As a special case of interest, we investigate the possibility of describing a given polyhedral translated cone by a system of linear inequalities and, from a reversed point of view, derive conditions for which a given system defines a polyhedral translated cone. We also mention how these results relate to and extend certain concepts from linear algebra and convex analysis.

In spite of the theoretical analogy of allowing arbitrary ordering cones, most practical applications adopt the Pareto order and hence motivate efforts to find relationships between minimal elements with respect to the Pareto and more general ordering cones. Very satisfying results have been established for polyhedral cones for which the set of minimal elements can be transformed into a set of minimal elements with respect to a Pareto cone. Restricted to minimal elements with respect to polyhedral cones, we derive the corresponding results for minimal elements with respect to arbitrary polyhedral sets and, as a special case, for epsilon-minimal elements with respect to polyhedral translated cones.

We hope that these results enable and stimulate further investigation of approximate solutions within the rigorous formal framework proposed in this paper.

As one such particular research question of interest and directly motivated by parallel methodological developments in Engau and Wiecek (2005a), we finally address the general problem of optimizing over the set of minimal or epsilon-minimal elements, or more precisely, the identification of (weakly) maximal elements among the set of (weakly) epsilon-minimal elements. Facing the challenge that for most practical situations the latter is not given explicitly and hence that actual optimization is in general not possible, we offer an alternative characterization of these maximal elements which as a by-product leads to an insightful result relating epsilon-minimal and weakly epsilon-minimal elements.

While parts of this research have already been put in practice for our personal work, we believe that some of the results presented in this paper also continue to be of interest and benefit to others for further use and study of approximate solutions in real-vector optimization problems.

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