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Alexander Engau

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Clemson University, Department of Mathematical Sciences
O-110 Martin Hall, P.O. Box 340975, Clemson, SC 29634-0975, USA
(+ 001) 864-656-3434, (+ 001) 864-656-5230 (fax), mathsci@clemson.edu

Variable Preference Modeling with Ideal-Symmetric Convex Cones

Alexander Engau

Department of Mathematical Sciences
Clemson University
Clemson, SC 29634, USA
E-mail: aengau@clemson.edu

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Abstract

Based on the concept of general domination structures, this paper presents an approach to model variable preferences for multicriteria optimization and decision making problems. The preference assumptions for using a constant convex cone are given, and, in remedy of some immanent model limitations, a new set of assumptions is presented. The underlying preference model is derived as a variable domination structure that is defined by a collection of ideal-symmetric convex cones. Necessary and sufficient conditions for nondominance are established, and the problem of finding corresponding nondominated solutions is addressed and solved on examples.

Keywords: Multicriteria optimization – Multicriteria decision making – Preference models – Variable domination structures – Convex cones

1 Introduction

In both theory and practical applications of multicriteria optimization and decision making, the subjective nature of making decisions necessitates the formulation of some simplified yet realistic preference model. The notion and modeling of preferences also play an important role in many other fields such as economy, sociology, psychology, or mathematical programming, and is extensively researched during the past century (for a comprehensive recent survey, see Öztürk et al., 2005).

First described in the economic literature, probably still the most commonly used preference model in multicriteria optimization and decision making is based on the Edgeworth-Pareto Principle (Edgeworth, 1881; Pareto, 1896), also known as the concept of Pareto dominance. One of its main characteristics is that, in contrast to problems with only one criterion, in general there does not exist a unique optimal solution as best overall outcome, but a solution set of Pareto nondominated points. By using the concept of cones, the Pareto concept can be generalized to other domination cones (Yu, 1973, 1974), domination structures (Yu and Leitmann, 1974; Yu, 1975; Bergstresser et al., 1976; Yu, 1985) and domination sets (Weidner, 1985, 2003). Other authors compare these new concepts with Pareto dominance (Lin, 1976), extend the notions of proper efficiency from traditional multiobjective programming (Kuhn and Tucker, 1951; Geoffrion, 1968) to cone extreme points (Borwein, 1977; Benson, 1979, 1983; Coladas Uría, 1981; Henig, 1982, 1990), and propose generalizations of domination structures for more abstract spaces (Chew, 1979). Primarily focusing on closed and convex domination cones, several other properties of the nondominated set are investigated, including existence (Corley, 1980), connectedness (Naccache, 1978), stability (Tanino and Sawaragi, 1978, 1980), duality (Tanino and Sawaragi, 1979; Corley, 1981; Hsia and Lee, 1988), and optimality conditions for polyhedral cones (Tamura and Miura, 1979; Corley, 1985; Fujita, 1996).

Although some early papers also anticipate the use of domination structures in multicriteria games (Bergstresser and Yu, 1977) and decision making (Takeda and Nishida, 1980; Tanino et al., 1980), Ramesh et al. (1988, 1989) finally draw the attention to preference modeling using domination structures and develop a methodology for representing a decision-maker's preference structure using convex and polyhedral cones. In the following decade, however, the main focus switches from using domination sets to rough and fuzzy sets (Słowiński, 1998; Fodor et al., 2000), before Hunt and Wiecek (2003) and Hunt (2004) follow Noghin (1997) and again propose polyhedral cones to model preferences of the decision maker. Most recently, Wu (2004) combines the two approaches of fuzzy sets and domination cones, and Yun et al. (2004) suggest a generalized model that incorporates various preference structures of decision makers in the context of data envelopment analysis.

In all reviewed papers that use the concept of domination structures for preference modeling, the chosen model is described by a constant domination set, most often by a constant convex cone. The only current papers explicitly addressing variable domination structures are found in the context of nonlinear scalarization for multicriteria decision making problems and variational inequalities (Chen and Yang, 2002; Chen et al., 2005), but do not discuss their possible roles in preference modeling. Therefore, the objective of this paper is first to highlight some shortcomings of the current models and second to propose a new preference model in remedy of the recognized limitations.

The remaining paper is organized as follows. In Section 2, some common terminology and basic definitions are introduced. Section 3 formulates a set of preference assumptions, that are subsequently used to derive the corresponding preference models, and introduces the concept of ideal-symmetric convex cones to define the variable domination structure studied in this paper. The nondominated set with respect to this new model is characterized in Section 4, necessary and sufficient conditions for nondominance are derived, and the problem of finding corresponding nondominated solutions is addressed. In Section 5, the main results are illustrated on three examples, and Section 6 summarizes and finally concludes the paper.

2 Terminology and Definitions

Let $\mathbb{R}^m$ be a Euclidean space equipped with the Euclidean norm, and let the nonnegative orthant of $\mathbb{R}^m$ be denoted by $\mathbb{R}^m_{\geqq} = \{y \in \mathbb{R}^m : y \geqq 0\}$. A nonempty set $C \subset \mathbb{R}^m$ is called a *cone* if $c \in C \Rightarrow \lambda c \in C$ for all $\lambda > 0$, and it may or may not contain the origin $0 \in \mathbb{R}^m$. A cone $C \subset \mathbb{R}^m$ is said to be *convex* if $c^1, c^2 \in C \Rightarrow c^1 + c^2 \in C$, and *pointed* if $\sum_{i=1}^k c^i = 0 \Rightarrow c^i = 0$ for all $i = 1, \ldots, k$, where the $c^i \in C$ are any k elements of C. If C is convex, then C is pointed if and only if $C \cap -C \subset \{0\}$. The *dual cone* of C is defined by $C^+ = \{n \in \mathbb{R}^m : \langle n, c\rangle \geq 0 \text{ for all } c \in C\}$ with interior $\operatorname{int} C^+ = \{n \in \mathbb{R}^m : \langle n, c\rangle > 0 \text{ for all } c \in C \setminus \{0\}\}$, where $\langle n, c\rangle = \sum_{i=1}^m n_i c_i$, and C is called *self-dual* if $C = C^+$. Finally, a set $S \subset \mathbb{R}^m$ is said to be *C-compact* if $(s - C) \cap S$ is compact for all $s \in S$. It is *C-convex* if $S + C$ is a convex set, so $s^1, s^2 \in S + C \Rightarrow \lambda s^1 + (1-\lambda)s^2 \in S + C$ for all $0 \leq \lambda \leq 1$, and *C-concave* if it is $-C$-convex, or if $S - C$ is convex.

Remark 1. The set $\mathbb{R}^m_{\geqq}$ is a convex, pointed, and self-dual cone that contains the origin.

Now let $Y \subset \mathbb{R}^m$ be a nonempty set of outcomes subject to minimization. For two points $y^1, y^2 \in \mathbb{R}^m$, the notation $y^1 \prec y^2$ is used to denote that y^1 is preferred to y^2, or equivalently, that y^1 dominates y^2. Accordingly, $y^1 \not\prec y^2$ is used to denote that y^1 is not preferred to, or equivalently, that y^2 is not dominated by y^1.

Definition 1. Let $Y \subset \mathbb{R}^m$ be nonempty. The *multicriteria optimization* (MCO) and the *multicriteria decision making* (MCDM) problems are defined as

$$\text{MCO: Find } y^\circ \in Y \text{ such that } y \not\prec y^\circ \text{ for all } y \in Y \setminus \{y^\circ\}$$
$$\text{MCDM: Find } y^* \in Y \text{ such that } y^* \prec y \text{ for all } y \in Y \setminus \{y^*\}$$

If an outcome $y^\circ \in Y$ is a solution to MCO, then there does not exist another outcome that is preferred to, or dominates y°. Therefore, y° is also called a *nondominated outcome* for MCO. If an outcome $y^* \in Y$ is a solution to MCDM, then y^* is preferred to all other outcomes. Therefore, y^* is also called the *preferred outcome* for MCDM.

Remark 2. It follows immediately that the preferred outcome $y^* \in Y$ for MCDM is also nondominated for MCO, but not vice versa.

Definition 2. Let $y \in Y \subset \mathbb{R}^m$ and $y' \in \mathbb{R}^m$. If $y' \prec y$, then the vector $d = y - y' \in \mathbb{R}^m$ is called a *dominated direction* at y, and the set of all dominated directions at y is denoted by $D(y) = \{d = y - y' \in \mathbb{R}^m : y' \prec y\}$.

Equivalently, a direction $d \in \mathbb{R}^m$ is dominated at $y \in Y$ if and only if deviation d from y is preferred to the original y, or $y - d \prec y$ (Yu, 1974). Although $d = 0$, in principle, is not a direction and, in particular, $y \not\prec y$, due to technical reasons $d = 0 \in D(y)$ is permissible as special case.

Remark 3. If $D(y) = D$ for all $y \in Y$, and if $y + d \in Y$, then $D(y) = D(y + d)$ and, in particular, $y - d \prec y$ is equivalent to $y \prec y + d$. In general, however, if $D(y) \neq D(y + d)$, then it is possible that $y - d \prec y$, but $y \not\prec y + d$.

If the sets of dominated directions vary for different outcomes, then the collection $\mathcal{D} = \{D(y) : y \in Y\}$ is also called a *variable domination structure* for Y. If $D(y) = D$ for all $y \in Y$, then $\mathcal{D} = D$ is written instead of $\mathcal{D} = \{D\}$, and the domination structure is said to be *constant*.

Definition 3. Let $Y \subset \mathbb{R}^m$ be nonempty, and $\mathcal{D} = \{D(y) : y \in Y\}$ be a domination structure for Y. An outcome $y^\circ \in Y$ is said to be *nondominated with respect to* $\mathcal{D}$ if there does not exist a dominated direction $d \in D(y^\circ)$ such that $y^\circ - d \in Y \setminus \{y^\circ\}$, and the set of all nondominated outcomes of Y with respect to $\mathcal{D}$ is denoted by

$$\mathrm{N}(Y, \mathcal{D}) = \{y^\circ \in Y : (y^\circ - D(y^\circ)) \cap Y \subset \{y^\circ\}\}$$

If $d \in D(y)$ is a dominated direction, then the vector $c = -d$ is also called a *preferred direction*, and the set of all preferred directions at y is denoted by $C(y) = -D(y)$.

Remark 4. It follows that, equivalent to Definition 3, an outcome $y^\circ \in Y$ is nondominated with respect to $\mathcal{D}$ if there does not exist a preferred direction $c \in C(y)$ such that $y^\circ + c \in Y \setminus \{y^\circ\}$. From Definition 1, then $y^\circ \in Y$ is, in particular, a solution for MCO.

Based on Remark 2, however, this does not imply that the solution y° for MCO is also the preferred outcome for MCDM, because the domination structure $\mathcal{D}$, in general, does not capture all the preferences by the decision maker that are needed to obtain a unique nondominated solution for MCO. Only in the ideal case, the nondominated set for MCO reduces to a singleton and then also characterizes the preferred outcome for MCDM.

Definition 4. Let $Y \subset \mathbb{R}^m$ be nonempty. The point $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$ with $z_i = \inf\{y_i : y \in Y\}$ for all $i = 1, \ldots, m$ is called the *ideal point* of Y. An outcome $y \in Y$ with $y_i = z_i$ for some index i is also called *partially ideal*.

The ideal point as defined in Definition 4 may, in general, be finite or infinite. In this paper, however, it is assumed that the ideal point is finite.

Remark 5. If $Y \subset \mathbb{R}^m$ is $\mathbb{R}^m_{\geq}$-compact, then the infimum in Definition 4 can be replaced by the minimum and $z \in \mathbb{R}^m$ is, in particular, finite.

Clearly, if $z \in Y$, then $\mathrm{N}(Y, \mathcal{D}) = \{z\}$ and z is both a unique nondominated outcome for MCO and, thus, also preferred for MCDM. Since, in this case, both MCO and MCDM reduce to the computation of z, in this paper it is assumed that $z \notin Y$.

Notation 1. Throughout the remaining paper, the notation $\bar{y} = y - z$ is used to denote the direction from the ideal point $z \in \mathbb{R}^m$ to any $y \in Y$.

Since all outcomes are dominated by the ideal point, it follows that $\bar{y} \in D(y)$ for all $y \in Y$. In particular, Definition 4 implies that $\bar{y} \geq 0$ for all $y \in Y$, and $\bar{y} > 0$ if y is not partially ideal. The special case in which exactly the nonnegative directions $d \geq 0$ belong to the set of dominated directions defines the classical concept of Pareto dominance (Pareto, 1896).

Definition 5. Let $Y \subset \mathbb{R}^m$ be nonempty. An outcome $y \in \mathrm{N}(Y, \mathbb{R}^m_{\geq})$ is called a *Pareto outcome*, and $\mathrm{N}(Y, \mathbb{R}^m_{\geq})$ is called the *Pareto set* of Y. The cone $\mathbb{R}^m_{\geq}$ is also called the m-dimensional *Pareto cone*.

3 Preference and Model Assumptions

This section first presents two preference assumptions for a constant preference structure $D \subset \mathbb{R}^m$.

Assumption 1 (Global Preferences)**.** Let $y^1, y^2, y^3, y^4 \in \mathbb{R}^m$ and $\lambda > 0$.

(i). Multiplicativity: If $y^1 \prec y^2$, then $\lambda y^1 \prec \lambda y^2$.

(ii). Additivity: If $y^1 \prec y^3$ and $y^2 \prec y^4$, then $y^1 + y^2 \prec y^3 + y^4$.

Remark 6. In particular, if $y^1 = y^2$ and $y^3 = y^4$, then Assumption 1 (ii) reduces to Assumption 1 (i) with $\lambda = 2$. Moreover, if $y^i \in \mathbb{R}^m$ for $i = 1, \ldots, 2k$ and $y^j \prec y^{j+k}$ for $j = 1, \ldots, k$, then Assumption 1 (ii) implies that also $\sum_{j=1}^k y^j \prec \sum_{j=1}^k y^{j+k}$.

In Assumption 1, multiplicativity can be assumed based on the argument that preferences should not depend on criterion scaling, while additivity holds under the assumption that separate preferences remain valid upon simultaneous consideration and combination.

Assumption 2 (Minimization)**.** Let $y \in \mathbb{R}^m$ and $e^i \in \mathbb{R}^m$ be the ith unit vector. Then $y - e^i \prec y$.

The following two results derive that the set of dominated directions for a preference model satisfying Assumptions 1 and 2 is described by a constant convex cone that contains the Pareto cone.

Proposition 1. *Assumption 1 implies that the set of dominated directions $D = \{d = y^2 - y^1 \in \mathbb{R}^m : y^1 \prec y^2\}$ is a convex cone.*

Proof. To show that the set D is a cone, let $d \in D$ and $\lambda > 0$. Then there exist $y^1, y^2 \in \mathbb{R}^m$ so that $d = y^2 - y^1$ and $y^1 \prec y^2$, and Assumption 1 (i) implies that also $\lambda y^1 \prec \lambda y^2$ and, thus, $\lambda y^2 - \lambda y^1 = \lambda(y^2 - y^1) = \lambda d \in D$, showing that D is a cone.

To show that the cone D is convex, let $d^1, d^2 \in D$. Then there exist $y^1, y^2, y^3, y^4 \in \mathbb{R}^m$ so that $d^1 = y^3 - y^1$, $d^2 = y^4 - y^2$ and $y^1 \prec y^3$, $y^2 \prec y^4$, and Assumption 1 (ii) implies that also $y^1 + y^2 \prec y^3 + y^4$ and, thus $y^3 + y^4 - (y^1 + y^2) = (y^3 - y^1) + (y^4 - y^2) = d^1 + d^2 \in D$, showing that the cone D is convex. $\square$

The second result uses that, if $D \subset \mathbb{R}^m$ is a convex cone, then $D \cup \{0\}$ is also a convex cone.

Proposition 2. *Together with Assumption 1, Assumption 2 implies that the convex cone $D = \{d = y^2 - y^1 \in \mathbb{R}^m : y^1 \prec y^2\} \cup \{0\}$ contains the Pareto cone, $\mathbb{R}^m_{\geq} \subset D$.*

Proof. To show that the convex cone D contains the Pareto cone, let $d \in \mathbb{R}^m_{\geq}$, so $d = \sum_{i=1}^m d_i e^i = (d_1, \ldots, d_m) \geq 0$. From Assumption 2, $y - e^i \prec y$ and, thus, $d = y - (y - e^i) = e^i \in D$. If $d_i = 0$, then $d_i e^i = 0 \in D$, otherwise $d_i > 0$ and $d_i e^i \in D$ also, because D is a cone. Convexity of D then implies that $d = \sum_{i=1}^m d_i e^i \in D$, showing that D contains the Pareto cone, $\mathbb{R}^m_{\geq} \subset D$. $\square$

Although most preference models, that define nondominated solutions using the concept of a domination structure, are described by a constant convex cone, there exist some immanent model limitations and shortcomings.

Example 1. Let $Y = \{y \in \mathbb{R}^2 : y_1 + y_2 \geq 1, y_1 \geq 0, y_2 \geq 0\}$, and $D \subset \mathbb{R}^m$ be a constant convex cone that contains the Pareto cone, $\mathbb{R}^m_{\geq} \subset D$. Note that the ideal point $z = (0,0) \notin Y$, and that all outcomes $y = (y_1, y_2) \in Y$ with $y_1 \geq 1$ and $y_2 = 0$, or $y_1 = 0$ and $y_2 \geq 1$, are partially ideal. In particular, denote $z^1 = (1,0)$, $z^2 = (0,1)$, and let $d^1 = z^2 - z^1 = (-1,1)$ and $d^2 = z^1 - z^2 = (1,-1)$.

(i). If $d^1, d^2 \in D$ or, equivalently, $D = \{y \in \mathbb{R}^2 : y_1 + y_2 \geq 0\}$, then $\mathrm{N}(Y, D) = \emptyset$.

(ii). If $d^1 \in D$ and $d^2 \notin D$, or $d^1 \notin D$ and $d^2 \in D$, then $\mathrm{N}(Y, D) = \{z^1\}$ or $\mathrm{N}(Y, D) = \{z^2\}$, respectively.

(iii). If $d^1, d^2 \notin D$ or, equivalently, $D \subset \mathrm{int}\{y \in \mathbb{R}^2 : y_1 + y_2 \geq 0\}$, then $\mathrm{N}(Y, D) = \{y \in Y : y_1 + y_2 = 1\}$.

Hence, using a constant convex cone D that contains the Pareto cone, the nondominated set of Y is either (i) empty, (ii) a singleton, or (iii) the complete line segment $\{y \in \mathbb{R}^2 : y_1 + y_2 = 1, y_1 \geq 0, y_2 \geq 0\}$. In particular, it is not possible to define a preference model that excludes the two extreme points z^1, z^2 while maintaining a set of nondominated outcomes in the middle region of the Pareto set, $\emptyset \neq \mathrm{N}(Y, D) \subset \{y \in \mathbb{R}^2 : y_1 + y_2 = 1, y_1 > 0, y_2 > 0\}$, for further consideration by a decision maker.

To remove the model limitation illustrated by Example 1, the global preference assumptions in Assumption 1 need to be modified to eventually allow for the definition of a variable domination structure $\mathcal{D} = \{D(y) : y \in Y\}$, based on a set of corresponding local preferences.

Assumption 3 (Local Preferences)**.** Let $Y \subset \mathbb{R}^m$ be nonempty, $y \in Y$, $d, d^1, d^2 \in \mathbb{R}^m$, and $\lambda > 0$.

(i). Multiplicativity: If $y - d \prec y$, then $y - \lambda d \prec y$.

(ii). Additivity: If $y - d^1 \prec y$ and $y - d^2 \prec y$, then $y - (d^1 + d^2) \prec y$.

Proposition 3. *Let $Y \subset \mathbb{R}^m$ be nonempty, $y \in Y$, and $D(y) = \{d \in \mathbb{R}^m : y - d \prec y\} \cup \{0\}$. Assumption 3 implies that $D(y)$ is a convex cone and, together with Assumption 2, that $D(y)$ contains the Pareto cone, $\mathbb{R}^m_{\geq} \subset D(y)$.*

Proof. To show that the set $D(y)$ is a cone, let $d \in D(y)$ and $\lambda > 0$. If $d = 0$, then $\lambda d = 0 \in D(y)$, otherwise $y - d \prec y$ and Assumption 3 (i) implies that also $y - \lambda d \prec y$ and, thus, $\lambda d \in D(y)$, showing that $D(y)$ is a cone.

To show that the cone $D(y)$ is convex, let $d^1, d^2 \in D(y)$. If $d^1 = 0$ or $d^2 = 0$, then $d^1 + d^2 \in D(y)$, otherwise $y - d^1 \prec y$, $y - d^2 \prec y$ and Assumption 3 (ii) implies that also $y - (d^1 + d^2) \prec y$ and, thus $d^1 + d^2 \in D(y)$, showing that the cone $D(y)$ is convex.

To show that $D(y)$ contains the Pareto cone, repeat the proof of Proposition 2. □

Proposition 3 shows that the domination structure for a preference model that satisfies Assumptions 2 and 3 is described by a collection of convex cones that contain the Pareto cone. The final assumption is motivated by another example that also prepares the subsequent notion of ideal-symmetric cones.

Example 2. Let $Y \subset \mathbb{R}^2$ and $y^1, y^2, y^3, y^4 \in Y$ with $y^1 + y^4 = y^2 + y^3$ be as depicted in Figure 1. Restricting consideration to these four outcomes, y^1, y^2, and y^3 are nondominated with respect to the Pareto cone, while y^4 is dominated by y^1. In particular, y^4 is neither dominated by nor preferred to y^2 and y^3. Although arguable, in principle, it seems reasonable that in a practical decision making context, y^1 would be preferred to y^2 and y^3 and, thus, be the overall best outcome. Hence, the underlying preference model should give that y^1 is preferred to all the three other outcomes, but it should not introduce any additional preference relationships between y^2, y^3 and y^4. Using a constant convex cone $D \subset \mathbb{R}^2$, however, $y^1 \prec y^2$ and $y^1 \prec y^3$ are equivalent with $y^2 - y^1$ and $y^3 - y^1 \in D$ and, thus, also imply that $y^4 - y^2 = y^3 - y^1$ and $y^4 - y^3 = y^2 - y^1 \in D$, or $y^2 \prec y^4$ and $y^3 \prec y^4$, respectively. In particular, it is not possible to define a preference model that allows to individually specify one or both of the preference relationships $y^1 \prec y^2$ and $y^1 \prec y^3$ between y^1, y^2, and y^3, without affecting the preference relationships $y^2 \not\prec y^4$ and $y^3 \not\prec y^4$ between y^2, y^3 and y^4.

To remove the model limitations illustrated by Examples 1 and 2, this paper suggests to define a new model that allows for variable preferences and, thus, can be described by a variable domination structure $\mathcal{D} = \{D(y) : y \in Y\}$. Variability of $\mathcal{D}$ is introduced by the assumption that the set of preferred directions at any $y \in Y$ is symmetric with respect to the direction leading to the ideal point $z \in \mathbb{R}^2$, and Figure 1 provides insight into how this assumption is capable to model preference relationships similar to the ones discussed in Example 2. Equivalently, then $D(y)$ is symmetric with respect to $\bar{y} = y - z \geq 0$, and the corresponding notion of ideal-symmetry is introduced in the following assumption.

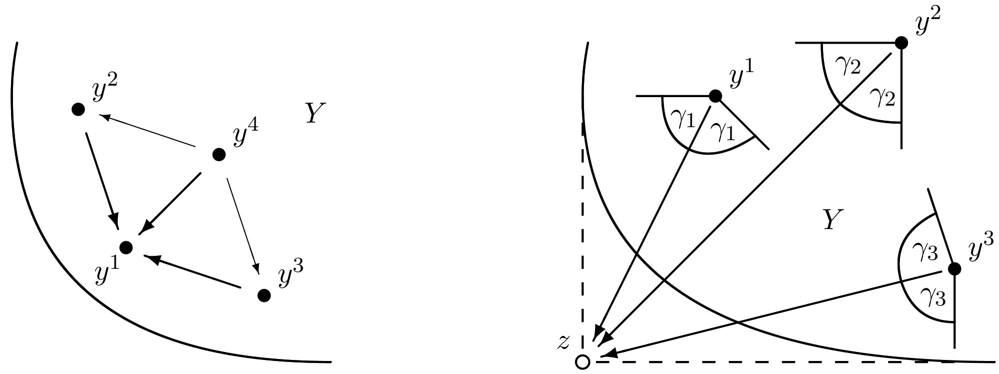

Figure 1: Illustration of Example 2 (on the left) and Assumption 4 (on the right)

Assumption 4 (Ideal-Symmetry)**.** Let $Y \subset \mathbb{R}^m$ be nonempty and $y \in Y$. If $d, d' \in \mathbb{R}^m$ with $\langle d, \bar{y} \rangle = \langle d', \bar{y} \rangle$ and $\|d\| = \|d'\|$, then $y - d \prec y$ if and only if $y - d' \prec y$.

Remark 7. In particular, if $d \neq 0$, $\langle d, \bar{y} \rangle = \langle d', \bar{y} \rangle$, and $\|d\| = \|d'\|$, then $\langle d, \bar{y} \rangle \|d\|^{-1} \|\bar{y}\|^{-1} = \langle d', \bar{y} \rangle \|d'\|^{-1} \|\bar{y}\|^{-1}$. For the bicriteria or tricriteria case, $m = 2$ or $m = 3$, results from analytical geometry give that $\cos \measuredangle(d, \bar{y}) = \cos \measuredangle(d', \bar{y})$, or $\measuredangle(d, \bar{y}) = \measuredangle(d', \bar{y})$, as depicted in Figure 1. For $m > 3$, Assumption 4 gives the natural generalization.

Based on the notion of ideal-symmetry in Assumption 4, the following lemma calls a cone $C \subset \mathbb{R}^m$ *symmetric with respect to* $s \in \mathbb{R}^m_{\geq}$, $s \neq 0$, if $c \in C$ implies that $c' \in C$ for all $c' \in \mathbb{R}^m$ with $\langle c, s \rangle = \langle c', s \rangle$ and $\|c\| = \|c'\|$. For further convenience, although ambivalent to Figure 1, now the parameter γ is used to denote the cosine of any corresponding angle $\gamma_1, \gamma_2, \gamma_3$.

Lemma 1. *Let* $\gamma \in \mathbb{R}$, $s \in \mathbb{R}^m_{\geq}$, $s \neq 0$, *and define*

$$C = \left\{ c \in \mathbb{R}^m : \frac{\langle c, s \rangle}{\|c\| \|s\|} \geq \gamma \right\} \cup \{0\}$$

Then C *is a cone that is symmetric with respect to* s.

(i) If $\gamma \geq 0$, *then* C *is convex.*

(ii) If $\gamma > 0$, *then* C *is convex and pointed.*

(iii) If $\gamma \leq \min_i \{s_i\} \|s\|^{-1}$, *then* C *contains the Pareto cone.*

Proof. To show that C is a cone, let $c \in C$ and $\lambda > 0$. If $c = 0$, then $\lambda c = 0 \in C$, otherwise

$$\frac{\langle \lambda c, s \rangle}{\|\lambda c\| \|s\|} = \frac{\lambda \langle c, s \rangle}{\lambda \|c\| \|s\|} = \frac{\langle c, s \rangle}{\|c\| \|s\|} \geq \gamma$$

and, thus, $\lambda c \in C$, showing that C is a cone. To show that the cone C is symmetric with respect to s, let $c \in C$ and $c' \in \mathbb{R}^m$ with $\langle c, s \rangle = \langle c', s \rangle$ and $\|c\| = \|c'\|$. If $c = 0$, then $c' = 0 \in C$, otherwise

$$\frac{\langle c', s \rangle}{\|c'\| \|s\|} = \frac{\langle c, s \rangle}{\|c\| \|s\|} \geq \gamma$$

and, thus, $c' \in C$, showing the the cone C is symmetric with respect to s.

(i) Let $\gamma \geq 0$. To show that the cone C is convex, let $c^1, c^2 \in C$. If $c^1 = c^2 = 0$, then $c^1 + c^2 = 0 \in C$, otherwise

$$\frac{\langle c^1 + c^2, s\rangle}{\|c^1 + c^2\|\|s\|} \geq \frac{\langle c^1, s\rangle + \langle c^2, s\rangle}{(\|c^1\| + \|c^2\|)\|s\|} \geq \frac{\gamma\|c^1\| + \gamma\|c^2\|}{(\|c^1\| + \|c^2\|)} = \gamma$$

and, thus, $c^1 + c^2 \in C$, showing that the cone C is convex.

(ii) Let $\gamma > 0$, then $\gamma \geq 0$ and the cone C is convex. To show that the convex cone C is pointed, let $c \in C \setminus \{0\}$, then

$$\frac{\langle -c, s\rangle}{\|-c\|\|s\|} = -\frac{\langle c, s\rangle}{\|c\|\|s\|} \leq -\gamma < \gamma$$

and, thus, $-c \notin C$, $c \notin -C$, or $C \cap -C \subset \{0\}$, showing that the convex cone C is pointed.

(iii) Let $\gamma \leq \min_i\{s_i\}\|s\|^{-1}$. To show that the cone C contains the Pareto cone, let $c = \sum_{i=1}^m c_i e^i = (c_1, \ldots, c_m) \geq 0$. If $c = 0$, then $c \in C$, otherwise

$$\frac{\langle c, s\rangle}{\|c\|\|s\|} = \frac{\langle \sum_{i=1}^m c_i e^i, s\rangle}{\|\sum_{i=1}^m c_i e^i\|\|s\|} \geq \frac{\sum_{i=1}^m c_i \langle e^i, s\rangle}{\sum_{i=1}^m c_i \|e^i\|\|s\|} = \frac{\sum_{i=1}^m c_i s_i}{\sum_{i=1}^m c_i \|s\|} \geq \frac{\min_i\{s_i\}}{\|s\|} \geq \gamma$$

and, thus, $c \in C$, showing that the cone C contains the Pareto cone. □

Notation 2. Throughout the remaining paper, the notation $\bar{y}_{\min} = \min_i\{\bar{y}_i\}$ is used to denote the minimal component of $\bar{y} = y - z$ for any $y \in Y$.

From Definition 4, it then follows that $\bar{y}_{min} > 0$ if and only if y is not partially ideal. The concluding result now follows immediately from Lemma 1 and Proposition 3.

Proposition 4. *Let $Y \subset \mathbb{R}^m$ be nonempty. Assumptions 2, 3 and 4 imply that the domination structure $\mathcal{D}$ of Y is variable and described by a collection of ideal-symmetric convex cones that contain the Pareto cone. Moreover, this variable domination structure can be modeled by $\mathcal{D} = \{D(y) : y \in Y\}$, where*

$$D(y) = \{d \in \mathbb{R}^m : \langle d, \bar{y}\rangle \geq \|d\|\bar{y}_{min}\}$$

for all $y \in Y$. In particular, the cone $D(y)$ is pointed if and only if y is not partially ideal.

Although, in principle, any choice $0 \leq \gamma \leq \bar{y}_{min}\|\bar{y}\|^{-1}$ in Lemma 1 would define a suitable domination cone at y for a preference model satisfying Assumptions 2, 3 and 4, the choice $\gamma = \bar{y}_{min}\|\bar{y}\|^{-1}$ in Proposition 4 is particularly convenient for the further characterization of the associated nondominated set now pursued in Section 4.

4 Characterization of the Nondominated Set

The first result is a direct application of the definition.

Lemma 2. *Let $Y \subset \mathbb{R}^m$ be nonempty, and $\mathcal{D} = \{D(y) : y \in Y\}$ be defined by $D(y) = \{d \in \mathbb{R}^m : \langle d, \bar{y}\rangle \geq \|d\|\bar{y}_{min}\}$ for all $y \in Y$. Then $y^\circ \in Y$ is nondominated with respect to $\mathcal{D}$, $y^\circ \in \mathrm{N}(Y, \mathcal{D})$, if and only if $\langle y^\circ - y, \bar{y}^\circ\rangle < \|y^\circ - y\|\bar{y}^\circ_{\min}$ for all $y \in Y \setminus \{y^\circ\}$.*

Proof. Let $y^\circ \in \mathrm{N}(Y, \mathcal{D})$, then, by definition, $(y^\circ - D(y^\circ)) \cap Y \subset \{y^\circ\}$ and, thus, there does not exist $d \in D(y^\circ)$ such that $y^\circ - d = y \in Y \setminus \{y^\circ\}$. Equivalently, then there does not exist $y \in Y \setminus \{y^\circ\}$ such that $d = y^\circ - y \in D(y^\circ)$, or $y^\circ - y \notin D(y^\circ)$ for all $y \in Y \setminus \{y^\circ\}$. Then, by definition of $D(y^\circ)$, this implies that $y^\circ \in \mathrm{N}(Y, \mathcal{D})$ if and only if $\langle y^\circ - y, \bar{y}^\circ \rangle < \|y^\circ - y\| \bar{y}^\circ_{\min}$ for all $y \in Y \setminus \{y^\circ\}$. □

Now consider the single criterion optimization problem

$$\text{SCO1: Minimize } \langle y, \bar{y}^\circ \rangle \text{ subject to } y \in Y$$

which corresponds to the weighted sum generating method from multiobjective programming (Gass and Saaty, 1955; Zadeh, 1963; Geoffrion, 1968) with weighting vector $\bar{y}^\circ \in \mathbb{R}^m$, here generalized for the variable domination structure $\mathcal{D} = \{D(y) : y \in Y\}$.

Proposition 5. *Let $Y \subset \mathbb{R}^m$ be nonempty, and $\mathcal{D} = \{D(y) : y \in Y\}$ be defined by $D(y) = \{d \in \mathbb{R}^m : \langle d, \bar{y} \rangle \geq \|d\| \bar{y}_{min}\}$ for all $y \in Y$.*

(i) If $y^\circ \in Y$ is a unique optimal solution to SCO1, then $y^\circ \in \mathrm{N}(Y, \mathcal{D})$.

(ii) If $y^\circ \in Y$ is an optimal solution to SCO1 and not partially ideal, then $y^\circ \in \mathrm{N}(Y, \mathcal{D})$.

Proof. Let $y^\circ \in Y$ be an optimal solution to SCO1, then $\langle y^\circ, \bar{y}^\circ \rangle \leq \langle y, \bar{y}^\circ \rangle$ for all $y \in Y \setminus \{y^\circ\}$.

(i) If y° is unique, then $\langle y^\circ, \bar{y}^\circ \rangle < \langle y, \bar{y}^\circ \rangle$ for all $y \in Y \setminus \{y^\circ\}$. In particular, it follows that $\langle y^\circ - y, \bar{y}^\circ \rangle = \langle y^\circ, \bar{y}^\circ \rangle - \langle y, \bar{y}^\circ \rangle < 0 \leq \|y^\circ - y\| \bar{y}^\circ_{min}$, and Lemma 2 implies that $y^\circ \in \mathrm{N}(Y, \mathcal{D})$.

(ii) If y° is not partially ideal, $\bar{y}^\circ_{min} > 0$, then it follows that $\langle y^\circ - y, \bar{y}^\circ \rangle = \langle y^\circ, \bar{y}^\circ \rangle - \langle y, \bar{y}^\circ \rangle \leq 0 < \|y^\circ - y\| \bar{y}^\circ_{min}$ for all $y \in Y \setminus \{y^\circ\}$, and again Lemma 2 implies $y^\circ \in \mathrm{N}(Y, \mathcal{D})$. □

Hence, to verify if an outcome $y^\circ \in Y$ is nondominated with respect to $\mathcal{D}$, SCO1 can be solved, and, if y° is a unique optimal solution, or if y° is an optimal solution and not partially ideal, then $y^\circ \in \mathrm{N}(Y, \mathcal{D})$.

Remark 8. From multiobjective programming, a solution generated by the weighted sum method is only known to be nondominated if the solution is unique and if the weighting vector is chosen from the dual cone $D(y^\circ)^+$, or from its interior $\operatorname{int} D(y^\circ)^+$ (see Sawaragi et al., 1985, among others). Since, by definition, $\langle d, \bar{y}^\circ \rangle \geq \|d\| \bar{y}^\circ_{min} \geq 0$ for all $d \in D(y^\circ)$, this shows that $y^\circ \in D(y^\circ)^+$. Moreover, if y° is not partially ideal, then $\bar{y}^\circ_{min} > 0$ and, thus, $\langle d, y^\circ \rangle \geq \|d\| \bar{y}^\circ_{min} > 0$ for all $d \in D(y^\circ) \setminus \{0\}$, showing that $\bar{y}^\circ \in \operatorname{int} D(y^\circ)^+$.

It is also known that problem SCO1 can only generate nondominated points that occur in convex regions of the nondominated frontier. However, by the slight modification of introducing an additional reference point $r \in Y$ (Wendell and Lee, 1977; Corley, 1980; Guddat et al., 1985), the problem

$$\text{SCO2: Minimize } \langle y, \bar{y}^\circ \rangle \text{ subject to } r - y \in D(y^\circ) \text{ and } y \in Y$$

is capable of generating the complete nondominated set.

Proposition 6. *Let $Y \subset \mathbb{R}^m$ be nonempty, and $\mathcal{D} = \{D(y) : y \in Y\}$ be defined by $D(y) = \{d \in \mathbb{R}^m : \langle d, \bar{y} \rangle \geq \|d\| \bar{y}_{min}\}$ for all $y \in Y$.*

(i) If $y^\circ \in Y$ is a unique optimal solution to SCO2, then $y^\circ \in \mathrm{N}(Y, \mathcal{D})$.

(ii) If $y^\circ \in Y$ is an optimal solution to SCO2 and not partially ideal, then $y^\circ \in \mathrm{N}(Y, \mathcal{D})$.

Proof. Let $y^\circ \in Y$ be an optimal solution to SCO2 and, by contradiction, assume that $y^\circ \notin \mathrm{N}(Y, \mathcal{D})$. Then there exists $y \in (y^\circ - D(y^\circ)) \cap Y \setminus \{y^\circ\}$, so $y^\circ - y = d \in D(y^\circ)$, or equivalently, $\langle y^\circ - y, \bar{y}^\circ \rangle \geq \|y^\circ - y\| \bar{y}^\circ_{min}$. In particular, $y \in Y$ is feasible for SCO2, because $r - y = r - y^\circ + d \in D(y^\circ)$ by feasibility of y° for SCO2 and convexity of $D(y^\circ)$.

(i) If $y^\circ \in Y$ is a unique optimal solution to SCO2, then $\langle y^\circ, \bar{y}^\circ \rangle < \langle y, \bar{y}^\circ \rangle$ and, thus, $\langle y^\circ - y, \bar{y}^\circ \rangle < 0 \leq \|y^\circ - y\| \bar{y}^\circ_{min} \leq \langle y^\circ - y, \bar{y}^\circ \rangle$ a contradiction, so $y^\circ \in \mathrm{N}(Y, \mathcal{D})$.

(ii) If $y^\circ \in Y$ is not partially ideal, $\bar{y}^\circ_{min} > 0$, then $\langle y^\circ, \bar{y}^\circ \rangle \leq \langle y, \bar{y}^\circ \rangle$ and, thus, $\langle y^\circ - y, \bar{y}^\circ \rangle \leq 0 < \|y^\circ - y\| \bar{y}^\circ_{min} \leq \langle y^\circ - y, \bar{y}^\circ \rangle$ a contradiction, so $y^\circ \in \mathrm{N}(Y, \mathcal{D})$. □

Remark 9. If $r = y^\circ \in \mathrm{N}(Y, D)$, then y° is a unique optimal solution to SCO2, so all nondominated outcomes of Y can be generated as solution to problem SCO2.

Based on Remark 9, problem SCO2 also provides a method to verify if any outcome $y \in Y$ is nondominated with respect to the variable domination structure $\mathcal{D}$. To restrict the initial set of points that would actually need to be checked, the next results show that all nondominated outcomes can be found within the Pareto set.

Lemma 3. *Let $Y \subset \mathbb{R}^m$ be nonempty, and $\mathcal{D}^1 = \{D^1(y) : y \in Y\}$ and $\mathcal{D}^2 = \{D^2(y) : y \in Y\}$ be two domination structures with $D^2(y) \subset D^1(y)$ for all $y \in Y$. Then $\mathrm{N}(Y, \mathcal{D}^1) \subset \mathrm{N}(Y, \mathcal{D}^2)$.*

The proof of Lemma 3 follows analogously to the corresponding result for two constant cones $D^2 \subset D^1$ (Sawaragi et al., 1985). In particular, since $\mathbb{R}^m_{\geq} \subset D(y)$ for all $y \in Y$ by Proposition 4, Lemma 3 now implies the following result.

Proposition 7. *Let $Y \subset \mathbb{R}^m$ be nonempty, and $\mathcal{D} = \{D(y) : y \in Y\}$ be defined by $D(y) = \{d \in \mathbb{R}^m : \langle d, \bar{y} \rangle \geq \|d\| \bar{y}_{min}\}$ for all $y \in Y$. Then $\mathrm{N}(Y, \mathcal{D}) \subset \mathrm{N}(Y, \mathbb{R}^m_{\geq})$.*

Hence, to solve the multicriteria optimization problem MCO under the preference model given by Assumptions 2, 3 and 4, the previous discussion suggests to first find the Pareto set and then check which Pareto points remain nondominated with respect to the variable domination structure $\mathcal{D}$. The latter can be accomplished by solving one of the single criterion optimization problems SCO1 or SCO2 with weighting vector $\bar{y}^\circ$ and, primarily for nonconvex problems and SCO2, additional reference point y°. Instead of solving another optimization problem, alternative optimality conditions can be established for the special cases of both convex or concave bicriteria problems. The former is based on the following application of the supporting hyperplane theorem (Rockafellar, 1970).

Lemma 4. *Let $Y \subset \mathbb{R}^m$ be $\mathbb{R}^m_{\geq}$-convex and $y^\circ \in \mathrm{N}(Y, \mathbb{R}^m_{\geq})$. Then there exists a supporting hyperplane of Y at y° with normal vector $n \in \mathbb{R}^m$, $n \neq 0$, so that $\langle n, y - y^\circ \rangle \geq 0$ for all $y \in Y$.*

Remark 10. If $n \in \mathbb{R}^m$ is the normal vector from Lemma 4, then it follows that $\langle n, y \rangle \geq \langle n, y^\circ \rangle$ for all $y \in Y$. In particular, similar to Proposition 5 and Remark 8, this implies that $y^\circ \in \mathrm{N}(Y, \mathcal{D})$ if $n \in \operatorname{int} D(y^\circ)^+$, or equivalently, if $\langle n, d \rangle > 0$ for all $d \in D(y^\circ) \setminus \{0\}$.

The condition $n \in \operatorname{int} D(y^\circ)^+$, in general, cannot be directly verified, but requires to solve the single criterion mathematical cone program (for details, see Alizadeh and Goldfarb, 2003)

$$\text{SCO3: Minimize } \langle n, d\rangle \text{ subject to } d \in D(y^\circ) = \{d \in \mathbb{R}^m : \langle d, \bar{y}^\circ\rangle > \|d\|\bar{y}^\circ_{min}\}$$

so that $n \in \operatorname{int} D(y^\circ)^+$ if and only if the optimal objective function value for SCO3 is positive. For the bicriteria and tricriteria case, an equivalent condition can also be derived using arguments from analytic geometry. For simplicity, the following theorem is formulated and proven only for the case $m = 2$.

Theorem 1. *Let $Y \subset \mathbb{R}^2$ be nonempty, and $\mathcal{D} = \{D(y) : y \in Y\}$ be defined by $D(y) = \{d \in \mathbb{R}^2 : \langle d, \bar{y}\rangle \geq \|d\|\bar{y}_{min}\}$ for all $y \in Y$. Assume that $\mathrm{N}(Y, \mathbb{R}^2_{\geq})$ is $\mathbb{R}^2_{\geq}$-convex, let $y^\circ \in \mathrm{N}(Y, \mathbb{R}^2_{\geq})$, and $n \in \mathbb{R}^m$, $n \neq 0$, be the normal vector of a supporting hyperplane at y° with $\langle n, y - y^\circ\rangle \geq 0$ for all $y \in Y$.*

(i) If $\langle n, \bar{y}^\circ\rangle^2 + \|n\|^2\bar{y}^{\circ 2}_{\min} > \|n\|^2\|\bar{y}^\circ\|^2$, then $y^\circ \in \mathrm{N}(Y, \mathcal{D})$.

Furthermore, let n satisfy that $\langle n, y - y^\circ\rangle > 0$ for all $y \in Y \setminus \{y^\circ\}$.

(ii) If $\langle n, \bar{y}^\circ\rangle^2 + \|n\|^2\bar{y}^{\circ 2}_{\min} \geq \|n\|^2\|\bar{y}^\circ\|^2$, then $y^\circ \in \mathrm{N}(Y, \mathcal{D})$.

Proof. Let $\mathrm{N}(Y, \mathbb{R}^2_{\geq})$ be $\mathbb{R}^2_{\geq}$-convex, and let $y^\circ \in \mathrm{N}(Y, \mathbb{R}^2_{\geq})$. As shown in Figure 2, let $\eta = \measuredangle(n, \bar{y}^\circ)$ be the positive angle between n and $\bar{y}^\circ$, so

$$0 \leq \cos\eta = \cos\measuredangle(n, \bar{y}^\circ) = \frac{\langle n, \bar{y}^\circ\rangle}{\|n\|\|\bar{y}^\circ\|} \leq 1$$

and $0 \leq \sin\delta \leq 1$. Let $\delta = \max\{\measuredangle(d, \bar{y}^\circ) : d \in D(y^\circ)\}$ be the maximal positive angle between any $d \in D(y^\circ) \setminus \{0\}$ and $\bar{y}^\circ$, so

$$0 \leq \cos\delta = \min\left\{\frac{\langle d, \bar{y}^\circ\rangle}{\|d\|\|\bar{y}^\circ\|} : \langle d, \bar{y}^\circ\rangle \geq \|d\|\bar{y}^\circ_{min}, d \neq 0\right\} = \frac{\|d\|\bar{y}^\circ_{min}}{\|d\|\|\bar{y}^\circ\|} = \frac{\bar{y}^\circ_{\min}}{\|\bar{y}^\circ\|} \leq 1$$

and $0 \leq \sin\mu \leq 1$. Finally, let $\mu = \max\{\measuredangle(n, d) : d \in D(y^\circ)\}$ be the maximal positive angle between n and any $d \in D(y^\circ) \setminus \{0\}$ at y, so $\mu = \eta + \delta$ and

$$\cos\mu = \min\left\{\frac{\langle n, d\rangle}{\|n\|\|d\|} : d \in D(y^\circ)\right\}$$

(i) Since the assumption $\langle n, \bar{y}^\circ\rangle^2 + \|n\|^2\bar{y}^{\circ 2}_{\min} > \|n\|^2\|\bar{y}^\circ\|^2$ is equivalent to

$$\frac{\langle n, \bar{y}^\circ\rangle^2}{\|n\|^2\|\bar{y}^\circ\|^2} + \frac{\bar{y}^{\circ 2}_{\min}}{\|\bar{y}^\circ\|^2} = \cos^2\eta + \cos^2\delta > 1$$

it follows that $\cos^2\eta > 1 - \cos^2\delta = \sin^2\delta$ and $\cos^2\delta > 1 - \cos^2\eta = \sin^2\eta$. In particular, this implies that $\cos\eta > \sin\delta$ and $\cos\delta > \sin\eta$ and, thus, $\cos\eta\cos\delta - \sin\eta\sin\delta = \cos(\eta + \delta) = \cos\mu > 0$. Equivalence follows from repeating the same argument with $<$ instead of $>$. Then, by definition of μ, it is shown that $\langle n, \bar{y}^\circ\rangle^2 + \|n\|^2\bar{y}^{\circ 2}_{\min} > \|n\|^2\|\bar{y}^\circ\|^2$ is equivalent to

$$\cos\mu = \min\left\{\frac{\langle n, d\rangle}{\|n\|\|d\|} : d \in D(y^\circ)\right\} > 0$$

or equivalently, $\langle n, d\rangle\|n\|^{-1}\|d\|^{-1} > 0$ and, thus, $\langle n, d\rangle > 0$ for all $d \in D(y^\circ) \setminus \{0\}$. Since, by assumption, $\langle n, y - y^\circ\rangle \geq 0$ for all $y \in Y$, or $\langle n, y^\circ - y\rangle \leq 0$, it follows that $y^\circ - y \notin D(y^\circ)\setminus\{0\}$, showing that $y^\circ \in \mathrm{N}(Y, \mathcal{D})$.

(ii) Furthermore, if $\langle n, y - y^\circ \rangle > 0$ for all $y \in Y \setminus \{y^\circ\}$, or $\langle n, y^\circ - y \rangle < 0$, the same conclusion already follows for $\langle n, d \rangle \geq 0$ for all $d \in D(y^\circ)$, or $\langle n, \bar{y} \rangle^2 + \|n\|^2 \bar{y}_{\min}^2 \geq \|n\|^2 \|\bar{y}\|^2$. □

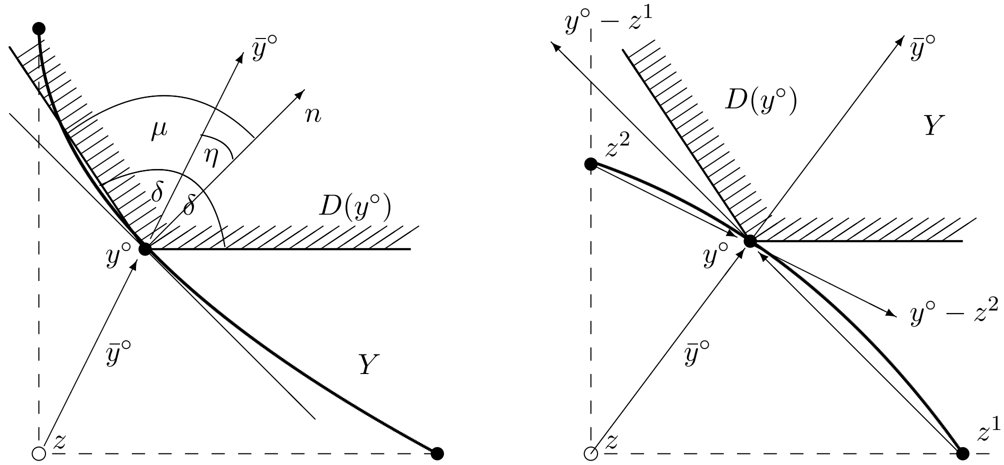

Figure 2: Illustration of Theorems 1 (convex case) and 2 (concave case)

Remark 11. The proof of Theorem 1 readily extends to the tricriteria case, as the interpretation of angles remains valid and, in particular, preserves the same geometric meaning as in the bicriteria case. For $m > 3$, however, the proof loses its geometric character and, thus, the theorem might not hold anymore.

A similar result can be established for the concave bicriteria case, under the additional assumption that Y is $\mathbb{R}^2_{\geq}$-compact. In particular, by Remark 5, then the ideal point $z = \{z_1, z_2\} \in \mathbb{R}^2$ can be defined using the minimum, $z_i = \min\{y_i : y \in Y\}$ for $i = 1, 2$.

Theorem 2. *Let $Y \subset \mathbb{R}^2$ be nonempty, and $\mathcal{D} = \{D(y) : y \in Y\}$ be defined by $D(y) = \{d \in \mathbb{R}^2 : \langle d, \bar{y} \rangle \geq \|d\| \bar{y}_{min}\}$ for all $y \in Y$. Assume that Y is $\mathbb{R}^2_{\geq}$-compact and $\mathrm{N}(Y, \mathbb{R}^2_{\geq})$ is $\mathbb{R}^2_{\geq}$-concave, let $y^\circ \in \mathrm{N}(Y, \mathbb{R}^2_{\geq})$, and $j \in \{1, 2\}$ be so that $\bar{y}^\circ_j = \bar{y}^\circ_{min}$. Denote $z^1 = (z^1_1, z_2)$ and $z^2 = (z_1, z^2_2)$, where $z^1_1 = \min\{y_1 : y_2 = z_2, y \in Y\}$, $z^2_2 = \min\{y_2 : y_1 = z_1, y \in Y\}$, and $z = (z_1, z_2) \in \mathbb{R}^2$ is the ideal point of Y. Then $y^\circ \in \mathrm{N}(Y, \mathcal{D})$ if and only if $y^\circ - z^j \notin D(y^\circ)$.*

Proof. Let Y be $\mathbb{R}^2_{\geq}$-compact. Since z^1 and z^2 are the two optimal lexicographic solutions to the bicriteria problem $\mathrm{N}(Y, \mathbb{R}^2_{\geq})$, it follows that $z^1, z^2 \in \mathrm{N}(Y, \mathbb{R}^2_{\geq})$, and $z^1 \neq z^2$ as the ideal point $z \notin Y$. Moreover, $\bar{z}^1 = (z^1_1 - z_1, 0) \geq 0$ implies that $D(z^1) = \{d = (d_1, d_2) \in \mathbb{R}^2 : d_1 \geq 0\}$ and, thus, $z^2 \in (z^1 - D(z^1)) \cap Y \setminus \{z^1\}$. This shows that $z^1 \notin \mathrm{N}(Y, \mathcal{D})$ and, by repeating the analogous argument for z^2, that $z^2 \notin \mathrm{N}(Y, \mathcal{D})$. Now let $\mathrm{N}(Y, \mathbb{R}^2_{\geq})$ be convex and $y^\circ \in \mathrm{N}(Y, \mathbb{R}^2_{\geq}) \setminus \{z^1, z^2\}$, so, in particular, $z_1 < y^\circ_1 < z^1_1$ and $z_2 < y^\circ_2 < z^2_2$. By $\mathbb{R}^2_{\geq}$-concavity, then there does not exist $y \in Y$ that falls below the two line segments from y° to z^1 and z^2, as indicated in Figure 2. In particular, if $y^\circ - z^1, y^\circ - z^2 \notin D(y^\circ)$, then there does not exist $y \in (y^\circ - D(y^\circ)) \cap Y \setminus \{y^\circ\}$, showing that $y^\circ \in \mathrm{N}(Y, \mathcal{D})$. Without loss of generality, let $\bar{y}^\circ_{min} = \bar{y}^\circ_1 \leq \bar{y}^\circ_2$, then, by assumption, $y^\circ - z^1 \notin D(y^\circ)$, and it only remains to show that $y^\circ - z^2 \notin D(y^\circ)$. But this follows, because

$$\langle y^\circ - z^2, \bar{y}^\circ \rangle - \|y^\circ - z^2\| \bar{y}^\circ_1 = \bar{y}^{\circ 2}_1 + (y^\circ_2 - z^2_2)\bar{y}^\circ_2 - \|y^\circ - z^2\| \bar{y}^\circ_1 \leq (y^\circ_2 - z^2_2)(y^\circ_2 - z_2) < 0$$

from $z_2 < y^\circ_2 < z^2_2$. The reverse direction is clear and follows because $z^1, z^2 \in Y$. □

The point $(z_1^1, z_2^2) \in \mathbb{R}^2$ in Theorem 2 is also called the *nadir point*. More general, for a set $Y \subset \mathbb{R}^m$, the nadir point is defined by $z^{nad} = \{z_1^{nad}, \ldots, z_m^{nad}\}$, where $z_i^{nad} = \sup\{y_i : y \in \mathrm{N}(Y, \mathbb{R}^m_{\geq})\}$.

Remark 12. The proof of Theorem 2 only holds for the bicriteria case $m = 2$, based on the exploited characterization of the nadir point using the two optimal lexicographic solutions for $\mathrm{N}(Y, \mathbb{R}^2_{\geq})$. For $m > 2$, however, the nadir point must be found through optimization over the Pareto set and, therefore, in general is not readily available (Yamamoto, 2002; Ehrgott and Tenfelde-Podehl, 2003).

5 Examples

The two theorems in Section 4 are illustrated for the three sets depicted in Figure 3. Each set $Y \subset \mathbb{R}^2$ has the ideal point $z = (0, 0) \in \mathbb{R}^2$ at the origin, so $\bar{y} = y$ for all $y \in Y$. In particular, then let $\mathcal{D} = \{D(y) : y \in Y\}$ be defined by $D(y) = \{d \in \mathbb{R}^2 : \langle d, y \rangle\} \geq \|d\| y_{min}\}$ for all $y \in Y$.

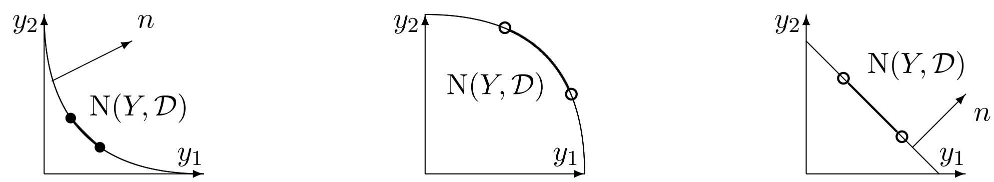

Figure 3: Illustration of Examples 3 (convex case), 4 (concave case) and 5 (linear case)

Example 3. Let $Y = \{y \in \mathbb{R}^2 : (1 - y_1)^2 + (1 - y_2)^2 \leq 1\}$. Then $\mathrm{N}(Y, \mathbb{R}^2_{\geq}) = \{y \in Y : (1 - y_1)^2 + (1 - y_2)^2 = 1, y_1 \leq 1, y_2 \leq 1\}$ is $\mathbb{R}^2_{\geq}$-convex, and Theorem 1 can be used to find $\mathrm{N}(Y, \mathcal{D})$. Hence, let $y \in \mathrm{N}(Y, \mathbb{R}^2_{\geq})$, so $(1 - y_1)^2 + (1 - y_2)^2 = 1$, $0 \leq y_1, y_2 \leq 1$, and, without loss of generality, assume that $y_{min} = y_1 \leq y_2$, so $0 \leq y_1 \leq 1 - \frac{1}{2}\sqrt{2}$ or $1 - y_1 \geq \frac{1}{2}\sqrt{2} > 0$. Since the supporting hyperplane at y has the normal vector $n = (1 - y_1, 1 - y_2) \in \mathbb{R}^2$ that satisfies $\|n\|^2 = 1$ and $\langle n, y - y' \rangle > 0$ for all $y' \in Y \setminus \{y\}$, the condition in Theorem 1 becomes

$$\langle n, y \rangle^2 + \|n\|^2 y_{\min}^2 - \|n\|^2 \|y\|^2 = [(1 - y_1)y_1 + (1 - y_2)y_2]^2 + y_1^2 - (y_1^2 + y_2^2) \geq 0$$

and solving this inequality yields that $\frac{1}{5} \leq y_1 \leq y_2 \leq \frac{2}{5}$ and, by symmetry of Y in y_1 and y_2,

$$\mathrm{N}(Y, \mathcal{D}) = \left\{y \in Y : (1 - y_1)^2 + (1 - y_2)^2 = 1, \tfrac{1}{5} \leq y_1, y_2 \leq \tfrac{2}{5}\right\}$$

If the set Y is $\mathbb{R}^2_{\geq}$-concave instead of $\mathbb{R}^2_{\geq}$-convex, Theorem 2 has to be used instead of Theorem 1.

Example 4. Let $Y = \{y \in \mathbb{R}^2 : y_1^2 + y_2^2 \geq 1, y_1 \geq 0, y_2 \geq 0\}$. Then $\mathrm{N}(Y, \mathbb{R}^2_{\geq}) = \{y \in Y : y_1^2 + y_2^2 = 1\}$ is $\mathbb{R}^2_{\geq}$-concave, and Theorem 2 can be used to find $\mathrm{N}(Y, \mathcal{D})$. Hence, let $y \in \mathrm{N}(Y, \mathbb{R}^2_{\geq})$, so $y_1^2 + y_2^2 = 1$ and $0 \leq y_1, y_2 \leq 1$, and, without loss of generality, assume that $y_{min} = y_1 \leq y_2$, so $0 \leq y_1 \leq \frac{1}{2}\sqrt{2}$ or $1 - y_1 \geq 1 - \frac{1}{2}\sqrt{2}$. Since $z^1 = (1, 0)$ and $z^2 = (0, 1)$, the condition in Theorem 2 becomes

$$\langle y - z^1, y \rangle - \|y - z^1\| y_1 = (y_1 - 1)y_1 + y_2^2 - \sqrt{(y_1 - 1)^2 + y_2^2} \cdot y_1 > 0$$

and solving this inequality yields $\frac{1}{2} \leq y_1 \leq y_2 \leq \frac{1}{2}\sqrt{3}$ and, by symmetry of Y in y_1 and y_2,

$$\mathrm{N}(Y, \mathcal{D}) = \left\{y \in Y : y_1^2 + y_2^2 = 1, \tfrac{1}{2} < y_1, y_2 < \tfrac{1}{2}\sqrt{3}\right\}$$

The concluding Example 5 considers the same set previously defined in Example 1 and shows how the new variable preference model resolves the limitations highlighted in the earlier discussion.

Example 5. Let $Y = \{y \in \mathbb{R}^2 : y_1 + y_2 \geq 1,\ y_1 \geq 0,\ y_2 \geq 0\}$. Then $\mathrm{N}(Y, \mathbb{R}^2_{\geq}) = \{y \in Y : y_1 + y_2 = 1\}$ is both $\mathbb{R}^2_{\geq}$-convex and $\mathbb{R}^2_{\geq}$-concave, and both Theorems 1 and 2 can be used to find $\mathrm{N}(Y, \mathcal{D})$. Hence, let $y \in \mathrm{N}(Y, \mathbb{R}^2_{\geq})$, so $y_1 + y_2 = 1$, $0 \leq y_1, y_2 \leq 1$, and, without loss of generality, assume that $y_{min} = y_1 \leq y_2$, so $0 \leq y_{min} = y_1 \leq \frac{1}{2} \leq y_2$. Since the supporting hyperplane at y has the normal vector $n = (1, 1)$ that satisfies $\langle n, y - y' \rangle \geq 0$ for all $y' \in Y$, the condition in Theorem 1 becomes

$$\langle n, y \rangle^2 + \|n\|^2 y_{\min}^2 - \|n\|^2 \|y\|^2 = (y_1 + y_2)^2 + 2y_1^2 - 2(y_1^2 + y_2^2) = 1 - 2y_2^2 > 0$$

and solving this inequality yields $1 - \frac{1}{2}\sqrt{2} \leq y_1 \leq y_2 \leq \frac{1}{2}\sqrt{2}$ and, by symmetry of Y in y_1 and y_2,

$$\mathrm{N}(Y, \mathcal{D}) = \left\{ y \in Y : y_1 + y_2 = 1, 1 - \tfrac{1}{2}\sqrt{2} < y_1, y_2 < \tfrac{1}{2}\sqrt{2} \right\}$$

Alternatively, with $z^1 = (1, 0)$ and $z^2 = (0, 1)$, the condition in Theorem 2 becomes

$$\langle y - z^1, y \rangle - \|y - z^1\| y_1 = (y_1 - 1)y_1 + y_2^2 - \sqrt{(y_1 - 1)^2 + y_2^2} \cdot y_1 > 0,$$

and solving this inequality yields the same set as Theorem 1.

In particular, the new variable preference model excludes parts of the Pareto frontier while maintaining a set of nondominated outcomes in the middle region for presentation to and further consideration by a decision maker. Furthermore, by choosing a different cone from Lemma 1 for Proposition 4, other collections of ideal-symmetric cones can be defined and, by Lemma 4, will further reduce the nondominated set with respect to the resulting variable domination structure. Investigation of this and other related aspects are planned as future research and, together with some final remarks, outlined in the following concluding section.

6 Conclusion

This paper presents an approach to variable preference modeling for multicriteria optimization and decision making problems, based on the concept of general domination structures. The relevant literature indicates that the majority of such preference models is described by constant convex or polyhedral cones, and it is first shown how these models are valid under the assumptions of global multiplicativity and additivity and subsume Pareto dominance as a special case.

Two examples then illustrate some undesirable restrictions of preference models that are described by constant convex cones and, in remedy of the recognized model limitations, motivate the formulation of a new set of preference assumptions. In particular, the previous assumption of global preferences is replaced by its local counterpart, and the new assumption of ideal-symmetry is introduced to steer variability of the associated domination structure.

To characterize the nondominated set with respect to the particular model chosen, two associated single criterion optimization problems are formulated and used to derive both a necessary and a sufficient condition for corresponding nondominated solutions. Together with the additional result that these solutions are necessarily Pareto nondominated, this provides a methodology to either generate the complete set, or to verify if any given outcome is nondominated with respect to the defined variable domination structure.

Using results from analytic geometry and relying on the geometrical character of the ideal-symmetry assumption, two further conditions are established for the bicriteria case and subsequently used to illustrate the new preference model on three examples. In particular, it is thereby shown how the new variable cone model resolves the previously recognized shortcomings of constant cones and, in general, finds a subset of Pareto outcomes that are located in the middle region of the original Pareto frontier.

Several further research questions are motivated by this paper. First, other conditions for nondominance can be derived in generalization of the results for the convex and concave bicriteria case established in this paper, preferably independent of a restricting geometrical character. Second, the assumption of ideal-symmetry also gives the possibility to describe the variable domination structure by a different collection of ideal-symmetric convex cones. The characterization of the corresponding nondominated sets, especially in comparison to the results obtained here, can be examined, for example with respect to effects of changing preferences on the nondominated set. Third, different sets of preference assumptions can be proposed to obtain variable domination structures other than the one derived in this paper, eventually producing a variety of new approaches to variable preference modeling in multicriteria optimization and decision making.

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