

A Quantile Implementation of the EM Algorithm and Applications to Parameter Estimation with Interval Data

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Abstract

The expectation-maximization (EM) algorithm is a powerful computational technique for finding the maximum likelihood estimates for parametric models when the data are not fully observed. The EM is best suited for situations where the s -expectation in each E-step and the maximization in each M-step are straightforward. A difficulty with the implementation of the EM algorithm is that each E-step requires the integration of the posterior log-likelihood function. This can be overcome by the Monte Carlo EM (MCEM) algorithm. This MCEM uses a random sample to estimate the integral in each E-step. But this MCEM converges very slowly to the true integral, which causes computational burden and instability. In this paper we present a quantile implementation of the expectation-maximization (QEM) algorithm. This proposed method shows a faster convergence and greater stability. The performance of the proposed method and its applications are numerically illustrated through Monte Carlo simulations and several examples.

Index Terms

EM algorithm, Grouped data, Incomplete data, Interval data, Maximum likelihood, MCEM, Missing data, Quantile.

ACRONYMS¹

BLUE	best linear unbiased estimate
EM	expectation-maximization
MCEM	Monte Carlo implementation of EM
QEM	quantile implementation of EM

¹The singular and plural of an acronym are always spelled the same.

MLE	maximum likelihood estimate
MSE	mean square error
SRE	simulated relative efficiency
Cdf	cumulative distribution function
pdf	probability density function
iid	s-independent and identically distributed

NOTATION

δ_i	censoring indicator variable
\mathbf{y}	(y_1, \dots, y_m) : observed data
\mathbf{z}	(z_{m+1}, \dots, z_n) : missing data
$\mathbf{z}^{(k)}$	$(z_{m+1,k}, \dots, z_{n,k})$
$\boldsymbol{\theta}$	vector of parameters
$\boldsymbol{\theta}^{(s)}$	estimate of $\boldsymbol{\theta}$ at the s th EM sequence
$\phi(\cdot)$	pdf of the standard s -normal distribution
$\Phi(\cdot)$	Cdf of the standard s -normal distribution
$L^c(\boldsymbol{\theta} \mathbf{x})$	$\prod_{i=1}^n f(x_i)$: complete-data likelihood
$L(\boldsymbol{\theta} \mathbf{y})$	$\int L^c(\boldsymbol{\theta} \mathbf{y}, \mathbf{z})d\mathbf{z}$: observed-data likelihood
$p(\mathbf{z} \boldsymbol{\theta}, \mathbf{y})$	pdf of \mathbf{z} conditional on $\boldsymbol{\theta}$ and \mathbf{y}
$Q(\boldsymbol{\theta} \boldsymbol{\theta}^{(s)})$	s -expected posterior log-likelihood
$\hat{Q}(\boldsymbol{\theta} \boldsymbol{\theta}^{(s)})$	Monte Carlo (or QEM) approximation of $Q(\boldsymbol{\theta} \boldsymbol{\theta}^{(s)})$

I. INTRODUCTION

THE ANALYSIS of lifetime or failure time data has been of considerable interest in many branches of statistical applications including reliability engineering, biological sciences, etc. In reliability analysis, censoring is very common due to inherent limitations, or time & cost considerations on experiments. The data are said to be censored when, for observations,

only a lower or upper bound on lifetime is available. Thus, the problem of parameter estimation from censored samples is very important for reliability analysis. To obtain the parameter estimate, some numerical optimization methods are required to find the MLE. However, ordinary numerical methods such as the Gauss-Seidel iterative method and the Newton-Raphson gradient method may be very ineffective for complicated likelihood functions and these methods can be sensitive to the choice of starting values used. In this paper, unless otherwise specified, “MLE” refers to the estimate obtained by direct maximization of the likelihood function.

For censored sample problems, several approximations of the MLE and the best linear unbiased estimate (BLUE) have been studied instead of direct calculation of the MLE. For example, the problem of parameter estimation from censored samples has been treated by several authors. Gupta [1] has studied the MLE and provided the BLUE for Type-I and Type-II censored samples from an s -normal distribution. Govindarajulu [2] has derived the BLUE for a symmetrically Type-II censored sample from a Laplace distribution for sample size up to $N = 20$. Balakrishnan [3] has given an approximation of the MLE of the scale parameter of the Rayleigh distribution with censoring. Sultan [4] has given an approximation of the MLE for a Type-II censored sample from an s -normal distribution. Balakrishnan [5] has given the BLUE for a Type-II censored sample from a Laplace distribution. The BLUE needs the coefficients a_i & b_i , which were tabulated in [5], but the table is provided only for sample size up to $N = 20$. In addition, the approximate MLE and the BLUE are not guaranteed to converge to the preferred MLE. The methods above are also restricted only to Type-I or Type-II (symmetric) censoring for sample size up to $N = 20$ only.

These aforementioned deficiencies can be overcome by the EM algorithm. In many practical problems, however, the implementation of the ordinary EM algorithm is very difficult. Thus, Wei & Tanner [6] [7] proposed to use the MCEM when the EM algorithm is not available. However, the MCEM algorithm presents a serious computational burden because in the E-step of the MCEM algorithm, Monte Carlo sampling is used to obtain the s -expected posterior log-likelihood. Thus, it is natural to look for a better method. The proposed method using the quantile function instead of Monte Carlo sampling has greater stability and also much faster convergence properties with smaller sample sizes.

Moreover, in many industrial experiments, more general incomplete observations are often encountered along with the fully observed data, where incompleteness arises due to censoring, grouping, quantal responses, etc. The general form of an incomplete observation is of interval form. That is, a lifetime of a subject X_i is specified as $a_i \leq X_i \leq b_i$. In this paper, we deal with computing the MLE for this general form of incomplete data using the EM algorithm and its variants, MCEM and QEM. This interval form can handle right-censoring, left-censoring, quantal responses and fully-observed observations. The proposed method includes the aforementioned existing methods as a special case. This proposed method can also handle the data from intermittent inspection which are referred to as *grouped data* which provide only the number of failures in each inspection period. Seo & Yum [8] and Shapiro & Gulati [9] have given an approximation of the MLE under the exponential distribution only. Nelson [10] described maximum likelihood methods, but the MLE should be obtained by ordinary numerical methods. The proposed method enables us to obtain the MLE through the EM or QEM sequences under a variety of distribution models.

The rest of the paper is organized as follows. In Section II, we introduce the basic concept of the EM & MCEM algorithms. In Section III, we present the quantile implementation of the EM algorithm. In Section IV, we provide the likelihood construction with interval data and its EM implementation issues. Section V deals with the parameter estimation procedure of exponential, s -normal, Laplace, Rayleigh, and Weibull distributions with interval data. In order to compare the performance of the proposed method with the EM & MCEM methods, Monte Carlo simulation study is presented in Section VI followed up with examples of various applications in Section VII. This paper ends with concluding remarks in Section VIII.

II. THE EM & MCEM ALGORITHMS

In this section, we give a brief introduction of the EM & MCEM algorithms. The EM algorithm is a powerful computational technique for finding the MLE of parametric models when there is no closed-form MLE, or the data are incomplete. The EM algorithm was introduced by Dempster, Laird, and Rubin [11] to overcome the above difficulties. For more details about this

EM algorithm, good references are Little & Rubin [12], Tanner [13], Schafer [14], and Hunter & Lange [15].

When the closed-form MLE from the likelihood function is not available, numerical methods are required to find the maximizer (*i.e.*, MLE). However, ordinary numerical methods such as the Gauss-Seidel iterative method and the Newton-Raphson gradient method may be very ineffective for complicated likelihood functions and these methods can be sensitive to the choice of starting values used. In particular, if the likelihood function is flat near its maximum, the methods will stop before reaching the maximizer. These potential problems can be overcome by using the EM algorithm.

The EM algorithm consists of two iterative steps: (i) Expectation step (E-step) and (ii) Maximization step (M-step). The advantage of the EM algorithm is that it solves a difficult incomplete-data problem by constructing two easy steps. The E-step of each iteration only needs to compute the conditional s -expectation of the log-likelihood with respect to the incomplete data given the observed data. The M-step of each iteration only needs to find the maximizer of this s -expected log-likelihood constructed in the E-step, which only involves handling “complete-data” log-likelihood function. Thus, the EM sequences repeatedly maximize the posterior log-likelihood function of the complete data given the incomplete data instead of maximizing the potentially complicated likelihood function of the incomplete data directly. An additional advantage of this method compared to other direct optimization techniques is that it is very simple and it converges reliably. In general, if it converges, it converges to a local maximum. Hence in the case of the unimodal & concave likelihood function, the EM sequences converge to the global maximizer from any starting value. We can employ this methodology for parameter estimation from interval data because interval data models are special cases of incomplete (missing) data models.

Here, we give a brief introduction of the EM & MCEM algorithms. Denote the vector of unknown parameters by $\boldsymbol{\theta} = (\theta_1, \dots, \theta_p)$. Then the complete-data likelihood is

$$L^c(\boldsymbol{\theta}|\mathbf{x}) = \prod_{i=1}^n f(x_i),$$

where $\mathbf{x} = (x_1, \dots, x_n)$ and we denote the observed part of \mathbf{x} by $\mathbf{y} = (y_1, \dots, y_m)$ and the incomplete (missing) part by $\mathbf{z} = (z_{m+1}, \dots, z_n)$.

- **E-step:** Compute $Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)})$
where $Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)}) = \int \log L^c(\boldsymbol{\theta}|\mathbf{y}, \mathbf{z}) p(\mathbf{z}|\mathbf{y}, \boldsymbol{\theta}^{(s)}) d\mathbf{z}$.
- **M-step:** Find $\boldsymbol{\theta}^{(s+1)}$
which maximizes $Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)})$ in $\boldsymbol{\theta}$.

In some problems, the implementation of the E-step is difficult. Wei & Tanner [6] [7] propose to use the MCEM to avoid this difficulty. The E-step is approximated by using Monte Carlo integration. Simulating z_{m+1}, \dots, z_n from the conditional distribution $p(\mathbf{z}|\mathbf{y}, \boldsymbol{\theta}^{(s)})$, we can approximate the s -expected posterior log-likelihood as follows:

$$\hat{Q}(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)}) = \frac{1}{K} \sum_{k=1}^K \log L^c(\boldsymbol{\theta}|\mathbf{y}, \mathbf{z}^{(k)}),$$

where $\mathbf{z}^{(k)} = (z_{m+1,k}, \dots, z_{n,k})$. This method is called the Monte Carlo EM (MCEM) algorithm. Major drawback to MCEM is that it is very slow because it requires a large sample size in order to possess stable convergence properties. This problem can be overcome by the proposed method using the quantile function.

III. THE QUANTILE IMPLEMENTATION OF THE EM ALGORITHM

The key idea underlying the quantile implementation of the EM algorithm can be easily illustrated by the following example. The data in the example were first presented by Freireich *et al.* [16] and have since then been used very frequently for illustration in the reliability engineering and survival analysis literature including Leemis [17] and Cox & Oakes [18].

A. Example: Length of Remission of Leukemia Patients

An experiment is conducted to determine the effect of a drug named 6-mercaptopurine (6-MP) on leukemia remission times. A sample of size $n = 21$ leukemia patients is treated with 6-MP and the times of remission are recorded. There are $m = 9$ individuals for whom the remission time is fully observed, and the remission times for the remaining 12 individuals are randomly

censored on the right. Letting a plus (+) denote a censored observation, the remission times (in weeks) are

$$\begin{array}{ccccccc} 6 & 6 & 6 & 6^+ & 7 & 9^+ & 10 \\ 10^+ & 11^+ & 13 & 16 & 17^+ & 19^+ & 20^+ \\ 22 & 23 & 25^+ & 32^+ & 32^+ & 34^+ & 35^+. \end{array}$$

Using an exponential model, we can obtain the complete likelihood function and the conditional pdf

$$\log L^c(\boldsymbol{\theta}|\mathbf{y}, \mathbf{z}) = -n \log \sigma - \frac{1}{\sigma} \sum_{i=1}^m y_i - \frac{1}{\sigma} \sum_{i=m+1}^n z_i,$$

$$p(\mathbf{z}|\mathbf{y}, \boldsymbol{\theta}^{(s)}) = \prod_{i=m+1}^n p_{z_i}(z_i|\sigma^{(s)}) = \prod_{i=m+1}^n \frac{1}{\sigma^{(s)}} e^{-(z_i - R_i)/\sigma^{(s)}}, \quad z_i > R_i,$$

where R_i is a right-censoring time of test unit i . Using the above conditional pdf, we have the s -expected posterior log-likelihood

$$\begin{aligned} Q(\sigma|\sigma^{(s)}) &= \int \log L^c(\sigma|\mathbf{y}, \mathbf{z}) p(\mathbf{z}|\mathbf{y}, \boldsymbol{\theta}^{(s)}) d\mathbf{z} \\ &= -n \log \sigma - \frac{1}{\sigma} \sum_{i=1}^m y_i - \frac{1}{\sigma} \sum_{i=m+1}^n \int z_i p_{z_i}(z_i|\sigma^{(s)}) dz_i \\ &= -n \log \sigma - (n - m) \frac{\sigma^{(s)}}{\sigma} - \frac{1}{\sigma} \sum_{i=1}^m y_i - \frac{1}{\sigma} \sum_{i=m+1}^n R_i. \end{aligned}$$

Then the Monte Carlo approximation of the s -expected posterior log-likelihood is given by

$$\hat{Q}(\sigma|\sigma^{(s)}) = -n \log \sigma - \frac{1}{\sigma} \sum_{i=1}^m y_i - \frac{1}{\sigma} \frac{1}{K} \sum_{k=1}^K \sum_{i=m+1}^n z_{i,k}.$$

where a random sample $z_{i,k}$ is from $p_{z_i}(z_i|\sigma^{(s)}) = \frac{1}{\sigma^{(s)}} e^{-(z_i - R_i)/\sigma^{(s)}}$. In the Monte Carlo approximation, the term $\int z_i p_{z_i}(z_i|\sigma^{(s)}) dz_i$ is approximated by

$$\int z_i p_{z_i}(z_i|\sigma^{(s)}) dz_i \approx \frac{1}{K} \sum_{k=1}^K z_{i,k}.$$

This approximation can be improved by using the quantile function (inverse Cdf). For the conditional pdf $p_{z_i}(z_{i,k}|\sigma^{(s)})$, the quantiles of ξ_k , denoted by $q_{i,k}$, are given by

$$q_{i,k} = F^{-1}(\xi_k|R_i, \sigma^{(s)}) = R_i - \sigma^{(s)} \log(1 - \xi_k),$$

for $0 < \xi_k < 1$. We can choose ξ_k from any of the fractions, k/K , $k/(K+1)$, $(k - \frac{1}{2})/K$, etc. Using the quantile function, we have the following approximation

$$\int z_i p_{z_i}(z_i|\sigma^{(s)}) dz_i \approx \frac{1}{K} \sum_{k=1}^K q_{i,k}.$$

It is noteworthy that that a random sample $z_{i,k}$ in the Monte Carlo approximation is usually generated by using the above quantile function with ξ_k from a random sample having a uniform distribution between 0 and 1.

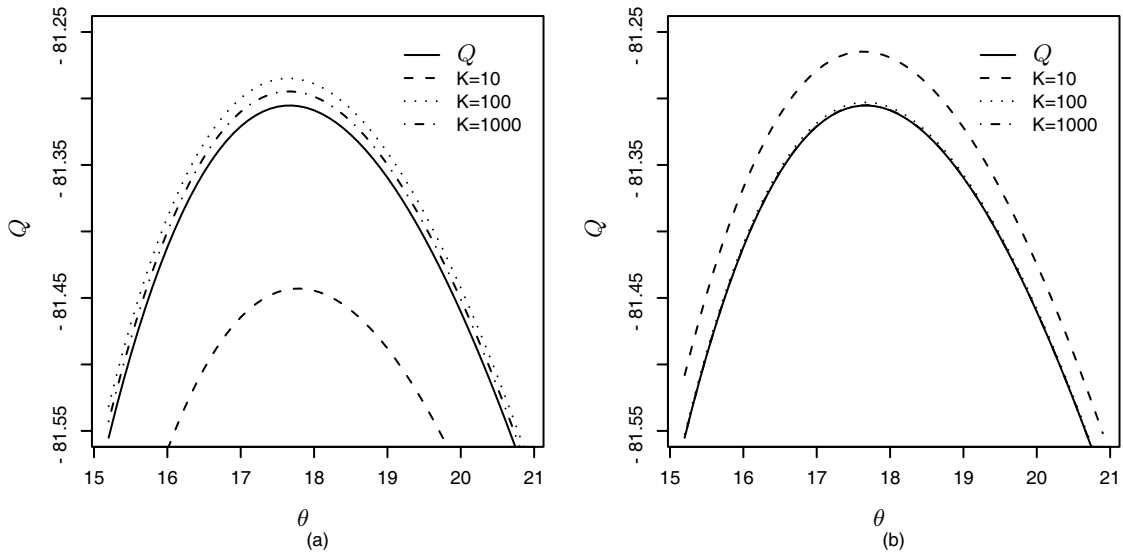


Fig. 1. The s -expected posterior log-likelihood functions and approximations. (a) Monte Carlo approximations. (b) Quantile approximations.

Fig. 1 presents the MCEM & QEM approximations of s -expected posterior log-likelihood functions for $K = 10$ (dashed curve), 100 (dotted curve) and 1000 (dot-dashed curve) at the first step ($s = 1$), along with the exact s -expected posterior log-likelihood (solid curve). The MCEM & QEM algorithms were run with starting value $\sigma^{(0)} = 1$. As can be seen in the figure, the MCEM & QEM both successfully converge to the s -expected posterior log-likelihood as K

gets larger. Note that the QEM is much closer to the true s -expected posterior log-likelihood for smaller values of K .

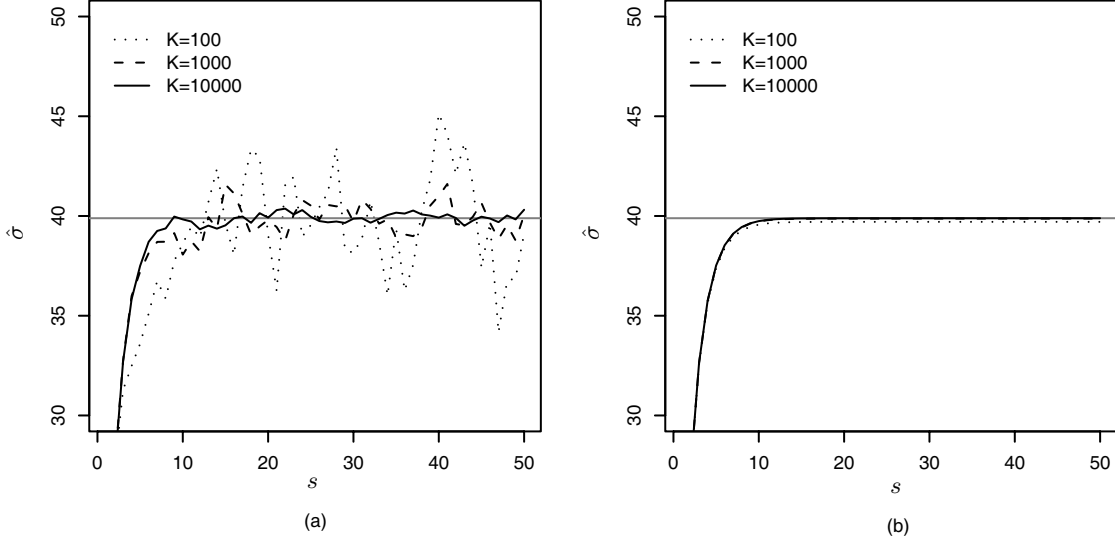


Fig. 2. Successive parameter estimates using (a) the MCEM and (b) the QEM. The horizontal solid lines indicate the MLE ($\hat{\sigma} = 39.89$).

Fig. 2 displays the iterations of the EM & QEM sequences in the example from the starting value $\sigma^{(0)} = 1$. The horizontal solid lines indicate the MLE ($\hat{\sigma} = 39.89$). The figures clearly show that the QEM is stable and converges very fast to the MLE. Even with very small quantile sizes, the QEM outperforms the MCEM. It should be noted that the QEM with $K = 100$ performs better than the MCEM with $K = 10,000$.

B. Convergence Properties of the MCEM & QEM Algorithms

Another way to view the quantile implementation idea is by looking at the Riemann-Stieltjes integral. For simplicity of presentation, we consider the case where \mathbf{z} is one-dimensional. Denote $h(\boldsymbol{\theta}, z) = \log L^c(\boldsymbol{\theta}|\mathbf{y}, z)$. Let us consider a following Riemann-Stieltjes sum,

$$\frac{1}{K} \sum_{k=1}^K h(\boldsymbol{\theta}, F^{-1}(\xi_k)).$$

In the limit as $K \rightarrow \infty$, we have

$$\int h(\boldsymbol{\theta}, F^{-1}(\xi)) d\xi.$$

Using a change-of-variable integration technique with $z = F^{-1}(\xi)$, we have

$$\int h(\boldsymbol{\theta}, z)dF(z) = \int h(\boldsymbol{\theta}, z)f(z)dz.$$

Hence the quantile approximation of the s -expectation posterior log-likelihood is a Riemann-Stieltjes sum which converges to the true integration. With $\xi_k = (k - \frac{1}{2})/K$, this sum is also known as the extended midpoint rule and it is accurate to the order of $O(1/K^2)$ (see Press *et al.* [19]), that is,

$$\int h(\boldsymbol{\theta}, z)f(z)dz = \frac{1}{K} \sum_{k=1}^K h(\boldsymbol{\theta}, q_k) + O\left(\frac{1}{K^2}\right),$$

where $q_k = F^{-1}(\xi_k)$.

On the other hand, the accuracy of the Monte Carlo approximation

$$\bar{h}_K = \frac{1}{K} \sum_{k=1}^K h(\boldsymbol{\theta}, z_k),$$

can be assessed as follows. By the central limit theorem, we have

$$\sqrt{K} \frac{\bar{h}_K - E(h(\boldsymbol{\theta}, Z))}{\sqrt{\text{Var}(h(\boldsymbol{\theta}, Z))}} \xrightarrow{D} N(0, 1), \quad (1)$$

and this is accurate to the order of $O_p(1)$. It is immediate from the weak law of large numbers that $\bar{h}_K \xrightarrow{p} E(h(\boldsymbol{\theta}, Z))$. Using this and (1), we have

$$\int h(\boldsymbol{\theta}, z)f(z)dz = \frac{1}{K} \sum_{k=1}^K h(\boldsymbol{\theta}, z_k) + O_p\left(\frac{1}{\sqrt{K}}\right).$$

From this, it is clear that the QEM is much more accurate than the MCEM.

We can generalize the above result as follows. In the E-step, we replace the Monte Carlo approximation with the quantile approximation

$$\hat{Q}(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)}) = \frac{1}{K} \sum_{k=1}^K \log L^c(\boldsymbol{\theta}|\mathbf{y}, \mathbf{q}^{(k)}),$$

where $\mathbf{q}^{(k)} = (q_{m+1,k}, \dots, q_{n,k})$ with $q_{i,k} = F^{-1}(\xi_k|z_i, \boldsymbol{\theta}^{(s)})$, and ξ_k is any fraction. In this paper, we use $\xi_k = (k - \frac{1}{2})/K$.

IV. LIKELIHOOD CONSTRUCTION

In this section, we develop the likelihood functions which can be conveniently used for the EM, MCEM and QEM algorithms.

The general form of an incomplete observation is often of interval form. That is, the lifetime of a subject X_i may not be observed exactly, but is known to fall in an interval: $a_i \leq X_i \leq b_i$. This interval form includes censored, grouped, quantal-response, and fully-observed observations. For example, a lifetime is left-censored when $a_i = -\infty$ and a lifetime is right-censored when $b_i = \infty$. The lifetime is fully observed when $X_i = a_i = b_i$.

Suppose that $\mathbf{x} = (x_1, \dots, x_n)$ are observations on random variables which are independent and identically distributed and have a continuous distribution with the pdf $f(x)$ and Cdf $F(x)$. Interval data from experiments can be conveniently represented by pairs (w_i, δ_i) with $w_i = [a_i, b_i]$,

$$\delta_i = \begin{cases} 0 & \text{if } a_i < b_i \\ 1 & \text{if } a_i = b_i \end{cases} \quad \text{for } i = 1, \dots, n,$$

where δ_i is an indicator variable and a_i and b_i are lower and upper ends of interval observations of test unit i , respectively. If $a_i = b_i$, then the lifetime of the i th test unit is fully observed. Denote the observed part of $\mathbf{x} = (x_1, \dots, x_n)$ by $\mathbf{y} = (y_1, \dots, y_m)$ and the incomplete (missing) part by $\mathbf{z} = (z_{m+1}, \dots, z_n)$ with $a_i \leq z_i \leq b_i$. Denote the vector of unknown distribution parameters by $\boldsymbol{\theta} = (\theta_1, \dots, \theta_d)$. Then ignoring a normalizing constant, we have the complete-data likelihood function

$$L^c(\boldsymbol{\theta}|\mathbf{y}, \mathbf{z}) \propto \prod_{i=1}^n f(x_i|\boldsymbol{\theta}). \quad (2)$$

Integrating $L^c(\boldsymbol{\theta}|\mathbf{x})$ with respect to \mathbf{z} , we obtain the observed-data likelihood

$$L(\boldsymbol{\theta}|\mathbf{y}) \propto \int L^c(\boldsymbol{\theta}|\mathbf{y}, \mathbf{z}) d\mathbf{z} = \prod_{i=1}^m f(y_i|\boldsymbol{\theta}) \prod_{j=m+1}^n \{F(b_j|\boldsymbol{\theta}) - F(a_j|\boldsymbol{\theta})\},$$

where in general an empty product is taken to be one. Using the (w_i, δ_i) notation, we have

$$L(\boldsymbol{\theta}|\mathbf{w}, \boldsymbol{\delta}) \propto \prod_{i=1}^n f(w_i|\boldsymbol{\theta})^{\delta_i} \{F(b_i|\boldsymbol{\theta}) - F(a_i|\boldsymbol{\theta})\}^{1-\delta_i}, \quad (3)$$

where $\mathbf{w} = (w_1, \dots, w_n)$ and $\boldsymbol{\delta} = (\delta_1, \dots, \delta_n)$. Here, although we provided the likelihood function for the interval-data case, it is easily extended to more general forms of incomplete data. For more details, the reader is referred to Heitjan [20] and Heitjan & Rubin [21].

Thus, the goal is inference about $\boldsymbol{\theta}$ given the complexity of the likelihood, and the EM algorithm is a tool that can be used to accomplish this goal. Then the issue here is how to implement the EM algorithm when there are interval data in the sample. By treating the interval data as incomplete (missing) data, it is possible to write the complete-data likelihood. This treatment allows one to find the *closed-form* maximizer in the M-step. For convenience, assume that all the data are of interval form with $a_i \leq w_i \leq b_i$ and $a_i < b_i$. Then the likelihood function in (3) can be rewritten as

$$L(\boldsymbol{\theta}|\mathbf{w}) \propto \prod_{i=1}^n \{F(b_i|\boldsymbol{\theta}) - F(a_i|\boldsymbol{\theta})\}. \quad (4)$$

Then the complete-data likelihood function corresponding to (4) becomes

$$L^c(\boldsymbol{\theta}|\mathbf{z}) \propto \prod_{i=1}^n f(z_i|\boldsymbol{\theta}), \quad (5)$$

where the pdf of Z_i is given by

$$p_{z_i}(z|\boldsymbol{\theta}) = \frac{f(z|\boldsymbol{\theta})}{F(b_i|\boldsymbol{\theta}) - F(a_i|\boldsymbol{\theta})},$$

for $a_i < z < b_i$. Using this, we have the following Q -function in the E-step,

$$Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)}) = \sum_{i=1}^n \int_{a_i}^{b_i} \log f(z_i|\boldsymbol{\theta}) \cdot p_{z_i}(z_i|\boldsymbol{\theta}^{(s)}) dz_i.$$

It is worth looking at the integration when $b_i \rightarrow a_i$. For convenience, omitting the subject index i and letting $b = a + \epsilon$, we have

$$\int_a^{a+\epsilon} \log f(z|\boldsymbol{\theta}) \cdot p_z(z|\boldsymbol{\theta}^{(s)}) dz. \quad (6)$$

It follows from the integration by parts that the above integral becomes

$$\left[\log f(z|\boldsymbol{\theta}) \cdot P_z(z|\boldsymbol{\theta}^{(s)}) \right]_a^{a+\epsilon} - \int_a^{a+\epsilon} \frac{f'(z|\boldsymbol{\theta})}{f(z|\boldsymbol{\theta})} \cdot P_z(z|\boldsymbol{\theta}^{(s)}) dz, \quad (7)$$

where

$$P_z(z|\boldsymbol{\theta}^{(s)}) = \frac{F(z|\boldsymbol{\theta})}{F(a + \epsilon|\boldsymbol{\theta}^{(s)}) - F(a|\boldsymbol{\theta}^{(s)})}.$$

Applying l'Hospital rule to (6) with (7), we obtain

$$\lim_{\epsilon \rightarrow 0} \int_a^{a+\epsilon} \log f(z|\boldsymbol{\theta}) \cdot P_z(z|\boldsymbol{\theta}^{(s)}) dz = \log f(a|\boldsymbol{\theta}).$$

Hence, for full observation, we simply use the interval $[a_i, a_i]$ notation which implies $[a_i, a_i + \epsilon]$ with the limit as $\epsilon \rightarrow 0$. All kinds of data considered in this paper can be denoted by the interval-data form without the indicator variable, δ_i . Thus, we use (4) or (5) for the likelihood function or complete-data likelihood function instead of (2) or (3), respectively.

For many distributions including Weibull and Laplace distributions, it is extremely difficult or may be impossible to implement the EM algorithm with interval data. This is because, during the E-step, the Q -function does not integrate easily and this causes computational difficulties in the M-step. In order to avoid this problem, one can use MCEM algorithm [6] which reduces the difficulty in the E-step through the use of a Monte Carlo integration. As aforementioned, although it can make some problems tractable, the MCEM involves a serious computational burden and can often lead to unstable estimates. Thus, we proposed a quantile implementation of the EM algorithm which alleviates some of the computational burden of the MCEM and leads to more stable estimates.

For stopping criterion for the EM, MCEM or QEM algorithm, the algorithm stops if the changes are all relatively small compared to a given precision ϵ . As an example for s -normal distribution, the QEM algorithm stops if $|\mu^{(s+1)} - \mu^{(s)}| < \epsilon\mu^{(s+1)}$ and $|\sigma^{(s+1)} - \sigma^{(s)}| < \epsilon\sigma^{(s+1)}$ as well. In what follows, we obtain the EM (if available), MCEM, and QEM sequences for a variety of distributions, which maximize the likelihood function in (3).

V. PARAMETER ESTIMATION

In this section, we briefly provide the inferential procedure for the parameter estimation of exponential, s -normal, Laplace, Rayleigh, and Weibull distributions from random samples in interval form. For the exponential and s -normal distributions, the ordinary EM algorithm applies, so the MCEM & QEM are not needed. To compare the performance of the MCEM & QEM, however, we include the exponential and s -normal distributions although the ordinary

EM algorithms are available. For the Laplace distribution, the computation of the E-step is very complex, so either the MCEM or the QEM is more appropriate. For the Rayleigh and Weibull distributions, the calculation of the integration in the E-step does not have a closed form. So, it is not feasible to use the ordinary EM algorithm. As aforementioned, it is noteworthy that the QEM sequences are easily obtained by replacing a random sample $\mathbf{z}^{(k)}$ in the MCEM sequences with a quantile sample $\mathbf{q}^{(k)}$.

A. Exponential Distribution

We assume that Z_i are iid exponential random variables with the pdf given by $f(z|\lambda) = \lambda \exp(-\lambda z)$. Using (5), we have the complete-data log-likelihood of λ

$$\log L^c(\lambda) = \sum_{i=1}^n (\log \lambda - \lambda z_i),$$

where the pdf of Z_i is given by

$$p_{z_i}(z|\lambda) = \frac{\lambda \exp(-\lambda z)}{\exp(-\lambda a_i) - \exp(-\lambda b_i)}, \quad (8)$$

for $a_i < z < b_i$. When $a_i = b_i$, the above random variable Z_i degenerates at $Z_i = a_i$.

- **E-step:**

The $Q(\cdot)$ function is given by

$$Q(\lambda|\lambda^{(s)}) = n \log \lambda - \lambda \sum_{i=1}^n A_i^{(s)},$$

where for $a_i < b_i$

$$A_i^{(s)} = E[Z_i|\lambda^{(s)}] = \int_{a_i}^{b_i} z \cdot p_{z_i}(z|\lambda^{(s)}) dz = \frac{a_i \exp(-\lambda^{(s)} a_i) - b_i \exp(-\lambda^{(s)} b_i)}{\exp(-\lambda^{(s)} a_i) - \exp(-\lambda^{(s)} b_i)} + \frac{1}{\lambda^{(s)}}.$$

When $a_i = b_i$, we have $A_i^{(s)} = a_i$.

- **M-step:**

Differentiating $Q(\lambda|\lambda^{(s)})$ with respect to λ and setting this to zero, we obtain

$$\frac{\partial Q(\lambda|\lambda^{(s)})}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^n A_i^{(s)} = 0.$$

Solving for λ , we obtain the $(s + 1)$ st EM sequence in the M-step

$$\lambda^{(s+1)} = \frac{n}{\sum_{i=1}^n A_i^{(s)}}. \quad (9)$$

If we instead use the MCEM (or QEM) algorithm by simulating (or quantiling) z_1, \dots, z_n from the truncated s -normal distribution $p(\mathbf{z}|\boldsymbol{\theta}^{(s)})$, we then obtain the MCEM (or QEM) sequences

$$\lambda^{(s+1)} = \frac{n}{\sum_{i=1}^n \frac{1}{K} \sum_{k=1}^K z_{i,k}},$$

where $z_{i,k}$ for $k = 1, 2, \dots, K$ are from the truncated exponential distribution $p_{z_i}(z|\lambda^{(s)})$ defined in (8).

It is of interest to consider the case where the data are right-censored. In this special case, the closed-form MLE is known. If the data are fully observed (*i.e.*, $w_i = [a_i, a_i]$) for $i = 1, 2, \dots, r$, it is easily seen from l'Hospital rule that $A_i^{(s)} = a_i$. If the observation is right-censored (*i.e.*, $w_i = [a_i, \infty]$) for $i = r + 1, \dots, n$, we have $A_i^{(s)} = a_i + 1/\lambda^{(s)}$. Substituting these results into (9) leads to

$$\lambda^{(s+1)} = \frac{n}{\sum_{i=1}^n a_i + (n - r)/\lambda^{(s)}}. \quad (10)$$

Note that solving the stationary-point equation $\hat{\lambda} = \lambda^{(s+1)} = \lambda^{(s)}$ of (10) gives

$$\hat{\lambda} = \frac{r}{\sum_{i=1}^n a_i}.$$

As expected, this result is the same as the well-known closed-form MLE with the right-censored data.

B. s -Normal Distribution

We assume that Z_i are iid s -normal random variables with parameter vector $\boldsymbol{\theta} = (\mu, \sigma)$. Then the complete-data log-likelihood is

$$\log L^c(\boldsymbol{\theta}) \propto -\frac{n}{2} \log \sigma^2 - \frac{n}{2\sigma^2} \mu^2 - \frac{1}{2\sigma^2} \left\{ \sum_{i=1}^n z_i^2 - 2\mu \sum_{i=1}^n z_i \right\},$$

where the pdf of Z_i is given by

$$p_{z_i}(z|\mu, \sigma) = \frac{\frac{1}{\sigma} \phi\left(\frac{z-\mu}{\sigma}\right)}{\Phi\left(\frac{b_i-\mu}{\sigma}\right) - \Phi\left(\frac{a_i-\mu}{\sigma}\right)}, \quad (11)$$

for $a_i < z < b_i$. Similarly as before, if $a_i = b_i$, then the random variable Z_i degenerates at $Z_i = a_i$.

- **E-step:**

Denote the estimate of θ at the s th EM sequence by $\theta^{(s)} = (\mu^{(s)}, \sigma^{(s)})$. Ignoring constant terms, we have

$$Q(\theta|\theta^{(s)}) = -\frac{n}{2} \log \sigma^2 - \frac{n}{2\sigma^2} \mu^2 - \frac{1}{2\sigma^2} \sum_{i=1}^n A_i^{(s)} + \frac{\mu}{\sigma^2} \sum_{i=1}^n B_i^{(s)},$$

where $A_i^{(s)} = E[Z_i^2|\theta^{(s)}]$ and $B_i^{(s)} = E[Z_i|\theta^{(s)}]$. Using the following integral identities

$$\int \frac{z}{\sigma} \phi\left(\frac{z-\mu}{\sigma}\right) dz = \mu \Phi\left(\frac{z-\mu}{\sigma}\right) - \sigma \phi\left(\frac{z-\mu}{\sigma}\right),$$

$$\int \frac{z^2}{\sigma} \phi\left(\frac{z-\mu}{\sigma}\right) dz = (\mu^2 + \sigma^2) \Phi\left(\frac{z-\mu}{\sigma}\right) - \sigma(\mu + z) \phi\left(\frac{z-\mu}{\sigma}\right),$$

we have for $a_i < b_i$

$$A_i^{(s)} = \{\mu^{(s)}\}^2 + \{\sigma^{(s)}\}^2 - \sigma^{(s)} \cdot \frac{(\mu^{(s)} + b_i) \phi\left(\frac{b_i - \mu^{(s)}}{\sigma^{(s)}}\right) - (\mu^{(s)} + a_i) \phi\left(\frac{a_i - \mu^{(s)}}{\sigma^{(s)}}\right)}{\Phi\left(\frac{b_i - \mu^{(s)}}{\sigma^{(s)}}\right) - \Phi\left(\frac{a_i - \mu^{(s)}}{\sigma^{(s)}}\right)},$$

$$B_i^{(s)} = \mu^{(s)} - \sigma^{(s)} \cdot \frac{\phi\left(\frac{b_i - \mu^{(s)}}{\sigma^{(s)}}\right) - \phi\left(\frac{a_i - \mu^{(s)}}{\sigma^{(s)}}\right)}{\Phi\left(\frac{b_i - \mu^{(s)}}{\sigma^{(s)}}\right) - \Phi\left(\frac{a_i - \mu^{(s)}}{\sigma^{(s)}}\right)}.$$

When $a_i = b_i$, we have $A_i^{(s)} = a_i^2$ and $B_i^{(s)} = a_i$.

- **M-step:**

Differentiating the s -expected log-likelihood $Q(\theta|\theta^{(s)})$ with respect to μ and σ^2 and solving for μ and σ^2 , we obtain the EM sequences

$$\mu^{(s+1)} = \frac{1}{n} \sum_{i=1}^n B_i^{(s)}, \quad (12)$$

$$\sigma^{2(s+1)} = \frac{1}{n} \sum_{i=1}^n A_i^{(s)} - \{\mu^{(s+1)}\}^2. \quad (13)$$

If we instead use the MCEM (or QEM) algorithm by simulating (or quantiling) z_1, \dots, z_n from the truncated s -normal distribution $p(\mathbf{z}|\theta^{(s)})$, we then obtain the MCEM (or QEM) sequences

$$\mu^{(s+1)} = \frac{1}{n} \sum_{i=1}^n \frac{1}{K} \sum_{k=1}^K z_{i,k}, \quad (14)$$

$$\sigma^{2(s+1)} = \frac{1}{n} \sum_{i=1}^n \frac{1}{K} \sum_{k=1}^K z_{i,k}^2 - \{\mu^{(s+1)}\}^2, \quad (15)$$

where $z_{i,k}$ are from the truncated s -normal distribution $p_{z_i}(z_{i,k}|\mu^{(s)}, \sigma^{(s)})$ defined in (11).

C. Laplace Distribution

We assume that Z_i are iid Laplace random variables with parameter $\boldsymbol{\theta} = (\mu, \sigma)$ whose pdf is given by

$$f(x|\mu, \sigma) = \frac{1}{2\sigma} \exp\left(-\frac{|x - \mu|}{\sigma}\right).$$

Using (5), we have the complete-data log-likelihood

$$\log L^c(\boldsymbol{\theta}|\mathbf{z}) = C - n \log \sigma - \frac{1}{\sigma} \sum_{i=1}^m |y_i - \mu| - \frac{1}{\sigma} \sum_{i=m+1}^n |z_i - \mu|,$$

where the pdf of Z_i is given by

$$p_{z_i}(z|\boldsymbol{\theta}) = \frac{f(z|\boldsymbol{\theta})}{F(b_i|\boldsymbol{\theta}) - F(a_i|\boldsymbol{\theta})} \quad (16)$$

for $a_i < z < b_i$. Similarly as before, if $a_i = b_i$, then the random variable Z_i degenerates at $Z_i = a_i$.

- **E-step:**

At the s th step in the EM sequence denoted by $\boldsymbol{\theta}^{(s)} = (\mu^{(s)}, \sigma^{(s)})$, we have the s -expected log-likelihood

$$\begin{aligned} Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)}) &= \int \log L^c(\boldsymbol{\theta}|\mathbf{z}) p(\mathbf{z}|\boldsymbol{\theta}^{(s)}) d\mathbf{z} \\ &= C - n \log \sigma - \frac{1}{\sigma} \sum_{i=1}^n \int_{a_i}^{b_i} |z_i - \mu| f(z_i|\boldsymbol{\theta}^{(s)}) dz_i. \end{aligned}$$

The computation of the above integration part is very complex. We can overcome this difficulty by using MCEM (or QEM) approach. The approximate s -expected log-likelihood is

$$\begin{aligned} \hat{Q}(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)}) &= \frac{1}{K} \sum_{k=1}^K \log L^c(\boldsymbol{\theta}|\mathbf{z}^{(k)}) \\ &= C - n \log \sigma - \frac{1}{\sigma} \sum_{i=1}^n \frac{1}{K} \sum_{k=1}^K |z_{i,k} - \mu|, \end{aligned}$$

where $\mathbf{z}^{(k)} = (z_{1,k}, z_{2,k}, \dots, z_{n,k})$, and $z_{i,k}$ for $k = 1, 2, \dots, K$ are from $p_{z_i}(z|\boldsymbol{\theta}^{(s)})$ defined in (16).

- **M-step:**

Then we have the MCEM (or QEM) sequences

$$\mu^{(s+1)} = \text{median}(\mathbf{z}^{(1)}, \dots, \mathbf{z}^{(K)}), \quad (17)$$

$$\sigma^{(s+1)} = \frac{1}{n} \sum_{i=1}^n \frac{1}{K} \sum_{k=1}^K |z_{i,k} - \mu^{(s+1)}|. \quad (18)$$

D. Rayleigh Distribution

Let Z_i be iid Rayleigh random variables with parameter β whose pdf is given by

$$f(z|\beta) = \frac{z}{\beta^2} \exp\left(-\frac{z^2}{2\beta^2}\right), \quad z > 0, \beta > 0.$$

Then the complete-data log-likelihood is

$$\log L^c(\beta|\mathbf{z}) = C - 2n \log \beta + \sum_{i=1}^n \log z_i - \frac{1}{2\beta^2} \sum_{i=1}^n z_i^2,$$

where the pdf of Z_i is given by

$$p_{z_i}(z|\beta) = \frac{\frac{z}{\beta^2} \exp\left(-\frac{z^2}{2\beta^2}\right)}{\exp\left(-\frac{a_i^2}{2\beta^2}\right) - \exp\left(-\frac{b_i^2}{2\beta^2}\right)} \quad (19)$$

for $a_i < z < b_i$. Similarly as before, if $a_i = b_i$, then the random variable Z_i degenerates at $Z_i = a_i$.

- **E-step:**

At the s th step in the EM sequence denoted by $\beta^{(s)}$, we have the s -expected log-likelihood

$$\begin{aligned} & Q(\beta|\beta^{(s)}) \\ &= \int \log L^c(\beta|\mathbf{z}) p(\mathbf{z}|\beta^{(s)}) d\mathbf{z} \\ &= C - 2n \log \beta + \sum_{i=1}^n \int_{a_i}^{b_i} \left(\log z_i - \frac{1}{2\beta^2} z_i^2\right) p_{z_i}(z_i|\beta^{(s)}) dz_i. \end{aligned}$$

The calculation of the above integration part does not have a closed form. Using the MCEM, we have the approximate s -expected log-likelihood

$$\begin{aligned}\hat{Q}(\beta|\beta^{(s)}) &= \frac{1}{K} \sum_{k=1}^K \log L^c(\beta|\mathbf{z}^{(k)}) \\ &= C - 2n \log \beta + \frac{1}{K} \sum_{k=1}^K \sum_{i=1}^n \log z_{i,k} - \frac{1}{2\beta^2} \frac{1}{K} \sum_{k=1}^K \sum_{i=1}^n z_{i,k}^2,\end{aligned}$$

where $\mathbf{z}^{(k)} = (z_{1,k}, \dots, z_{n,k})$ and $z_{i,k}$ for $k = 1, 2, \dots, K$ are from $p_{z_i}(z|\beta^{(s)})$ defined in (19).

- **M-step:**

We then have the following MCEM (or QEM) sequences by differentiating $\hat{Q}(\beta|\beta^{(s)})$:

$$\beta^{(s+1)} = \sqrt{\frac{1}{2n} \sum_{i=1}^n \frac{1}{K} \sum_{k=1}^K z_{i,k}^2}. \quad (20)$$

E. Weibull Distribution

We assume that X_i is iid Weibull random variables with the pdf and Cdf of X_i given by $f(x) = \lambda \beta x^{\beta-1} \exp(-\lambda x^\beta)$ and $F(x) = 1 - \exp(-\lambda x^\beta)$, respectively.

Using (5), we obtain the complete-data log-likelihood of $\boldsymbol{\theta} = (\lambda, \beta)$:

$$\log L^c(\boldsymbol{\theta}) = \sum_{i=1}^n \left\{ \log \lambda + \log \beta + (\beta - 1) \log z_i - \lambda z_i^\beta \right\},$$

where the pdf of Z_i is given by

$$p_{z_i}(z|\boldsymbol{\theta}) = \frac{\lambda \beta z^{\beta-1} \exp(-\lambda z^\beta)}{\exp(-\lambda a_i^\beta) - \exp(-\lambda b_i^\beta)}, \quad (21)$$

for $a_i < z < b_i$. Similarly as before, if $a_i = b_i$, then the random variable Z_i degenerates at $Z_i = a_i$.

- **E-step:**

Denote the estimate of $\boldsymbol{\theta}$ at the s th EM sequence by $\boldsymbol{\theta}^{(s)} = (\lambda^{(s)}, \beta^{(s)})$. It follows from

$Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)}) = E[\log L^c(\boldsymbol{\theta})]$ that

$$Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)}) = n \log \lambda + n \log \beta + (\beta - 1) \sum_{i=1}^n A_i^{(s)} - \lambda \sum_{i=1}^n B_i^{(s)},$$

where $A_i^{(s)} = E[\log Z_i|\boldsymbol{\theta}^{(s)}]$ and $B_i^{(s)} = E[Z_i^\beta|\boldsymbol{\theta}^{(s)}]$.

- **M-step:**

Differentiating $Q(\lambda|\lambda^{(s)})$ with respect to λ and β and setting this to zero, we obtain

$$\begin{aligned} \frac{\partial Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)})}{\partial \lambda} &= \frac{n}{\lambda} - \sum_{i=1}^n B_i^{(s)}(\beta) = 0, \\ \frac{\partial Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)})}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^n A_i^{(s)} - \lambda \sum_{i=1}^n \frac{\partial B_i^{(s)}(\beta)}{\partial \beta} = 0. \end{aligned}$$

Arranging for β , we have the equation of β

$$\frac{1}{\beta} + \frac{1}{n} \sum_{i=1}^n A_i^{(s)} - \frac{\sum_{i=1}^n \frac{\partial B_i^{(s)}(\beta)}{\partial \beta}}{\sum_{i=1}^n B_i^{(s)}(\beta)} = 0.$$

The $(s+1)$ st EM sequence of β is the solution of the above equation. After finding $\beta^{(s+1)}$, we obtain the $(s+1)$ st EM sequence of $\lambda^{(s+1)}$

$$\lambda^{(s+1)} = \frac{n}{\sum_{i=1}^n B_i^{(s)}(\beta^{(s+1)})}.$$

In this Weibull case, it is extremely difficult or may be impossible to find the explicit s -expectations of $E[\log Z_i|\boldsymbol{\theta}^{(s)}]$ and $E[Z_i^\beta|\boldsymbol{\theta}^{(s)}]$ in the E-step, but the quantile function of the random variable Z_i at the s th step can be easily obtained. From (21), we have

$$q_{i,k} = F_Z^{-1}(\xi_k|\boldsymbol{\theta}^{(s)}) = \left[-\frac{1}{\lambda^{(s)}} \log \left\{ (1 - \xi_k) \exp(-\lambda^{(s)} a_i^{\beta^{(s)}}) + \xi_k \exp(-\lambda^{(s)} b_i^{\beta^{(s)}}) \right\} \right]^{1/\beta^{(s)}}.$$

Using the above quantiles, we obtain the following QEM algorithm.

- **E-step:**

Denote the quantile approximation of $Q(\cdot)$ by $\hat{Q}(\cdot)$. Then, we have

$$\hat{Q}(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)}) = n \log \lambda + n \log \beta + (\beta - 1) \sum_{i=1}^n \frac{1}{K} \sum_{k=1}^K \log q_{i,k} - \lambda \sum_{i=1}^n \frac{1}{K} \sum_{k=1}^K q_{i,k}^\beta.$$

- M-step:

Differentiating $\hat{Q}(\lambda|\lambda^{(s)})$ with respect to λ and β and setting this to zero, we obtain

$$\begin{aligned}\frac{\partial \hat{Q}(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)})}{\partial \lambda} &= \frac{n}{\lambda} - \frac{1}{K} \sum_{i=1}^n \sum_{k=1}^K q_{i,k}^\beta = 0, \\ \frac{\partial \hat{Q}(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)})}{\partial \beta} &= \frac{n}{\beta} + \frac{1}{K} \sum_{i=1}^n \sum_{k=1}^K \log q_{i,k} - \lambda \frac{1}{K} \sum_{i=1}^n \sum_{k=1}^K q_{i,k}^\beta \log q_{i,k} = 0.\end{aligned}$$

Arranging for β , we have the equation of β

$$\frac{1}{\beta} + \frac{1}{nK} \sum_{i=1}^n \sum_{k=1}^K \log q_{i,k} - \frac{\sum_{i=1}^n \sum_{k=1}^K q_{i,k}^\beta \log q_{i,k}}{\sum_{i=1}^n \sum_{k=1}^K q_{i,k}^\beta} = 0. \quad (22)$$

The $(s+1)$ st QEM sequence of β is the solution of the above equation. After finding $\beta^{(s+1)}$, we obtain the $(s+1)$ st QEM sequence of $\lambda^{(s+1)}$

$$\lambda^{(s+1)} = \frac{nK}{\sum_{i=1}^n \sum_{k=1}^K q_{i,k}^{\beta^{(s+1)}}}.$$

In the M-step, we need to estimate the shape parameter β by numerically solving (22), but this is only a one-dimensional root search and the uniqueness of this solution is guaranteed. Lower and upper bounds for the root are explicitly obtained, so with these bounds we can find the root easily. We provide the proof of the uniqueness under quite reasonable conditions and give lower and upper bounds of β in the Appendix.

VI. SIMULATION STUDY

In order to examine the performance of the proposed method, we use the Monte Carlo simulations with 5000 replications. We present the performance of this new method with the EM and MCEM estimators by comparing their estimated biases and the mean square errors (MSE). The biases are calculated by the sample average (over 5000 replications) of the differences between the method under consideration and the MLE. The MSE are also obtained by the sample variance of the differences between the method under consideration and the MLE.

First, a random sample of size $n = 20$ was drawn from an s -normal distribution with $\mu = 50$ and $\sigma = 5$ and the largest five have been right-censored. All the algorithms were stopped after

TABLE I

ESTIMATED BIASES AND MSE, AND SRE OF ESTIMATORS UNDER CONSIDERATION WITH s -NORMAL DATA.

Estimator	$\hat{\mu}$			$\hat{\sigma}$		
	Bias	MSE	SRE	Bias	MSE	SRE
EM	1.342988×10^{-5}	1.779955×10^{-10}	—	3.706033×10^{-5}	1.143094×10^{-10}	—
MCEM						
$K = 10$	7.169276×10^{-2}	8.381887×10^{-3}	2.123573×10^{-8}	1.139404×10^{-1}	2.133204×10^{-2}	5.358577×10^{-9}
$K = 10^2$	2.223300×10^{-2}	8.170053×10^{-4}	2.178633×10^{-7}	3.540433×10^{-2}	2.069714×10^{-3}	5.522955×10^{-8}
$K = 10^3$	7.135135×10^{-3}	8.417492×10^{-5}	2.114590×10^{-6}	1.137090×10^{-2}	2.130881×10^{-4}	5.364419×10^{-7}
$K = 10^4$	2.265630×10^{-3}	8.433365×10^{-6}	2.110610×10^{-5}	3.621140×10^{-3}	2.150602×10^{-5}	5.315227×10^{-6}
QEM						
$K = 10$	2.511190×10^{-2}	2.558357×10^{-5}	6.957412×10^{-6}	5.580507×10^{-2}	1.272262×10^{-4}	8.984739×10^{-7}
$K = 10^2$	2.382535×10^{-3}	2.305853×10^{-7}	7.719289×10^{-4}	5.890585×10^{-3}	1.418158×10^{-6}	8.060413×10^{-5}
$K = 10^3$	2.349116×10^{-4}	2.432084×10^{-9}	7.318639×10^{-2}	6.315578×10^{-4}	1.644996×10^{-8}	6.948917×10^{-3}
$K = 10^4$	3.232357×10^{-5}	2.176558×10^{-10}	8.177841×10^{-1}	9.699447×10^{-5}	4.529568×10^{-10}	2.523627×10^{-1}

10 iterations ($s = 10$). The results are presented in Table I. To help compare the MSE, we also find the simulated relative efficiency (SRE) which is defined as

$$\text{SRE} = \frac{\text{MSE of the EM estimator}}{\text{MSE of the estimator under consideration}}.$$

From the result, the EM is as efficient as the MLE (the MSE of the EM is almost zero). Compared to the MCEM, the QEM has much smaller MSE and much higher efficiency. For example with $K = 10^4$, the SRE of the MCEM is only 2.110610×10^{-5} for $\hat{\mu}$ and 5.315227×10^{-6} for $\hat{\sigma}$. On the other hand, the SRE of the QEM is 0.8177841 for $\hat{\mu}$ and 0.2523627 for $\hat{\sigma}$. Comparing the results in Table I, the QEM with only $K = 100$ performs better than the MCEM with $K = 10,000$.

Next, we draw a random sample of size $n = 20$ from a Rayleigh distribution with $\beta = 10$ with the largest five being right-censored. We compare the QEM only with the MCEM because the EM is not available. The results are presented in Table II for Rayleigh data. The results also show that the QEM clearly outperforms the MCEM.

TABLE II

ESTIMATED BIASES, MSE, AND SRE OF ESTIMATORS UNDER CONSIDERATION WITH RAYLEIGH DATA.

Estimator	$\hat{\beta}$	
	Bias	MSE
MCEM		
$K = 10$	0.1457421790	3.419463×10^{-2}
$K = 10^2$	0.0450846748	3.260372×10^{-3}
$K = 10^3$	0.0142957976	3.283920×10^{-4}
$K = 10^4$	0.0045336330	3.269005×10^{-5}
QEM		
$K = 10$	0.0560471712	5.554717×10^{-5}
$K = 10^2$	0.0057055322	5.739379×10^{-7}
$K = 10^3$	0.0005675668	5.517117×10^{-9}
$K = 10^4$	0.0000527132	3.759086×10^{-11}

VII. EXAMPLES OF APPLICATION OF THE PROPOSED METHODS

This section provides four numerical examples of parameter estimation for a variety of distributions using the EM (if available), MCEM, and QEM algorithms.

A. Censored s -Normal Sample

Let us consider the data presented earlier by Gupta [1] in which, out of $N = 10$, the largest three observations have been censored. The Type-II right-censored observations are as follows:

$$1.613, 1.644, 1.663, 1.732, 1.740, 1.763, 1.778.$$

The MLE is $\hat{\mu} = 1.742$ and $\hat{\sigma} = 0.079$.

We use the EM sequences from (12) & (13) to compare with the MLE. Starting values are chosen by selecting arbitrary number (for example, $\mu^{(0)} = 0$ and $\sigma^{2(0)} = 1$). We obtain the same result as the MLE up to the third decimal point after around nine iterations.

Next, using the MCEM sequences from (14) & (15), we obtained the MCEM & QEM estimates. The algorithm was run with $K = 1,000$ for 10 iterations.

TABLE III

ITERATIONS OF THE EM, MCEM, AND QEM SEQUENCES WITH s -NORMAL DATA.

s	$\mu^{(s)}$			$\sigma^{(s)}$		
	EM	MCEM	QEM	EM	MCEM	QEM
0	0	0	0	1	1	1
1	1.8467	1.8475	1.8466	0.2968	0.3003	0.2962
2	1.8058	1.8071	1.8056	0.1931	0.1953	0.1927
3	1.7761	1.7766	1.7759	0.1370	0.1377	0.1367
4	1.7593	1.7605	1.7592	0.1070	0.1085	0.1067
5	1.7504	1.7505	1.7503	0.0919	0.0922	0.0918
6	1.7459	1.7460	1.7459	0.0848	0.0846	0.0847
7	1.7439	1.7436	1.7438	0.0816	0.0811	0.0816
8	1.7429	1.7428	1.7429	0.0802	0.0800	0.0802
9	1.7425	1.7426	1.7425	0.0796	0.0800	0.0796
10	1.7424	1.7426	1.7423	0.0793	0.0797	0.0793

Table III presents the iterations of the EM, MCEM, and QEM sequences for this problem. From the table, we can see that, when compared with the MCEM estimate, the QEM estimate is closer to the MLE and the EM estimate.

B. Censored Laplace Sample

Let us consider the data presented earlier by Balakrishnan [5] in which, out of $N = 20$ observations, the largest two have been censored. The Type-II right-censored sample thus obtained is as follows:

32.00692, 37.75687, 43.84736, 46.26761, 46.90651,
47.26220, 47.28952, 47.59391, 48.06508, 49.25429,
50.27790, 50.48675, 50.66167, 53.33585, 53.49258,
53.56681, 53.98112, 54.94154.

In this case, Balakrishnan [5] computed the best linear unbiased estimates (BLUE) of μ & σ to be $\hat{\mu} = 49.56095$ and $\hat{\sigma} = 4.81270$. The MLE is $\hat{\mu} = 49.76609$ and $\hat{\sigma} = 4.68761$.

We use the MCEM sequences from (17) & (18) for the MCEM & QEM estimates. The algorithms were run with $K = 1,000$ for 10 iterations with the starting value ($\mu^{(0)} = 0$ & $\sigma^{(0)} = 1$). Table IV presents the iterations of the MCEM & QEM sequences. When compared to the MCEM estimate, especially for σ , the QEM estimate is closer to the MLE. Note also that both MCEM and QEM estimates are closer to the MLE than the BLUE.

TABLE IV

ITERATIONS OF THE EM, MCEM AND QEM SEQUENCES WITH LAPLACE DATA.

s	$\mu^{(s)}$		$\sigma^{(s)}$	
	MCEM	QEM	MCEM	QEM
0	0	0	1	1
1	49.76609	49.76609	4.32098	4.31882
2	49.76609	49.76609	4.66901	4.65058
3	49.76609	49.76609	4.66958	4.68375
4	49.76609	49.76609	4.68236	4.68706
5	49.76609	49.76609	4.69325	4.68740
6	49.76609	49.76609	4.68779	4.68743
7	49.76609	49.76609	4.69379	4.68743
8	49.76609	49.76609	4.67895	4.68743
9	49.76609	49.76609	4.70283	4.68743
10	49.76609	49.76609	4.67191	4.68743

TABLE V
ITERATIONS OF THE MCEM WITH RAYLEIGH DATA.

s	$\beta^{(s)}$		$\beta^{(s)}$	
	MCEM	QEM	MCEM	QEM
0	1	1	10	10
1	5.3356	5.3358	7.2851	7.2946
2	5.9317	5.9444	6.4190	6.4435
3	6.0928	6.0870	6.2165	6.2126
4	6.1176	6.1221	6.1484	6.1536
5	6.1333	6.1309	6.1410	6.1387
6	6.1357	6.1330	6.1376	6.1350
7	6.1318	6.1336	6.1323	6.1341
8	6.1402	6.1337	6.1403	6.1338
9	6.1399	6.1338	6.1400	6.1338
10	6.1260	6.1338	6.1261	6.1338

C. Censored Rayleigh Sample

We simulated a data set with $\beta = 5$ in which, out of $N = 20$ observations, the largest five have been censored. The Type-II right censored sample thus obtained is as follows:

$$\begin{aligned}
 &1.950, 2.295, 4.282, 4.339, 4.411, \\
 &4.460, 4.699, 5.319, 5.440, 5.777, \\
 &7.485, 7.620, 8.181, 8.443, 10.627.
 \end{aligned}$$

We use the MCEM sequences from (20) for the MCEM & QEM estimates. The algorithms were run with $K = 1,000$ for 10 iterations with two different starting values ($\beta^{(0)} = 1, 10$). Table V presents the iterations of the MCEM & QEM sequences. These iteration sequences show that the QEM converges very fast. The MLE is $\hat{\beta} = 6.1341$. The QEM sequences (after rounding) are the same as the MLE up to third decimal place after the sixth iteration.

D. Weibull Interval Data

The previous examples have indicated that the QEM algorithm outperforms the MCEM. In this example, we consider a real-data example of intermittent inspection. The data in this example were originally provided by Nelson [10]. The parts were intermittently inspected to obtain the number of cracked parts in each interval. The data from intermittent intermittent inspection are referred to as grouped data which provide only the number of failures in each inspection period. Table VI gives the data on cracked parts. Other examples of grouped and censored are in Seo & Yum [8], Shapiro & Gulati [9], Xiong & Ji [22], and Meeker [23]. These grouped data can be regarded as interval data. Thus, the proposed method is easily applicable to this case.

The QEM estimate under the exponential model is $\hat{\lambda} = 0.01209699$ while QEM estimate under the Weibull model is $\hat{\lambda} = 0.001674018$ and $\hat{\beta} = 1.497657$. We used $\epsilon = 10^{-5}$ for stopping criterion for the QEM algorithm.

TABLE VI
OBSERVED FREQUENCIES OF INTERMITTENT INSPECTION DATA

Inspection time	Observed failures
0 ~ 6.12	5
6.12 ~ 19.92	16
19.92 ~ 29.64	12
29.64 ~ 35.40	18
35.40 ~ 39.72	18
39.72 ~ 45.24	2
45.24 ~ 52.32	6
52.32 ~ 63.48	17
63.48 ~	73

VIII. CONCLUDING REMARKS

In this paper, we have shown that the QEM algorithm offers clear advantages over the MCEM. It reduces the computational burden required when using the MCEM because a much smaller size is required. Unlike the MCEM, the QEM also possesses very stable convergence properties at each step. The QEM algorithm provides a flexible and useful alternative when one is faced with a difficulty with the implementation of the ordinary EM algorithm. A variety of examples of application were also illustrated using the proposed method.

APPENDIX

SKETCH PROOF OF THE UNIQUENESS AND BOUNDS OF THE WEIBULL SHAPE PARAMETER

Analogous to the approach of [24], the uniqueness of the solution of (22) can be proved as follows. For convenience, letting

$$g(\beta) = \frac{1}{\beta}$$

$$h(\beta) = \frac{\sum_{i=1}^n \sum_{k=1}^K q_{i,k}^\beta \log q_{i,k}}{\sum_{i=1}^n \sum_{k=1}^K q_{i,k}^\beta} - \frac{1}{nK} \sum_{i=1}^n \sum_{k=1}^K \log q_{i,k},$$

we rewrite (22) by $g(\beta) = h(\beta)$. The function $g(\beta)$ is strictly decreasing from ∞ to 0 on $\beta \in [0, \infty]$, while $h(\beta)$ is increasing because it follows from the Jensen's inequality that

$$\frac{\partial h(\beta)}{\partial \beta} = \frac{1}{\{\sum_{i=1}^n \sum_{k=1}^K q_{i,k}^\beta\}^2} \left[\sum_{i=1}^n \sum_{k=1}^K q_{i,k}^\beta \log^2 q_{i,k} \sum_{i=1}^n \sum_{k=1}^K q_{i,k}^\beta - \left\{ \sum_{i=1}^n \sum_{k=1}^K q_{i,k}^\beta \log q_{i,k} \right\}^2 \right] \geq 0.$$

Now, it suffices to show that $h(\beta) > 0$ for some β . Since

$$\lim_{\beta \rightarrow \infty} h(\beta) = \frac{1}{nK} \sum_{i=1}^n \sum_{k=1}^K \left\{ \log q_{\max} - \log q_{i,k} \right\},$$

where $q_{\max} = \max_{i,k} \{q_{i,k}\}$, we have $h(\beta) > 0$ for some β unless $q_{i,k} = q_{\max}$ for all i and k . This condition is extremely unrealistic in practice.

Next, we provide upper and lower bounds of β . These bounds guarantee the unique solution in the interval and enable the root search algorithm to find the solution very stably and easily. Since $h(\beta)$ is increasing, we have $g(\beta) \leq \lim_{\beta \rightarrow \infty} h(\beta)$, that is,

$$\beta \geq \frac{nK}{\sum_{i=1}^n \sum_{k=1}^K (\log q_{\max} - \log q_{i,k})}.$$

Denote the above lower bound by β_L . Then, since $h(\beta)$ is again increasing, we have $g(\beta) = h(\beta) \geq h(\beta_L)$, which leads to

$$\beta \leq \frac{1}{h(\beta_L)}.$$

REFERENCES

- [1] A. K. Gupta, "Estimation of the mean and standard deviation of a normal population from a censored sample," *Biometrika*, vol. 39, pp. 260–273, 1952.
- [2] Z. Govindarajulu, "Best linear estimates under symmetric censoring of the parameters of a double exponential population," *Journal of the American Statistical Association*, vol. 61, pp. 248–258, 1966.
- [3] N. Balakrishnan, "Approximate MLE of the scale parameter of the rayleigh distribution with censoring," *IEEE Transactions on Reliability*, vol. 38, pp. 355–357, 1989.
- [4] A. M. Sultan, "New approximation for parameters of normal distribution using Type-II censored sampling," *Microelectronics Reliability*, vol. 37, pp. 1169–1171, 1997.
- [5] N. Balakrishnan, "BLUEs of location and scale parameters of Laplace distribution based on Type-II censored samples and associated inference," *Microelectronics Reliability*, vol. 36, pp. 371–374, 1996.
- [6] G. C. G. Wei and M. A. Tanner, "A Monte Carlo implementation of the EM algorithm and the poor man's data augmentation algorithm," *Journal of the American Statistical Association*, vol. 85, pp. 699–704, 1990.
- [7] —, "Posterior computations for censored regression data," *Journal of the American Statistical Association*, vol. 85, pp. 829–839, 1990.
- [8] S.-K. Seo and B.-J. Yum, "Estimation methods for the mean of the exponential distribution based on grouped & censored data," *IEEE Transactions on Reliability*, vol. 42, pp. 87–96, 1993.
- [9] S. S. Shapiro and S. Gulati, "Estimating the mean of an exponential distribution from grouped observations," *Journal of Quality Technology*, vol. 30, pp. 107–118, 1998.
- [10] W. Nelson, *Applied Life Data Analysis*. New York: John Wiley & Sons, 1982.
- [11] A. P. Dempster, N. M. Laird, and D. B. Rubin, "Maximum likelihood from incomplete data via the EM algorithm," *Journal of the Royal Statistical Society B*, vol. 39, pp. 1–22, 1977.
- [12] R. J. A. Little and D. B. Rubin, *Statistical Analysis with Missing Data*, 2nd ed. New York: John Wiley & Sons, 2002.
- [13] M. A. Tanner, *Tools for Statistical Inference: Methods for the Exploration of Posterior Distributions and Likelihood Functions*. Springer-Verlag, 1996.
- [14] J. L. Schafer, *Analysis of Incomplete Multivariate Data*. Chapman & Hall, 1997.
- [15] D. R. Hunter and K. Lange, "A tutorial on MM algorithms," *The American Statistician*, vol. 58, pp. 30–37, 2004.
- [16] E. J. Freireich, E. Gehan, E. Frei, L. R. Schroeder, I. J. Wolman, R. Anbari, E. O. Burgert, S. D. Mills, D. Pinkel, O. S. Selawry, J. H. Moon, B. R. Gendel, C. L. Spurr, R. Storrs, F. Haurani, B. Hoogstraten, and S. Lee, "The effect of 6-Mercaptopurine on the duration of steroid-induced remissions in acute leukemia: a model for evaluation of other potentially useful therapy," *Blood*, vol. 21, pp. 699–716, 1963.
- [17] L. M. Leemis, *Reliability*. Englewood Cliffs, N.J.: Prentice-Hall, 1995.
- [18] D. R. Cox and D. Oakes, *Analysis of Survival Data*. New York: Chapman & Hall, 1984.

- [19] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in C++: The Art of Scientific Computing*. Cambridge: Cambridge University Press, 2002.
- [20] D. F. Heitjan, "Inference from grouped continuous data: a review (with discussion)," *Statistical Science*, vol. 4, pp. 164–183, 1989.
- [21] D. F. Heitjan and D. B. Rubin, "Inference from coarse data via multiple imputation with application to age heaping," *Journal of the American Statistical Association*, vol. 85, pp. 304–314, 1990.
- [22] C. Xiong and M. Ji, "Analysis of grouped and censored data from step-stress life test," *IEEE Transactions on Reliability*, vol. 53, pp. 22–28, 2004.
- [23] W. Q. Meeker, "Planning life tests in which units are inspected for failure," *IEEE Transactions on Reliability*, vol. 35, pp. 571–578, 1986.
- [24] N. R. Farnum and P. Booth, "Uniqueness of maximum likelihood estimators of the 2-parameter Weibull distribution," *IEEE Transactions on Reliability*, vol. 46, pp. 523–525, 1997.

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