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Multi-Criteria Design Optimization

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2D Decision-Making for Multi-Criteria Design Optimization

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Abstract

The high dimensionality encountered in engineering design optimization due to large numbers of performance criteria and specifications leads to cumbersome and sometimes unachievable tradeoff analyses. To facilitate those analyses and enhance decision-making and design selection, we propose to decompose the original problem by considering only pairs of criteria at a time, thereby making tradeoff evaluation the simplest possible. For the final design integration, we develop a novel coordination mechanism that guarantees that the selected design is also preferred for the original problem. The solution of an overall large-scale problem is therefore reduced to solving a family of bi-criteria subproblems and allows designers to effectively use decision-making in merely two dimensions for multi-criteria design optimization.

Keywords: multi-criteria design optimization – interactive decision-making – decomposition – coordination – tradeoff visualization – sensitivity

1 Introduction and Literature Review

Structural design optimization deals with the development of complex systems and structures such as cars, airplanes, spaceships or satellites. Based on tremendous gain in experience and knowledge, together with the rapid progress in computing technologies, the underlying mathematical models and design simulations become better and better and provide designers with growing amounts of data that need to be analyzed for choosing a final optimal design. In particular, the steadily increasing number of specifications and criteria used to evaluate the performance of the simulated designs leads to cumbersome and sometimes unachievable tradeoff analyses, thus resulting in complex and difficult, if not unsolvable, decision-making problems.

Design optimization under multiple performance measures When evaluated by multiple criteria, it is long known (Zadeh, 1963) that an overall optimal design, in general, does not exist, but a set of nondominated or Pareto optimal solutions (Pareto, 1896). In principle, these solutions can be found in three different ways. In the traditional but still widely applied first approach,

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the design space is sampled and a certain number of feasible designs is evaluated using simulation codes that describe the underlying model (Verma et al., 2005). Thereafter, these designs are filtered based on pairwise comparisons of their associated performances so that only nondominated designs remain subject to further consideration (Mattson et al., 2004). In enhancement of the first, a second approach makes additional use of genetic or evolutionary algorithms to further improve the initial designs compared to a mere sampling (Narayanan and Azarm, 1999; Gunawan et al., 2004). Both approaches are similar in that they provide the designer with a set of nondominated solutions, but in general do not guarantee that any of these solution is also an optimal design.

Hence, opposed to the former two, the third approach is based on actual optimization and finds one Pareto point at a time by solving an auxiliary single objective problem. The typical formulation of this problem uses a linear combination (or weighted-sum) that aggregates all participating criteria into a single objective, so that, by varying the weights assigned to each criterion, different Pareto solutions can be generated. Although commonly used, numerous drawbacks of this approach, including its failure to generate points in nonconvex regions of the Pareto set, are well recognized (Das and Dennis, 1997). Recently, a family of aggregation functions more appropriate for engineering design is presented in (Scott and Antonsson, 2005). The concept of an aggregate objective function is also used in the context of physical programming for robust designs (Messac and Ismail-Yahaya, 2002), with the integration of the physical programming methodology within multidisciplinary design optimization described in (McAllister et al., 2005).

Decomposition of the design optimization problem The combination of multiple performance measures into one single index is used not only to reduce the number of criteria, but more importantly to reduce the complexity of the underlying optimization problem. Alternatively, a reduction in the complexity of most design problems is typically achieved by various decomposition strategies (Blouin et al., 2004). These are particularly well suited for design optimization as most complex engineering systems usually consist of many subsystems and components having smaller complexity (Chanron et al., 2005). Decomposition then means to divide the large and complex system into several smaller entities, while responsibilities for the various subsystems or components are assigned to different designers or design teams with autonomy in their local optimization and decision-making. In general, however, these subsystems will still be coupled so that the solution of each subsystem is dependent upon information from the others. Hence, along with the benefit of reduced complexity comes the difficulty of coordinating the different design decisions to eventually arrive at a single overall design solution that is feasible, thus meeting the design requirements, preferably optimal and acceptable for all participating designers or decision-makers. A survey on existing coordination approaches is given in (Coates et al., 2000; Whitfield et al., 2000) and, with a special focus on decentralized design, in (Whitfield et al., 2002). The issue of converging to a common overall design is addressed in (Chanron and Lewis, 2005; Chanron et al., 2005) who use game theoretic concepts to model and analyze the competing interest of the different decision-makers. Most recently, new strategies for the coordination between multi-objective systems are also proposed in the framework of collaborative optimization (Rabeau et al., 2006).

Relevance of this paper Adding to these recent results on the decomposition of decision-making problems, this paper presents an interactive decision-making procedure for replacing the intuitive selection of the overall design (Agrawal et al., 2004, 2005) by a more systematic design integration. This final design integration is based on a novel coordination mechanism that guarantees that

the selected design is also preferred for the original problem. Previous interactive methods are described in (Tappeta and Renaud, 1999; Tappeta et al., 2000; Azarm and Narayanan, 2000), but none of these methods makes use of a decomposition as suggested by the approach presented in this paper. Motivated by the several recent studies emphasizing the importance of visualizing the optimization process (Messac and Chen, 2000), the design data (Eddy and Lewis, 2002) and Pareto frontiers (Agrawal et al., 2004, 2005; Mattson and Messac, 2005), the new methodology includes the feature of visualizing the tradeoff curves for every subproblem to support the choice of an optimal design. In addition to the tradeoff between those objectives participating in the same subproblem, information on the tradeoffs between different subproblems is obtained from a sensitivity analysis and used for the subsequent coordination. This method thereby offers an alternative to the tradeoff analyses for the complete problem as suggested in (Tappeta et al., 2000; Kasprzak and Lewis, 2000) or, in the context of robust multi-criteria optimization, in (Gunawan and Azarm, 2005; Li et al., 2005).

Only one paper (Verma et al., 2005) relies on a similar decomposition and visualization strategy in order to successively filter solutions from a set of simulated designs. However, therein all decisions are based merely on the intuition of the designer, and as no optimization is involved, the method cannot guarantee that the final solution is also optimal for the overall problem. The procedure in this paper, on the other hand, is capable of revealing tradeoffs, generating new solutions based on the designer’s choices, and always guaranteeing Pareto optimality of the final and all intermediate solutions.

Objectives of this paper The objective of this paper is thus three-fold. First, assuming that Pareto optimal solutions can be found by either a traditional approach or genetic algorithms, an interactive procedure is proposed for the selection of an optimal design for a complex and multi-criteria design optimization problem. Second, this selection is facilitated using a decomposition strategy, so that all intermediate decisions are made on smaller-sized subproblems involving only two performance measures at a time. A new coordination mechanism then guarantees that the final selection leads to a common design that is optimal for the overall problem. By choosing this decomposition-coordination framework, the method is also well suited for decentralized design processes and decision-making situations involving multiple decision-makers. Finally, the third objective is to enhance the procedure by providing the designer or decision-maker with tradeoff information in form of sensitivities and a 2D representation of the tradeoff curves for every subproblem.

2 Problem Statement and Preliminaries

For the scope of this paper, we consider the mathematical model of a multi-objective optimization problem

$$\begin{aligned}
 \text{MOP: minimize } & f(x) = [f_1(x), f_2(x), \dots, f_p(x)] \\
 \text{subject to } & g(x) = [g_1(x), g_2(x), \dots, g_m(x)] \leq 0 \\
 & h(x) = [h_1(x), h_2(x), \dots, h_l(x)] = 0 \\
 & x_i^L \leq x_i \leq x_i^U, i = 1, \dots, n.
 \end{aligned}$$

In this formulation, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ is the vector of design variables that is evaluated by the performance function $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$ to produce the associated performance vector $y = f(x)$. This vector function f is composed of p criteria that are modeled as real-valued functions $f_i : \mathbb{R}^n \rightarrow \mathbb{R}, i = 1, 2, \dots, p$. Similarly, the constraint functions $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $h : \mathbb{R}^n \rightarrow \mathbb{R}^l$ consist of m inequality constraints $g_i : \mathbb{R}^n \rightarrow \mathbb{R}, i = 1, 2, \dots, m$, and l equality constraints $h_i : \mathbb{R}^n \rightarrow \mathbb{R}, i = 1, 2, \dots, l$, respectively. Finally, the values $x_i^L \leq x_i^U, i = 1, 2, \dots, n$ denote lower and upper bounds on the design variables, and then the set of all feasible designs in the design space \mathbb{R}^n is given by

$$X := \{x \in \mathbb{R}^n : g(x) \leq 0, h(x) = 0, x^L \leq x \leq x^U\}.$$

We denote the set of all realizable or attainable performance vectors in the performance space \mathbb{R}^p by

$$Y := f(X) := \{y \in \mathbb{R}^p : y = f(x) \text{ for some } x \in X\}.$$

Pareto optimal designs A design is a Pareto optimal design if it is not possible to improve its performance with respect to one criterion without deterioration in at least one other criterion. If it is possible to improve a design with respect to some but not all criteria at the same time, then this design is not Pareto optimal but only weakly Pareto optimal (Chankong and Haimes, 1983).

It then follows that a Pareto optimal solution for some set of criteria remains at least weakly Pareto optimal upon enlarging this set with additional performance indices. To see why this is true, note that for a Pareto optimal solution, no other design is available that is better in some but not worse in any other criterion. Then, in particular, no other solution can be better in *all* criteria, and clearly this must still be true after additional criteria are added. According to the above definition, this means that the solution is at last weakly Pareto optimal for the larger set of criteria, and thus the following result holds.

If a design is Pareto optimal for a subset of all performance measures, then it is (at least weakly) Pareto optimal for the overall problem involving all criteria.

Therefore, it is possible to identify Pareto optimal solutions for a large-scale problem by merely considering smaller sized problems with a reduced number of criteria. Clearly, such approach requires to decide which criteria to choose and which to drop. In (Matsumoto et al., 1993) and (Dym et al., 2002), it is shown how to develop a ranking among all criteria, and the related issue of decomposing a multiobjective design based on criteria influence is addressed in (Yoshimura et al., 2002, 2003). After these issues have been resolved, the reduced problem commonly takes the form shown in Figure 1.

Traditional decomposition and integration Figure 1 illustrates the typical decomposition-integration scheme for an overall MOP with p criteria that is divided into k subproblems of smaller size. Since all subproblems are completely uncoupled, each problem can be solved separately and then communicates one (or a set of) optimal solutions to the overall MOP. Since all subproblems are formulated over the same feasible design set, every such solution is also feasible for the other subproblems and, in particular, for the overall problem. Therefore, the only task remaining is to select a final solution among the designs proposed by the subproblems, and the above result then guarantees that this design is also (at least weakly) Pareto optimal for the overall problem.

This important observation can also be motivated using the following intuitive explanation. Suppose that the problem of interest is the design of a system that consists of several, say k

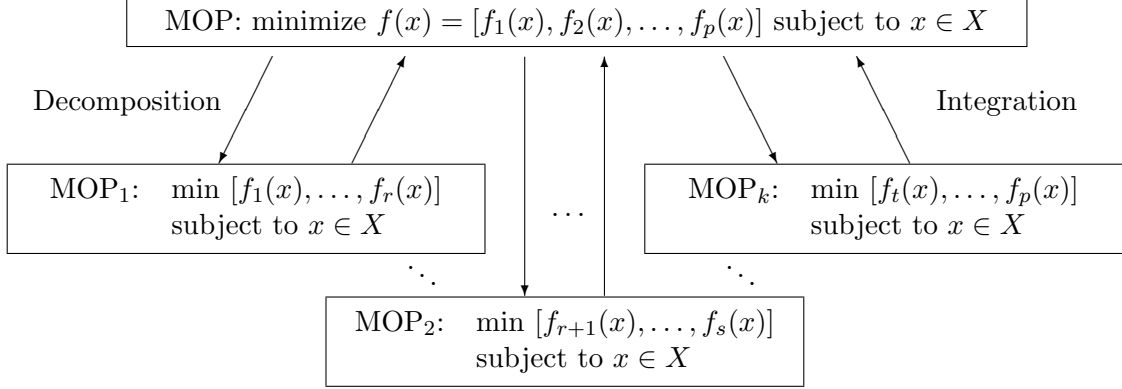


Figure 1: Traditional decomposition and integration without coordination

subsystems, and that we decide to design only one selected subsystem to optimality. Then we know that the overall system is still weakly Pareto optimal, because improvement in the weak Pareto sense would require to improve *all* subproblems, including the one that is already optimal.

Nevertheless, by focusing on only one subsystem at a time, we might risk to significantly degrade the performances of the remaining subsystems. For illustration assume that we have found four designs x^1 , x^2 , x^3 and x^4 that are evaluated by the criteria f_1 , f_2 and f_3 which we wish to minimize, as shown in Table 1.

Table 1: Drawback of traditional decomposition and integration without coordination

design	f_1	f_2	f_3	observation
x^1	1	1	9	unique optimal design for subproblem with criteria f_1 and f_2
x^2	1	9	1	unique optimal design for subproblem with criteria f_1 and f_3
x^3	9	1	1	unique optimal design for subproblem with criteria f_2 and f_3
x^4	2	2	2	not optimal for any subproblem, but overall min-max solution

It is easy to verify that all four designs are Pareto optimal for the complete problem with criteria f_1 , f_2 and f_3 . For every combination of two criteria, however, only one of the designs remains optimal for the associated subproblem. If we consider the subproblem with criteria f_1 and f_2 , then x^1 is better than x^2 with respect to f_2 (same for f_1), better than x^3 with respect to f_1 (same for f_2) and better than x^4 in both f_1 and f_2 . Hence, x^1 is better than all other designs and thus the unique optimal design for the subproblem with criteria f_1 and f_2 . Similarly, we find that x^2 and x^3 are the unique optimal designs for the subproblems with criteria f_1 and f_3 , and f_2 and f_3 , respectively. Note, however, that each of these designs performs worst for the criterion omitted from consideration in the respective subproblem in which it is optimal.

The remaining design x^4 is not Pareto optimal for any of these three subproblems, but it is also Pareto optimal for the overall problem. In particular, this design is the best compromise among all four designs and, moreover, constitutes the overall min-max solution, that is the best design for the strategy to minimize all worst performances (i.e., the worst performance of x^1 , x^2 , and x^3 is 9, of x^4 only 2). Hence, focusing on only one subsystem suffers from the major drawback of regularly missing the best compromise solution for the overall problem and is, therefore, highly

insufficient for the design of complex systems with several important subsystems. Hence, in such cases, it would be preferable to design an overall optimal system such that all subsystems perform comparably well, although maybe suboptimal when considered separately from the overall system.

Importance of tolerances and suboptimal designs The above observation indicates that solving the individual subsystems to optimality before selecting a final design generally risks to overemphasize one subsystem while significantly degrading the performances of the others. Moreover, it shows that the best compromise solution is not necessarily optimal for any subproblem and thus remains unknown to the designer who follows the traditional decomposition-integration approach. However, it also suggests that better solutions for the overall problem can be found by enlarging a subproblem’s solution set with slightly suboptimal designs. In other words, a solution is allowed to deviate by some tolerance from an optimal design for a subproblem, as long as the improvement with respect to another subproblem guarantees that this solution is still optimal for the overall problem. It then is better if a design meets all tolerances, although possibly being suboptimal in all subproblems, rather than it is optimal for one subproblem but violates the tolerances for some or all the others.

Again consider the situation in Table 1. The optimal value for each of the three performance criteria equals 1, and as we explained before, for every combination of two criteria there exists a unique optimal design that achieves these optimal performances. This also means that no design is optimal for all criteria pairs, or similarly, that no design is preferred in every subproblem. However, if we also accept suboptimal solutions and allow an additional tolerance of 1 that is added to the optimal performance values of 1, then the design x^4 satisfies all new performance expectations of at least 2. In fact, x^4 then is the only solution that is accepted for all pairs of criteria and, thus, the preferred overall design. Note how this analysis remains unchanged as long as the new performance is allowed to be less than 9, or equivalently, as long as all tolerances are chosen to be at most 8. In general, however, these additional tolerances should still be reasonably small, and in accordance with the literature and common practice to denote small variations by the Greek letter ε , or epsilon, we conform to this notation for the exposition of our new method in the subsequent section.

3 Decomposition and Integration with Coordination

Because of the broad familiarity of designers with the traditional decomposition-integration scheme in Figure 1, our procedure uses a similar setup and initially decomposes the overall performance function into subsets of criteria, thus reducing the complexity of the individual subproblems. In particular, as one of our objectives is to enable decision-making based on a 2D visualization of the underlying tradeoff curves, we suggest to divide the performance indices into pairs, thereby making tradeoff evaluation the simplest possible.

Coordination between different subproblems While the decomposition into pairs gives a convenient way to handle and reveal tradeoffs within every subproblem, the tradeoff between different subproblems then is to be accomplished by some other mechanism. For the coordination between subproblem, we use the lexicographical ordering approach for multicriteria optimization (Fishburn, 1974; Rentmeesters et al., 1996) but introduce the following two essential modifications to the original formulation. First, instead of ordering all single objectives, we account for the specific feature of our approach and apply the proposed ordering to pairs of criteria at a time (Ying,

1983). Second and based on the discussion in the previous section, we include additional epsilon tolerances to reflect the implicit tradeoff between two different subproblems. This substantially extends the original approach and, in particular, guarantees that all Pareto optimal solutions for the complete problem can still be found through a suitable coordination. To the authors' knowledge, a procedure combining these highly desirable benefits has not yet been formulated, so that we propose a novel decomposition and integration scheme with coordination to enable 2D decision-making for essentially every multicriteria design optimization problem. Consider Figure 2.

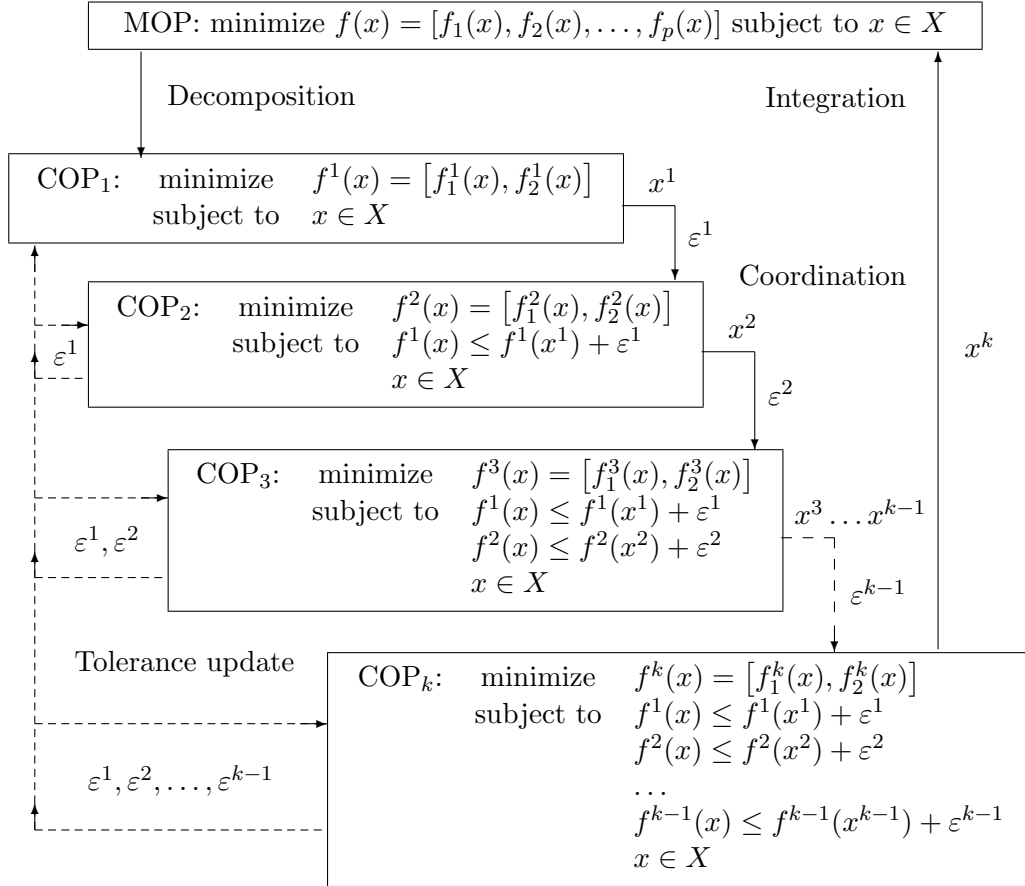


Figure 2: Decomposition and integration with coordination

Description of procedure The performance function $f = (f_1, f_2, \dots, f_p)$ for the overall MOP is decomposed into k pairs of criteria that are used as new performance functions $f^j = (f_1^j, f_2^j)$, $j = 1, \dots, k$. As described in the above paragraph, the canonical subproblem formulations

$$\begin{aligned} \text{MOP}_j: \text{ minimize } & f^j(x) = [f_1^j(x), f_2^j(x)] \\ \text{subject to } & x \in X \end{aligned}$$

are modified by additional epsilon-constraints to account for the additional tolerances imposed by the designer and to sequentially coordinate the performance tradeoffs between different subprob-

lems. Therefore, we also call these new subproblems *coordination problems*

$$\begin{aligned} \text{COP}_j: \text{ minimize } & f^j(x) = [f_1^j(x), f_2^j(x)] \\ \text{subject to } & f^i(x) \leq f^i(x^i) + \varepsilon^i \text{ for all } i = 1, \dots, j-1 \\ & x \in X, \end{aligned}$$

where $\varepsilon^i = (\varepsilon_1^i, \varepsilon_2^i) \in \mathbb{R}^2$, $i = 1, \dots, j-1$, are the performance tolerances specified by the designer.

Since each coordination problem COP_j is, in particular, a bi-criteria problem, we can visualize the tradeoff between optimal solutions in form of a two-dimensional Pareto frontier. The choice of a preferred solution x^j can, thus, be based on a visual representation and only requires decision-making with respect to two dimensions. The chosen design x^j is communicated as a baseline design to the subsequent coordination problem COP_{j+1} , together with two values $\varepsilon^j = (\varepsilon_1^j, \varepsilon_2^j)$ that specify acceptable tolerances for the two optimal performance values $f_1^j(x^j)$ and $f_2^j(x^j)$. This provides a mechanism to also accept solutions with slightly worse performances than the previously selected design and, eventually, to achieve better compromise solutions for the overall problem. We later discuss possible choices of ε and show how the designer, through the choice of ε , gains close control on the desired tradeoff between the different coordination problems. If no tradeoff is desirable, the designer may also choose $\varepsilon^j = 0$ to guarantee that all subsequently found designs meet the performances $f^j(x^j) = [f_1^j(x^j), f_2^j(x^j)]$ achieved by the previously selected baseline design x^j . Moreover, to provide the designer with maximal flexibility, all tolerances ε^j can still be updated throughout the remaining decision-making cycle, as indicated on the left of Figure 2.

As a special feature of this procedure, the designer does not actually need to solve all coordination problems. Based on the previous results, it is guaranteed that all intermediate designs x^j are already at least weakly Pareto optimal for the overall problem, although possibly in favor of those subproblems $\text{COP}_1, \dots, \text{COP}_j$, so far participating in the coordination process, compared to $\text{COP}_{j+1}, \dots, \text{COP}_k$ that are omitted due to early termination. Nevertheless and different from most other decomposition and decision-making schemes, the designer may thus choose to stop the decision-making process after every iteration and still obtain an (at least weakly) Pareto-optimal design for the overall problem.

Tolerance update To support the designer with the task of setting and changing the tolerance values ε , or to reveal remaining tradeoff benefits and decide upon early termination, we propose to perform a tradeoff and sensitivity analysis at the current design x^j with respect to its performance in different coordination problems.

To explain the details of this analysis, assume that the designer has solved COP_1 through COP_{j-1} , that is, the designer has selected designs x^1, x^2, \dots, x^{j-1} and tolerances $\varepsilon^1, \varepsilon^2, \dots, \varepsilon^{j-1}$ for all previous coordination problems and arrives at COP_j , as defined before. Based on a 2D visualization of the Pareto curve for COP_j and possibly supported by the underlying numerical data, the designer should then select a new design x^j and proceed to the next coordination problem. By feasibility, it is always guaranteed that this new design satisfies the tolerances $\varepsilon^1, \dots, \varepsilon^{j-1}$ specified for all previous designs x^1, \dots, x^{j-1} found in $\text{COP}_1, \dots, \text{COP}_{j-1}$. However, it might happen that the designer is not satisfied with the achievable performance of x^j with respect to the two objectives $f^j = (f_1^j, f_2^j)$ in COP_j . In other words, the designer might be willing to accept a further relaxation of one or more of the previous tolerances $\varepsilon^1, \dots, \varepsilon^{j-1}$ to improve one or both of the performance values $f_1^j(x^j)$ or $f_2^j(x^j)$. Denoting the selected tolerance by $\varepsilon^i = (\varepsilon_1^i, \varepsilon_2^i)$, $1 \leq i \leq j-1$, this situation

is depicted in Figure 3 that illustrates the idea of the tradeoff at the current design x^j between the criterion pairs $f^i = (f_1^i, f_2^i)$ in COP_i and $f^j = (f_1^j, f_2^j)$ in COP_j .

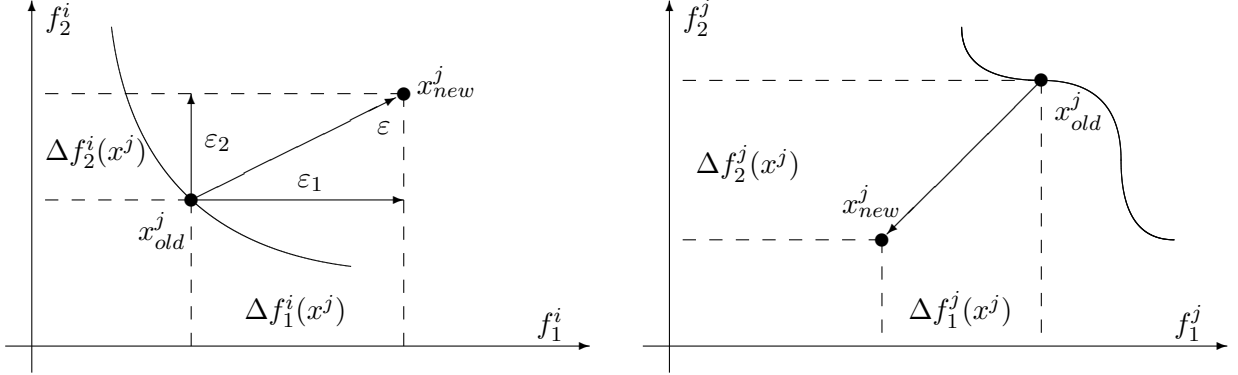


Figure 3: Pareto curve for COP_i with $f^i = (f_1^i, f_2^i)$ and its image for COP_j with $f^j = (f_1^j, f_2^j)$

At first, when $\varepsilon^i = 0$, the epsilon-constraint pair $f^i(x) \leq f^i(x^i) + \varepsilon^i$ in COP_j guarantees that any optimal solution x^j for COP_j still meets the performance of the optimal design x^i selected for coordination problem COP_i , $f^i(x^j) \leq f^i(x^i)$. In particular, then x^j must also lie on the Pareto curve for COP_i with the possible consequence that a satisfactory performance of x^j with respect to the two criteria $f^j = (f_1^j, f_2^j)$ of COP_j cannot anymore be achieved. Therefore, if we specify some additional tolerances $\varepsilon^i = (\varepsilon_1^i, \varepsilon_2^i)$ and allow x^j , but in a controlled way, to move away from the Pareto curve for COP_i , then we expect to gain improvement with the previously unsatisfactory performances in COP_j . Using the notation introduced in Figure 3, the resulting achieved tradeoffs can be computed as

$$\frac{\Delta f_1^j(x^j)}{\Delta f_1^i(x^j)} = \frac{f_1^j(x_{old}^j) - f_1^j(x_{new}^j)}{f_1^i(x_{new}^j) - f_1^i(x_{old}^j)} = \frac{f_1^j(x_{old}^j) - f_1^j(x_{new}^j)}{\varepsilon_1} \quad (1)$$

and similarly for $\frac{\Delta f_1^j(x^j)}{\Delta f_2^i(x^j)}$, $\frac{\Delta f_2^j(x^j)}{\Delta f_1^i(x^j)}$, and $\frac{\Delta f_2^j(x^j)}{\Delta f_2^i(x^j)}$. In general, however, it is not obvious how far we need to move away from the Pareto curve in COP_i to achieve a satisfactory improvement in COP_j , that is, how large we need to choose the value ε . To provide the designer with better intuition, we propose to examine the sensitivity at the current design with respect to the criteria f^i in COP_i and f^j in COP_j . Then, the higher the sensitivity, the higher the potential tradeoff, so that even small additional tolerances for $f^i = (f_1^i, f_2^i)$ are expected to yield a significant improvement with respect to $f^j = (f_1^j, f_2^j)$.

Sensitivity analysis Our approach uses sensitivity results from nonlinear programming (Fiacco, 1983; Luenberger, 2003), for which we first formulate the auxiliary sensitivity problem

$$\begin{aligned} \text{SEP}_{j1}: \quad & \text{minimize } f_1^j(x) \\ & \text{subject to } f^i(x) \leq f^i(x^i) + \varepsilon^i \text{ for all } i = 1, \dots, j-1 \\ & \quad f_2^j(x) \leq f_2^j(x^j) \\ & \quad x \in X. \end{aligned}$$

Other than COP_j , this problem minimizes only the first of the two objectives $f^j = (f_1^j, f_2^j)$ while including the second into the additional constraint $f_2^j(x) \leq f_2^j(x^j)$. This guarantees that improvement in f_1^j is not achieved through degradation of f_2^j and, thus, that the sensitivity analysis strictly distinguishes tradeoffs that, on the one hand, occur between the two criteria in the same subproblem and, on the other hand, occur between two criteria in two different coordination problems.

As the formulation of this problem is only auxiliary, we do not need to actually solve it. Instead, it can easily be shown that the design x^j , that is Pareto optimal for COP_j , must also be optimal for SEP_{j1} . Otherwise, if this were not the case, then there would exist a better design which improves the first performance measure, as this is the objective we minimize, but be also at least as good as the second objective, as we enforce that $f_2^j(x) \leq f_2^j(x^j)$. Then, however, this design would also be better than x^j in COP_j , which cannot be true as x^j is already Pareto optimal.

Therefore, we do not need to solve SEP_{j1} . The reason for that we still consider this problem is that it allows to compute the sensitivities between its objective f_1^j and its constraint values f^i at the optimal design x^j in terms of the associated Lagrangean multipliers λ^{ji} . Then, under some technical assumptions, the sensitivity theorem from nonlinear programming states that

$$\left. \frac{\partial f_1^j(x)}{\partial f_1^i(x)} \right|_{x=x^j} = -\lambda_{11}^{ji} \quad \text{and} \quad \left. \frac{\partial f_1^j(x)}{\partial f_2^i(x)} \right|_{x=x^j} = -\lambda_{12}^{ji} \quad \text{for all } i = 1, \dots, j-1 \quad (2a)$$

and similarly, if we change the roles of f_1^j and f_2^j and formulate the corresponding SEP_{j2} ,

$$\left. \frac{\partial f_2^j(x)}{\partial f_1^i(x)} \right|_{x=x^j} = -\lambda_{21}^{ji} \quad \text{and} \quad \left. \frac{\partial f_2^j(x)}{\partial f_2^i(x)} \right|_{x=x^j} = -\lambda_{22}^{ji} \quad \text{for all } i = 1, \dots, j-1. \quad (2b)$$

As almost all common optimization software provides Lagrangean multipliers together with a final solution, these sensitivities can be very efficiently computed using an arbitrary optimization routine and choosing x^j as initial design. Then, since x^j is known to be optimal, we obtain the desired sensitivities in at most one iteration and, in view of (2), the associated tradeoff estimates

$$\frac{\Delta f_1^j(x^j)}{\Delta f_1^i(x^j)} \approx -\lambda_{11}^{ji}, \quad \frac{\Delta f_1^j(x^j)}{\Delta f_2^i(x^j)} \approx -\lambda_{12}^{ji}, \quad \frac{\Delta f_2^j(x^j)}{\Delta f_1^i(x^j)} \approx -\lambda_{21}^{ji}, \quad \frac{\Delta f_2^j(x^j)}{\Delta f_2^i(x^j)} \approx -\lambda_{22}^{ji}. \quad (3)$$

As a rule of thumb, we can state that the larger the magnitude of a computed Lagrangean multiplier, the larger the tradeoff we can expect. For example, if $\lambda_{11}^{ji} \gg 1$, then this tells us that even a small additional tolerance ε_1^i on $f_1^i(x^j)$ can be expected to yield a significant improvement in $f_1^j(x^j)$. On the other hand, if, for example, $\lambda_{21}^{ji} < 1$, then the improvement in $f_2^j(x^j)$ is comparably smaller than the allowed tolerance ε_1^i that we would be willing to give up for $f_1^i(x^j)$. In general, a sensitivity value less than 1 indicates that an additional tolerance exceeds the expected improvement and, thus, that the corresponding tradeoff is not favorable at the current design. Then, if all sensitivities become less than 1, this suggests to terminate the procedure with the current design as final preferred design for the overall problem.

In general, however, we are not able to precisely predict the actual tradeoffs to be achieved. This is because all former sensitivities only provide us with local tradeoff information at x^j and, thus, are accurate only if the designer decides to allow a very small tolerance. Nevertheless, if the chosen tolerance ε_1^i is very small, by (1) and (3) it then can be expected that the resulting improvement in $f_1^j(x^j)$ or $f_2^j(x^j)$ is approximately

$$\Delta f_1^j(x^j) \approx -\lambda_{11}^{ji} \Delta f_1^i(x^j) = -\lambda_{11}^{ji} \varepsilon_1^i, \quad \text{or} \quad \Delta f_2^j(x^j) \approx -\lambda_{21}^{ji} \Delta f_1^i(x^j) = -\lambda_{21}^{ji} \varepsilon_1^i, \quad (4a)$$

respectively. Similarly, if the designer decides to choose a very small tolerance ε_2^i , then

$$\Delta f_1^j(x^j) \approx -\lambda_{12}^{j_i} \Delta f_2^i(x^j) = -\lambda_{12}^{j_i} \varepsilon_2^i, \text{ or } \Delta f_2^j(x^j) \approx -\lambda_{22}^{j_i} \Delta f_2^i(x^j) = -\lambda_{22}^{j_i} \varepsilon_2^i. \quad (4b)$$

Moreover, the designer may also choose to simultaneously change both tolerances $\varepsilon^i = (\varepsilon_1^i, \varepsilon_2^i)$, and, instead of restricting improvement to only one criterion f_1^j or f_2^j , choose a new design that achieves improvements in both criteria. The sensitivity analysis for these cases, however, is out of the scope of this paper.

Finally, it is always possible to replace this analytic sensitivity and tradeoff analysis by an alternative trial and error approach based on the designer's intuition. If the designer is expert, it should be expected that such an approach works perfectly fine. Otherwise, if the designer is not completely familiar with all relevant problem characteristics, we recommend to always perform the above tradeoff and sensitivity analysis to obtain better insight and knowledge that otherwise would not be available.

4 Examples

For a demonstration of the proposed 2D decision-making procedure and the underlying coordination mechanism, we adopt the role of a hypothetical decision-maker and apply this method to two examples from multiobjective programming and structural design optimization. By the nature of this approach, it is unavoidable that all our decisions made remain subjective and, in practice, would also depend on the expertise, preferences and performance expectations of the actual designer.

4.1 A mathematical programming problem

The problem chosen consists of four quadratic objective functions f_1, f_2, f_3 and f_4 which need to be minimized, subject to three inequality constraints g_1, g_2 and g_3 in the two variables x_1 and x_2 .

$$\begin{aligned} \text{Minimize} \quad & \begin{aligned} [f_1(x_1, x_2) &= (x_1 - 2)^2 + (x_2 - 1)^2, \\ f_2(x_1, x_2) &= x_1^2 + (x_2 - 3)^2, \\ f_3(x_1, x_2) &= (x_1 - 1)^2 + (x_2 + 1)^2, \\ f_4(x_1, x_2) &= (x_1 + 1)^2 + (x_2 - 1)^2] \end{aligned} \\ \text{subject to} \quad & \begin{aligned} g_1(x_1, x_2) &= x_1^2 - x_2 && \leq 0 \\ g_2(x_1, x_2) &= x_1 + x_2 - 2 && \leq 0 \\ g_3(x_1, x_2) &= -x_1 && \leq 0. \end{aligned} \end{aligned}$$

The feasible set for this problem is $X = \{x \in \mathbb{R}^2 : g_i(x) \leq 0, i = 1, 2, 3\}$, and in spite of the lack of an underlying physical meaning, we call X the set of all feasible designs. Without practical interpretation, however, the four objectives also lack relative importances and thus do not allow for priority ranking according to their relevance on the overall performance. Instead, we group the objectives into the canonical pairs $f^1 = (f_1^1, f_2^1) = (f_1, f_2)$ and $f^2 = (f_1^2, f_2^2) = (f_3, f_4)$ and, accordingly, reduce further notational burden by replacing all double indices $_{11}, _{12}, _{21}$ and $_{22}$ by $_1, _2, _3$ and $_4$, respectively.

To find a preferred design to the above problem using the proposed procedure, we then start by solving

$$\begin{aligned} \text{COP}_1: \text{ minimize } & f^1 = [f_1(x), f_2(x)] \\ \text{subject to } & x \in X. \end{aligned}$$

Recall that solving COP_1 means to find a set of Pareto solutions and, from among those, select one design x^1 that will be used as baseline design for the second coordination problem. Thus, using an optimization approach to solve COP_1 , we first find the ten Pareto solutions that are depicted on the left in Figure 4 and then select the highlighted middle point as first design $x^1 = (x_1^1, x_2^1) = (0.4471, 1.5529)$. In a practical context, this choice would be justified by the fact that this design yields comparable performances of $f_1(x^1) = 2.7173$ and $f_2(x^1) = 2.2939$ and, thus, is one of the best compromise designs for COP_1 .

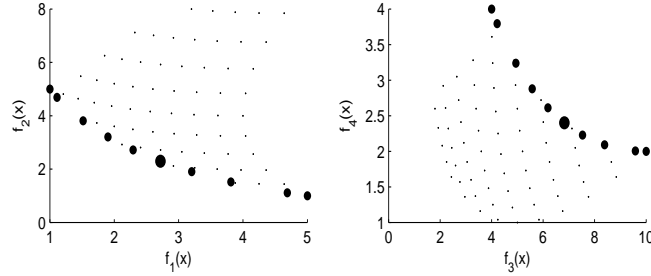


Figure 4: Pareto solutions for COP_1 with $f^1 = (f_1, f_2)$ and their images for $f^2 = (f_3, f_4)$

Note how, so far, our decision is merely based on the visualization of the two objectives f_1 and f_2 and, hence, found by 2D decision-making. The visualization of the Pareto designs for COP_1 with respect to the two other criteria f_3 and f_4 on the right in Figure 4 is not needed, but provided here for convenience. In particular, for later reference we report that $f_3(x^1) = 6.8232$ and $f_4(x^1) = 2.3997$.

Moreover, although usually not readily available, the two former plots depict a sample of the complete set of attainable outcomes to show how all Pareto solutions for the first subproblem are among the worst outcomes for the second. Recall that traditional decomposition methods that do not allow for a better coordination would already stop at this point and, although clearly undesirable, propose these solutions as final designs for the overall problem.

To formulate the coordination problem COP_2 , we first investigate the sensitivities of the current design x^1 with respect to the two criteria f_3 and f_4 by computing the Lagrangean multipliers for the sensitivity problems

$$\begin{array}{ll}
 \text{SEP}_3: & \text{minimize } f_3(x) \\
 & \text{subject to } f_1(x) \leq f_1(x^1) + \varepsilon_1 \\
 & f_2(x) \leq f_2(x^1) + \varepsilon_2 \\
 & f_4(x) \leq f_4(x^1) \\
 & x \in X
 \end{array}
 \qquad
 \begin{array}{ll}
 \text{SEP}_4: & \text{minimize } f_4(x) \\
 & \text{subject to } f_1(x) \leq f_1(x^1) + \varepsilon_1 \\
 & f_2(x) \leq f_2(x^1) + \varepsilon_2 \\
 & f_4(x) \leq f_4(x^1) \\
 & x \in X
 \end{array}$$

where, initially, $\varepsilon_1 = \varepsilon_2 = 0$. Then choosing the optimal design x^1 as initial design in our optimization routine, this immediately (after one iteration) confirms optimality of x^1 for both SEP_3 and SEP_4 and, according to the sensitivity theorem (2), provides us with the sensitivity information

$$\begin{array}{ll}
 \left. \frac{\partial f_3(x)}{\partial f_1(x)} \right|_{x=x^1} = -\lambda_{31} = 0.1706, & \left. \frac{\partial f_4(x)}{\partial f_1(x)} \right|_{x=x^1} = -\lambda_{41} = 1.1706, \\
 \left. \frac{\partial f_3(x)}{\partial f_2(x)} \right|_{x=x^1} = -\lambda_{32} = 1.8294, & \left. \frac{\partial f_4(x)}{\partial f_2(x)} \right|_{x=x^1} = -\lambda_{42} = 0.8294.
 \end{array}$$

Hence, we see that only two tradeoff values are greater than 1, thereby suggesting that improvement in f_3 , or f_4 , is best achieved by allowing some additional tolerance ε_2 on f_2 , or ε_1 on f_1 , respectively, and then solving the second coordination problem

$$\begin{aligned} \text{COP}_2: \text{ minimize } f^2 &= [f_3(x), f_4(x)] \\ \text{subject to } f_1(x) &\leq f_1(x^1) + \varepsilon_1 = 2.7173 + \varepsilon_1 \\ f_2(x) &\leq f_2(x^1) + \varepsilon_2 = 2.2939 + \varepsilon_2 \\ x &\in X. \end{aligned}$$

In particular, if we decide to focus on the more promising tradeoff between f_3 and f_2 , we might set the new tolerance $\varepsilon_2 = 1$ and, after solving COP_2 with $\varepsilon = (\varepsilon_1, \varepsilon_2) = (0, 1)$, select the new design $x_{new} = (0.4402, 1.2393)$ with performances $f(x_{new}) = (2.4903, 3.2939, 5.3278, 2.1314)$. In particular, the actually achieved tradeoff between f_3 and f_2 can now be computed as

$$\frac{\Delta f_3(x^1)}{\Delta f_2(x^1)} = \frac{f_3(x^1) - f_3(x_{new})}{f_2(x_{new}) - f_2(x^1)} = \frac{6.8232 - 5.3278}{3.2939 - 2.2939} = 1.4954.$$

Note that although the chosen tolerance $\varepsilon_2 = 1$ is rather large compared to the performance value $f^2(x^1) = 2.2939$, the local tradeoff $-\lambda_{32} = 1.8294$ at x^1 provides a quite reasonable estimate for the actual tradeoff of 1.4954. However and as emphasized before, the local tradeoff values should not be mistaken as predictions for the tradeoffs that are achieved globally, especially when we decide to change the tolerance values ε_1 and ε_2 simultaneously.

For illustration, assume that we now decide to set $\varepsilon = (\varepsilon_1, \varepsilon_2) = (1, 2)$. Based on the initial tradeoff values computed at x^1 and the tradeoff estimates (4), we might expect to gain improvements of $\Delta f_3(x^1) \approx -\lambda_{32}\varepsilon_2 = 1.8294 \cdot 2 = 3.6588$ and $\Delta f_4(x^1) \approx -\lambda_{41}\varepsilon_1 = 1.1706 \cdot 1 = 1.1706$. Then solving COP_2 for five new designs, we obtain the Pareto solutions depicted in Figure 5 and, together with their updated sensitivities, listed in Table 2. For convenience, we also circle all those sampled outcomes from Figure 4 that satisfy the new performance bounds of $f_1(x^1) + \varepsilon_1 = 3.7173$ and $f_2(x^1) + \varepsilon_2 = 4.2939$ and, thus, form the underlying set of attainable outcomes for COP_2 .

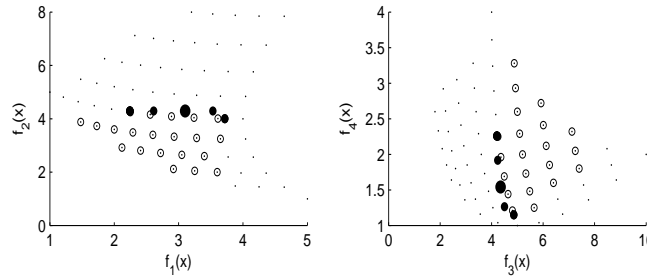


Figure 5: On the right, Pareto solutions for COP_2 with $f^2 = (f_3, f_4)$ and, on the left, their images for $f^1 = (f_1, f_2)$, with tolerances $\varepsilon = (\varepsilon_1, \varepsilon_2) = (1, 2)$

Note how the maximal improvements with respect to f^3 and f^4 ,

$$\Delta_{max} f^3(x^1) = f^3(x^1) - f^3(0.5026, 0.9897) = 6.8232 - 4.2064 = 2.6168 \text{ (expected: 3.6588)}$$

$$\Delta_{max} f^4(x^1) = f^4(x^1) - f^4(0.0720, 1.0000) = 2.3997 - 1.1491 = 1.2506 \text{ (expected: 1.1706)}$$

Table 2: Pareto solutions for COP₂ as depicted in Figure 5 and their updated sensitivities

x_1	x_2	f_1	f_2	f_3	f_4	λ_{31}	λ_{32}	λ_{41}	λ_{42}
0.5026	0.9897	2.2424	4.2939	4.2064	2.2578	0	0.9898	1.0034	0
0.3838	0.9637	2.6133	4.2939	4.2358	1.9163	0	0.9898	0.8558	0
0.2400	0.9418	3.1009	4.2939	4.3481	1.5410	0	0.9898	0.7036	0
0.1220	0.9314	3.5317	4.2939	4.5013	1.2635	0	0.9898	0.5964	0
0.0720	1.0000	3.7173	4.0052	4.8613	1.1491	0	1.0602	0.5560	0

are achieved by two different Pareto solutions and, thus, not achievable simultaneously. Therefore, a preferred solution must also compromise between these two improvements, which then can be accomplished by 2D decision-making for COP₂. Again selecting the best compromise design, we find the highlighted third design in Figure 5 or Table 2, $x^2 = (0.2400, 0.9418)$ with performances $f(x^2) = (3.1009, 4.2939, 4.3481, 1.5410)$, as preferred overall design. In particular, note that at this design all sensitivities and hence all remaining tradeoffs are reduced to values less than 1, suggesting to terminate the algorithm.

To conclude this example, also observe that while our final and, in fact, all highlighted designs in Figure 5 are Pareto optimal for the COP₂ and, thus, (at least weakly) Pareto optimal for the original MOP, none of these solutions actually lies on the Pareto curve for either the first or second subproblem. Hence, as expected, our methodology is capable to find a best compromise design that cannot be found using the traditional decomposition approach.

4.2 A structural optimization problem

While the discussion of the first example focused in detail on the evaluation of tradeoffs, we omit some of those details for our more practical second test problem to design a four bar plane truss structure (Koski, 1984, 1988; Stadler and Dauer, 1992).

The original mathematical model is a biobjective program with the two conflicting objectives of minimizing both the volume V of the truss and the displacement d_1 of the node joining bars 1 and 2, given the loading condition depicted for the leftmost truss in Figure 6.

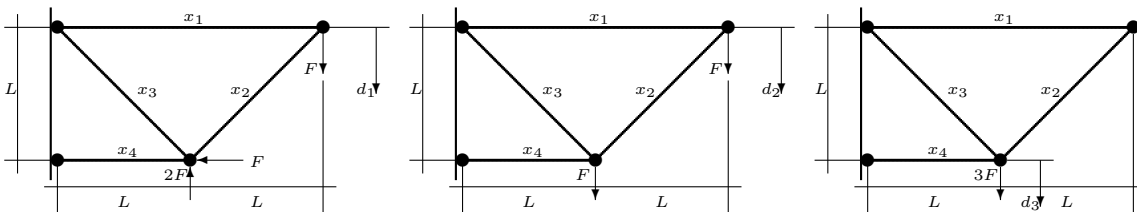


Figure 6: Three different loading conditions on a four bar plane truss structure

In addition, we use the two additional loading conditions (Koski, 1984) that are depicted for the middle and right truss in Figure 6, for which one can compute the two respective displacements d_2 and d_3 (Blouin, 2004). The length $L = 200$ cm of the structure, the acting force $F = 10$ kN, Young's modulus of elasticity $E = 2 \times 10^5$ kN/cm² and the only nonzero stress component $\sigma = 10$ kN/cm² are assumed to be constant. The cross sectional areas x_1 , x_2 , x_3 and x_4 of the four bars

are subject to physical restrictions yielding the feasible design set

$$X = \left\{ x = (x_1, x_2, x_3, x_4) : (F/\sigma) \leq x_1, x_4 \leq 3(F/\sigma), \sqrt{2}(F/\sigma) \leq x_2, x_3 \leq 3(F/\sigma) \right\}.$$

The overall problem then becomes to select a feasible preferred design that minimizes the structural volume $V(x)$ and joint displacements d_1, d_2, d_3 for the three different loading conditions

$$\begin{aligned} \text{Minimize } & \left[V(x) = L(2x_1 + \sqrt{2}x_2 + \sqrt{2}x_3 + x_4), \right. \\ & d_1(x) = \frac{FL}{E} \left(\frac{2}{x_1} + \frac{2\sqrt{2}}{x_2} - \frac{2\sqrt{2}}{x_3} + \frac{2}{x_4} \right), \\ & d_2(x) = \frac{FL}{E} \left(\frac{2}{x_1} + \frac{2\sqrt{2}}{x_2} + \frac{4\sqrt{2}}{x_3} + \frac{6}{x_4} \right), \\ & \left. d_3(x) = \frac{FL}{E} \left(\frac{6\sqrt{2}}{x_3} + \frac{3}{x_4} \right) \right] \text{ subject to } x \in X. \end{aligned}$$

This problem can be viewed as a multi-scenario multi-objective problem, with each loading condition acting as one of three possible scenarios. Based on the assumption that the first loading scenario occurs most often and the third scenario only very rarely, we decompose the overall problem into the three associated criteria pairs $f^1 = (f_1^1, f_2^1) = (d_1, V)$, $f^2 = (f_1^2, f_2^2) = (d_2, V)$ and $f^3 = (f_1^3, f_2^3) = (d_3, V)$. Note how the volume criterion participates in every scenario and, thus, is repeated in each subproblem. Using the proposed procedure to find a common design that is preferred for all three scenarios, we then start by solving

$$\begin{aligned} \text{COP}_1: \text{ minimize } & f^1(x) = [d_1(x), V(x)] \\ \text{subject to } & x \in X. \end{aligned}$$

Again, recall that solving COP_1 means to find a number of Pareto designs, from which we need to select a first baseline design for the subsequent coordination problems. As before, we choose an optimization approach and find the ten solutions that are highlighted in Figure 7.

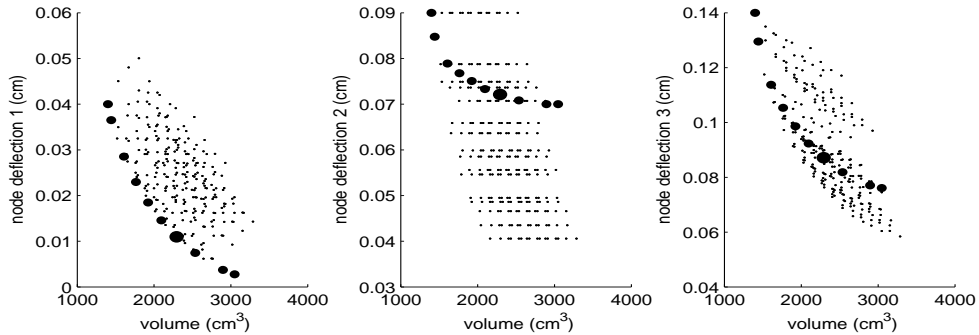


Figure 7: From left to right, Pareto solutions for COP_1 (first scenario) and their images for the second and third scenario

For convenience, we also plot a sample set of all attainable outcomes and show where the Pareto designs for COP_1 lie in the two other subproblems. In particular, provided we would be willing to accept maximal node deflections in all three scenarios, we see that we could choose the topmost design that is, in fact, Pareto optimal for all three scenarios. In practice, however, this design is not preferred because of its extremely poor performance with respect to all three node deflection criteria.

Therefore returning to the 2D decision-making as proposed by our procedure, we only focus on the first subproblem and select the highlighted point as preferred compromise design for COP_1 . The corresponding design $x^1 = (1.7459, 2.4732, 1.4142, 2.4730)$ is illustrated in Figure 8 and has performance values of $V(x^1) = 2292.5 \text{ cm}^3$, $d_1(x^1) = 0.0110 \text{ cm}$, $d_2(x^1) = 0.0721 \text{ cm}$ and $d_3(x^1) = 0.0872 \text{ cm}$.

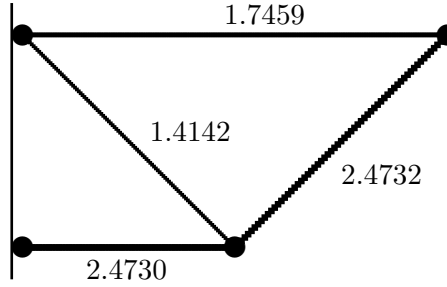


Figure 8: Selected Pareto design x^1 for COP_1 (first scenario)

Before we solve the next coordination problems, we formulate the two sensitivity problems

$$\begin{array}{ll}
 \text{SEP}_{21}: & \text{minimize } d_2(x) \\
 & \text{subject to } d_1(x) \leq d_1(x^1) + \varepsilon_{d_1} \\
 & \quad V(x) \leq V(x^1) + \varepsilon_V \\
 & \quad x \in X
 \end{array}
 \qquad
 \begin{array}{ll}
 \text{SEP}_{31}: & \text{minimize } d_3(x) \\
 & \text{subject to } d_1(x) \leq d_1(x^1) + \varepsilon_{d_1} \\
 & \quad V(x) \leq V(x^1) + \varepsilon_V \\
 & \quad x \in X
 \end{array}$$

but, for conceptual simplicity, restrict our analysis to the tradeoff between the different deflection criteria. Solving the two sensitivity problems SEP_{21} and SEP_{31} with x^1 as initial design, we obtain the associated Lagrangean multipliers

$$\left. \frac{\partial d_2(x)}{\partial d_1(x)} \right|_{x=x^1} = -\lambda_{21} = 413.77 \quad \text{and} \quad \left. \frac{\partial d_3(x)}{\partial d_1(x)} \right|_{x=x^1} = -\lambda_{31} = 538.03.$$

The very large magnitudes of these values clearly indicate that the currently selected design x^1 should be further improved. However, as emphasized before and quite obvious at this point, these values do not give us an actual prediction on the improvement that we should expect.

Considering that the current first node deflections of $d_1(x^1) = 0.01 \text{ cm}$ is significantly smaller than the second and third, $d_2(x^1) = 0.0721 \text{ cm}$ and $d_3(x^1) = 0.0872 \text{ cm}$, we assume that we would still be willing to accept a design that yields a first node deflection of up to 0.03 cm , provided a reasonable tradeoff with respect to d_2 or d_3 . Thus, we select the corresponding tolerance values $\varepsilon_{d_1} = 0.02 \text{ cm}$ and then solve the next coordination problem

$$\begin{aligned}
\text{COP}_2: \text{ minimize } f^2 &= [d_2(x), V(x)] \\
\text{subject to } d_1(x) &\leq d_1(x^1) + \varepsilon_{d_1} = 0.0310 \\
V(x) &\leq V(x^1) + \varepsilon_V = 2292.5 \\
x &\in X.
\end{aligned}$$

By solving COP_2 for the second scenario, we find a new set of Pareto optimal designs, that is depicted in the middle plot of Figure 9, with the corresponding performances for the first and third scenario depicted on the left and right, respectively.

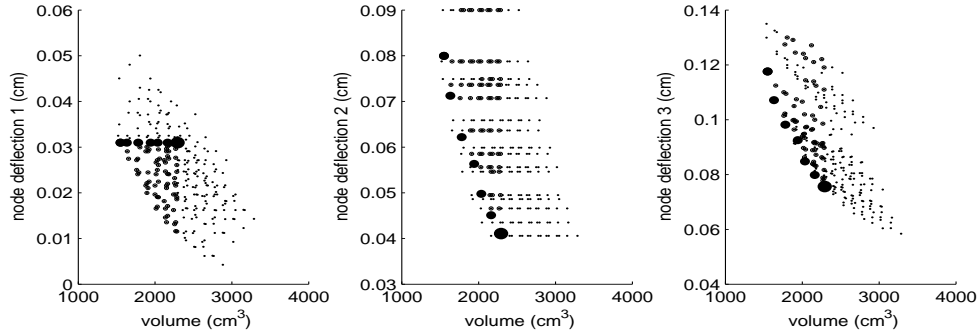


Figure 9: In the center, Pareto solutions for COP_2 (second scenario) and, on the left and right, their images for the first and third scenario, respectively

Note from the left plot that all these new solutions, in fact, meet the specified upper performance bound on d_1 for the first subproblem, while now visualizing a tradeoff between the volume and second node deflection in COP_2 . Assuming that main incentive is still the improvement with respect to node deflection d_2 , we decide not to further improve the structural volume and, thus, select the highlighted bottommost point as improved second design $x^2 = (1.1754, 1.6582, 2.7837, 2.8298)$, depicted in Figure 10.

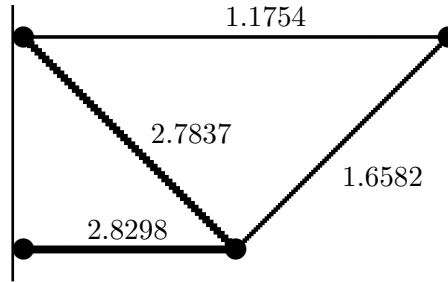


Figure 10: Selected Pareto design x^2 for COP_2 (second scenario)

The performances of this new design are given by $V(x^2) = 2292.5 \text{ cm}^3$, $d_1(x^2) = 0.0310 \text{ cm}$, $d_2(x^2) = 0.0411 \text{ cm}$ and $d_3(x^2) = 0.0756 \text{ cm}$. Hence, the actual tradeoffs achieved are

$$\begin{aligned}\frac{\Delta d_2(x)}{\Delta d_1(x)} &= \frac{d_2(x^1) - d_2(x^2)}{d_1(x^2) - d_1(x^1)} = \frac{0.0721 - 0.0411}{0.0310 - 0.0110} = 1.5500, \\ \frac{\Delta d_3(x)}{\Delta d_1(x)} &= \frac{d_3(x^1) - d_3(x^2)}{d_1(x^2) - d_1(x^1)} = \frac{0.0756 - 0.0872}{0.0310 - 0.0110} = 0.5800.\end{aligned}$$

As expected, the tradeoff between d^1 and d^2 achieved a value greater than 1 and, thus, was a favorable one. The smaller tradeoff value between d_1 and d_3 is not surprising either as we only solved COP₂ which, in fact, did not minimize with respect to d_3 . Nevertheless, from the right plot in Figure 9 we see that the current design x^2 already gives a reasonable compromise solution with respect to the third loading scenario that also involves d_3 . In particular, upon computing the updated sensitivity λ_{21} from SEP₂₁ at the new design x^2 and, in addition, including the additional constraint $d_2(x) \leq d_2(x^2) + \varepsilon_{d_2}$ with $\varepsilon_{d_2} = 0$ into a new SEP₃₁

$$\begin{aligned}&\text{minimize } d_3(x) \\ &\text{subject to } d_1(x) \leq d_1(x^1) + \varepsilon_{d_1} = d_1(x^2) = 0.0310 \\ &\quad d_2(x) \leq d_2(x^2) + \varepsilon_{d_2} = 0.0411 \\ &\quad V(x) \leq V(x^2) + \varepsilon_V = 2.2925 \\ &\quad x \in X,\end{aligned}$$

we obtain a set of new Lagrangean multipliers, yielding the updated sensitivities

$$\left. \frac{\partial d_2(x)}{\partial d_1(x)} \right|_{x=x^2} = -\lambda_{21} = 0.7877, \quad \left. \frac{\partial d_3(x)}{\partial d_1(x)} \right|_{x=x^2} = -\lambda_{31} = 0, \quad \left. \frac{\partial d_3(x)}{\partial d_2(x)} \right|_{x=x^2} = -\lambda_{32} = 0.2481.$$

In particular, since all Lagrangean multipliers are now less than 1, these sensitivities reveal that solving the third coordination problem is unlikely to yield significant further improvement and, thus, suggests to terminate this problem with x^2 from Figure 10 as final design.

5 Conclusion

In this paper, we have proposed an interactive decision-making procedure to select an overall preferred design for a complex and multi-criteria design optimization problem. Taking into account that these problems are usually solved by multiple designers or multi-disciplinary design teams, the selection of a final design is facilitated using a decomposition-integration framework, together with a novel coordination mechanism that guarantees that the chosen design is also commonly preferred for the overall problem.

The method requires that only one subproblem needs to be solved at a time, upon which the designer communicates a new baseline design and a set of new tolerances to the other design teams to sequentially coordinate the final design integration. Moreover, in order to not overload the individual designers with analytical analyses and to make tradeoff evaluation and decision-making the simplest possible, all decisions are reduced to merely two dimensions. As one of the main features of this procedure, this also enables the complete visualization of all design decisions in the form of 2D tradeoff curves.

The specification of tolerances for a selected design is supported by an additional tradeoff analysis that provides a very efficient way to compute design sensitivities with respect to different performance criteria. In particular, this extends the consideration of performance tradeoffs within one subproblem to tradeoffs that occur between different subproblems, in further enhancement of the proposed coordination and final design integration.

Based on several theoretical results and the illustration of the procedure on two examples, this method has been shown to

1. offer the capability of replacing the cumbersome tradeoff analysis for an overall multi-criteria problem with tradeoff analyses for a collection of smaller-sized bi-criteria problems;
2. make designers' judgment and knowledge about smaller-sized problems sufficient for the final design integration;
3. allow designers to visualize the Pareto curves in each subproblem and for each decision;
4. provide a general framework for the independent participation of multiple designers.

In our future work, we intend to further investigate the information that can be obtained from the proposed tradeoff and sensitivity analysis. In view of the current approach, we are aware of the remaining weakness that this information only allows a local tradeoff assessment and, thus, cannot be used for more accurate estimates in a larger region of the outcome space.

As for now, however, we believe that the proposed method is already well suited for performance-based decision-making, in particular because of its capability to allow for 2D decision-making with respect to all underlying tradeoff and design decisions. We are convinced that our future efforts will even further improve the recognized features of the current method and eventually provide a new and useful decision-making tool for finding better solutions in multi-criteria design optimization.

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