A Dual-Mixed Approximation Method for a Three-Field Model of a Nonlinear Generalized Stokes Problem

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Abstract. In this work a dual-mixed approximation of a nonlinear generalized Stokes problem is studied. The problem is analyzed in Sobolev spaces which arise naturally in the problem formulation. Existence and uniqueness results are given and error estimates are derived. It is shown that both lowest-order and higher-order mixed finite elements are suitable for the approximation method. Numerical experiments that support the theoretical results are presented.

Key words. generalized Stokes problem, dual-mixed method, twofold saddle point problem, Sobolev spaces

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1 Introduction

In this article we investigate the solution of a nonlinear generalized Stokes problem using a dual-mixed formulation. The nonlinear generalized Stokes problem arises in modeling flows of, for example, biological fluids, lubricants, paints, polymeric fluids, where the fluid viscosity is assumed to be a nonlinear function of the fluid's velocity gradient tensor. The generalized Stokes problem is given by: *Find* $(\mathbf{u}, p)$ *such that*

$$-\nabla \cdot (\nu(|\nabla \mathbf{u}|)\nabla \mathbf{u}) + \nabla p = \mathbf{f} \quad \text{in } \Omega\,, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega\,, \tag{1.2}$$

$$\mathbf{u} = \mathbf{u}_\Gamma \quad \text{on } \Gamma\,, \tag{1.3}$$

where Ω is a bounded open subset of $\mathbb{R}^n$ with Lipschitz continuous boundary Γ. The fluid velocity is denoted by $\mathbf{u}$, and $\nabla \mathbf{u} := (\nabla \mathbf{u})_{ij} = \partial u_i / \partial x_j$ is the tensor gradient of $\mathbf{u}$. Here and throughout the paper we use the following notation: for tensors $\boldsymbol{\sigma} = (\sigma_{ij})$, $\boldsymbol{\tau} = (\tau_{ij})$, $\boldsymbol{\sigma} : \boldsymbol{\tau} = \sum_{i,j} \sigma_{ij}\tau_{ij}$,

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$|\boldsymbol{\sigma}|^2 = \boldsymbol{\sigma} : \boldsymbol{\sigma}$. The pressure is denoted by p, and $\mathbf{f}$ describes the external forces on the fluid. The function ν describes the nonlinear kinematic viscosity of the fluid.

Some classical examples of ν are given by:

Power Law

$$\nu(|\mathbf{d}(\mathbf{u})|) \;=\; \nu_0 \, |\mathbf{d}(\mathbf{u})|^{r-2} \,, \quad \nu_0 > 0, \quad 1 < r < 2, \tag{1.4}$$

where $\mathbf{d}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}})$ denotes the fluid deformation tensor. The power law model has been used to model the viscosity of many polymeric solutions and melts over a considerable range of shear rates [17].

Ladyzhenskaya Law[20]:

$$\nu(|\nabla\mathbf{u}|) \;=\; (\nu_0 \,+\, \nu_1 \, |\nabla\mathbf{u}|)^{r-2} \,, \;\; \nu_0 \geq 0, \; \nu_1 > 0, \; r > 1 \;, \tag{1.5}$$

which has been used in modeling fluids with large stresses.

Carreau Law:

$$\nu(|\mathbf{d}(\mathbf{u})|) \;=\; \nu_0 \left(1 \,+\, |\mathbf{d}(\mathbf{u})|^2\right)^{(r-2)/2} \;, \;\; \nu_0 > 0, \; r \geq 1 \;, \tag{1.6}$$

used in modeling visco-plastic flows and creeping flow of metals.

General descriptions of (1.1) are often written in terms of the tensor $\boldsymbol{\sigma} = \nu(|\nabla\mathbf{u}|)\nabla\mathbf{u}$:

$$-\nabla \cdot \boldsymbol{\sigma} + \nabla p = \mathbf{f} \quad \text{ in } \Omega \,. \tag{1.7}$$

The work in this paper extends the investigates of [4, 22, 14]. In [4] Baranger, Najib, and Sandri provided an analysis for the existence and uniqueness of the modeling equations in appropriate Sobolev spaces and gave an error analysis of a finite element approximation method applied to the primitive variables ($\boldsymbol{\sigma}$, p, $\mathbf{u}$). Manouzi and Farhloul in [22] reformulated the modeling equations into a saddle point problem and used a *mixed formulation* to study the existence and uniqueness of the solution, again in appropriate Sobolev spaces. An error analysis for the finite element approximation was also given. In both [4] and [22] the analysis used the assumption that the equation describing $\boldsymbol{\sigma}$ in terms of $\mathbf{d}(\mathbf{u})$ or $\nabla\mathbf{u}$ was invertible to give $\mathbf{d}(\mathbf{u})$ or $\nabla\mathbf{u}$ as a function of $\boldsymbol{\sigma}$. Gatica, González, and Meddahi in [14] reformulated the modeling equations, using the tensor $\boldsymbol{\psi}$ in place of the $\boldsymbol{\sigma}$ ($\boldsymbol{\psi} = \boldsymbol{\sigma} - p\mathbf{I}$) and introducing an additional variable for $\nabla\mathbf{u}$. Doing so their formulation used the constitutive equation for $\boldsymbol{\sigma}$ as a function of $\nabla\mathbf{u}$ and reduced the regulatity requirement for the velocity. Advantages of this approach include: (i) more flexibility in choosing the approximating finite element space for $\mathbf{u}$, (ii) Dirichlet boundary conditions for $\mathbf{u}$ become *natural* boundary conditions and are easily incorporated into the variational formulations, (iii) avoids the assumption of expressing $\nabla\mathbf{u}$ was a function of $\boldsymbol{\sigma}$. A disadvantage in this formulation is that additional unknowns are introduced. The analysis of this approach was only studied in a Hilbert space setting.

In this paper we recast the formulation described in [14] in appropriate Sobolev spaces. Because of the nonlinearity in (1.7), this problem is more appropriately studied in Sobolev spaces which should result in tighter error estimates for the approximate solution. This extends the work of [22] by avoiding the assumption of expressing $\nabla \mathbf{u}$ as a function of $\boldsymbol{\sigma}$. In addition, we show that higher-order approxmiating spaces can be used in the mixed finite element method for this formulation and give the associated a priori error estimates.

A description of the notation used in this paper, the mathematical problem, and the dual-mixed variational formulation is given in Section 2. Existence and uniqueness of the variational formulation is studied in Section 3. In Section 4 the finite element approximation is presented and analyzed. Numerical results are given in Section 5.

2 Mathematical Setting

For $r > 1$ we denote its unitary conjugate by r', satisfying $r^{-1} + r'^{-1} = 1$. Used in the analysis below are the following function spaces and norms.

$$T := (L^r(\Omega))^{n\times n} \;=\; \{\boldsymbol{\tau} = (\tau_{ij});\; \tau_{ij} \in L^r(\Omega)\;;\; i,j = 1,\ldots,n\},$$

with norm $\|\boldsymbol{\tau}\|_T := \left(\int_\Omega |\boldsymbol{\tau}|^r \, d\Omega\right)^{1/r}$.

$$T^{'} := \left(L^{r'}(\Omega)\right)^{n\times n} \quad \text{and} \quad T^{'}_{div} := \left\{\boldsymbol{\tau} \in T^{'}\;;\; div\,\boldsymbol{\tau} \in \left(L^{r'}(\Omega)\right)^{n}\right\},$$

with norm $\|\boldsymbol{\tau}\|_{T^{'}_{div}} := \left(\int_\Omega (|\boldsymbol{\tau}|^{r'} + |div\,\boldsymbol{\tau}|^{r'})\, d\Omega\right)^{1/r'}$. Let $U := (L^r(\Omega))^n$, and $P := L^{r'}(\Omega)$.

For a Banach space X, X^* denotes its dual space with associated norm $\|\cdot\|_{X^*}$. Note that $T^* = T^{'}$, and $\left(T^{'}\right)^* = T$. The norm and seminorm associated with the Sobolev space $W^{m,r}(\Omega)$ will be denoted by $\|\cdot\|_{m,r,\Omega}$ and $|\cdot|_{m,r,\Omega}$, respectively, and the infinity norm will be denoted by $\|\cdot\|_\infty$.

Motivated by (1.4),(1.5),(1.6), we will assume that the extra stress tensor is a function of the velocity gradient, i.e.

$$\boldsymbol{\sigma} := \mathbf{g}(\nabla \mathbf{u}) = \nu(|\nabla \mathbf{u}|)\nabla \mathbf{u}\,. \tag{2.1}$$

Specifically, we assume

A1: $\mathbf{g} : T \to T^*$ is a bounded, continuous, strictly monotone operator [25];

and that there exist constants $\hat{C}_1$ and $\hat{C}_2$ such that, for $\mathbf{s}, \mathbf{t}, \mathbf{w} \in T$,

$$\textbf{A2:}\; \int_\Omega (\mathbf{g}(\mathbf{s}) - \mathbf{g}(\mathbf{t})) : (\mathbf{s} - \mathbf{t})\, d\Omega \geq \hat{C}_1 \left(\int_\Omega |\mathbf{g}(\mathbf{s}) - \mathbf{g}(\mathbf{t})||\mathbf{s} - \mathbf{t}|\, d\Omega + \frac{\|\mathbf{s} - \mathbf{t}\|_T^2}{\|\mathbf{s}\|_T^{2-r} + \|\mathbf{t}\|_T^{2-r}} \right), \tag{2.2}$$

$$\textbf{A3:}\; \int_\Omega (\mathbf{g}(\mathbf{s}) - \mathbf{g}(\mathbf{t})) : \mathbf{w}\, d\Omega \leq \hat{C}_2 \left\| \frac{|\mathbf{s} - \mathbf{t}|}{|\mathbf{s}| + |\mathbf{t}|} \right\|_\infty^{\frac{2-r}{r}} \left(\int_\Omega |\mathbf{g}(\mathbf{s}) - \mathbf{g}(\mathbf{t})||\mathbf{s} - \mathbf{t}|\, d\Omega \right)^{1/r'} \|\mathbf{w}\|_T\,, \tag{2.3}$$

with the convention that $\mathbf{g}(\mathbf{s}) = \mathbf{0}$ if $\mathbf{s} = \mathbf{0}$ and $|\mathbf{s}(\mathbf{x}) - \mathbf{t}(\mathbf{x})|/(|\mathbf{s}(\mathbf{x})| + |\mathbf{t}(\mathbf{x})|) = 0$ if $\mathbf{s}(\mathbf{x}) = \mathbf{t}(\mathbf{x}) = \mathbf{0}$. Properties **A1**–**A3** have been established for power law and Carreau law fluids [3]. (For the case of a power law fluid monotonicity is also shown in [27, 8].)

Remark 2.1 *From (1.2) it follows that* $\mathbf{u}_\Gamma$ *must satisfy the compatibility condition*

$$\int_\Gamma \mathbf{u}_\Gamma \cdot \mathbf{n} \, d\Gamma \; = \; 0 \, ,$$

where $\mathbf{n}$ *denotes the outward pointing unit normal vector to* Ω*.*

In order to obtain the dual-mixed formulation, introduce two new variables, $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$.

$$\begin{aligned} \boldsymbol{\phi} \; &:= \; \nabla \mathbf{u} \; , && (2.4) \\ \boldsymbol{\psi} \; &:= \; \boldsymbol{\sigma} \, - \, p\mathbf{I} \; , \; \text{the total stress tensor,} && (2.5) \\ &= \; \mathbf{g}(\boldsymbol{\phi}) \, - \, p\mathbf{I} \; , \; \text{using (2.1)} \, . && (2.6) \end{aligned}$$

With the definition of $\boldsymbol{\psi}$ a variational form for (1.1) can be written as

$$- \int_\Omega \mathbf{v} \cdot div \, \boldsymbol{\psi} \, d\Omega \; = \; \int_\Omega \mathbf{v} \cdot \mathbf{f} \, d\Omega \, , \; \text{for } \mathbf{v} \in T \, . \qquad (2.7)$$

Note that from the definition of $\boldsymbol{\phi}$ we have that, for sufficiently smooth functions,

$$\begin{aligned} 0 \; &= \; - \int_\Omega \boldsymbol{\phi} \, : \boldsymbol{\tau} \, d\Omega \; + \; \int_\Omega \nabla \mathbf{u} \, : \boldsymbol{\tau} \, d\Omega \\ &= \; - \int_\Omega \boldsymbol{\phi} \, : \boldsymbol{\tau} \, d\Omega \; + \; \int_\Gamma (\boldsymbol{\tau} \cdot \mathbf{n}) \cdot \mathbf{u}_\Gamma \, d\Gamma \; - \; \int_\Omega \mathbf{u} \, \cdot \, div \, \boldsymbol{\tau} \, d\Omega && (2.8) \end{aligned}$$

and the condition $div \, \mathbf{u} \, = \, 0$ is equivalent to

$$tr(\boldsymbol{\phi}) \; = \; 0 \; , \qquad (2.9)$$

where we use $tr(\boldsymbol{\phi})$ to denote the *trace* of $\boldsymbol{\phi}$.

Combining (1.4), (2.8), and (2.7) a variational formulation to (1.4), (2.8), and (2.7) is: *Given* $\mathbf{f} \in \left(L^{r'}(\Omega) \right)^n$*,* $\mathbf{u}_\Gamma \in \left(W^{1-1/r \, , r}(\Gamma) \right)^n$*, determine* $(\boldsymbol{\phi}, \boldsymbol{\psi}, p, \mathbf{u}) \in T \times T^{'}_{div} \times P \times U$ *such that*

$$\begin{aligned} \int_\Omega \mathbf{g}(\boldsymbol{\phi}) : \boldsymbol{\varsigma} \, d\Omega \; - \; \int_\Omega \boldsymbol{\psi} : \boldsymbol{\varsigma} \, d\Omega \; - \; \int_\Omega p \, tr(\boldsymbol{\varsigma}) \, d\Omega \; &= \; 0 \; , \forall \boldsymbol{\varsigma} \in T \, , && (2.10) \\ - \int_\Omega \boldsymbol{\tau} \, : \boldsymbol{\phi} \, d\Omega \; - \; \int_\Omega q \, tr(\boldsymbol{\phi}) \, d\Omega \; - \; \int_\Omega \mathbf{u} \, \cdot \, div \, \boldsymbol{\tau} \, d\Omega \; &= \; - \int_\Gamma (\boldsymbol{\tau} \cdot \mathbf{n}) \cdot \mathbf{u}_\Gamma \, d\Gamma \; , \forall (\boldsymbol{\tau}, q) \in T^{'}_{div} \times P \, , && (2.11) \\ - \int_\Omega \mathbf{v} \, \cdot \, div \, \boldsymbol{\psi} \, d\Omega \; &= \; \int_\Omega \mathbf{v} \cdot \mathbf{f} \, d\Omega \, , \forall \mathbf{v} \in U \, . && (2.12) \end{aligned}$$

Note that equations (2.10)-(2.12) do not uniquely define a solution; as adding $(0, c\mathbf{I}, -c, \mathbf{0})$ to a solution $(\boldsymbol{\phi}, \boldsymbol{\psi}, p, \mathbf{u})$, also satisfies (2.10)-(2.12) for any $c \in \mathbb{R}$. In order to guarantee uniqueness we proceed as in [2, 7, 14] and impose, via a Lagrange multiplier, the constraint $\int_\Omega tr(\boldsymbol{\psi}) \, d\Omega \; = \; 0$. The variational formulation may then be restated as: *Given* $\mathbf{f} \in \left(L^{r'}(\Omega) \right)^n$*,* $\mathbf{u}_\Gamma \in \left(W^{1-1/r \, , r}(\Gamma) \right)^n$*,* *determine* $(\boldsymbol{\phi}, \boldsymbol{\psi}, p, \mathbf{u}, \lambda) \in T \times T^{'}_{div} \times P \times U \times \mathbb{R}$ *such that*

$$\int_\Omega \mathbf{g}(\boldsymbol{\phi}) : \boldsymbol{\varsigma}\, d\Omega \;-\; \int_\Omega \boldsymbol{\psi} : \boldsymbol{\varsigma}\, d\Omega \;-\; \int_\Omega p\, tr(\boldsymbol{\varsigma})\, d\Omega \;=\; 0 \;, \forall \boldsymbol{\varsigma} \in T\,, \tag{2.13}$$

$$\begin{aligned} -\int_\Omega \boldsymbol{\tau} : \boldsymbol{\phi}\, d\Omega \;-\; \int_\Omega q\, tr(\boldsymbol{\phi})\, d\Omega \;-\; \int_\Omega \mathbf{u} \cdot div\, \boldsymbol{\tau}\, d\Omega \;&+\; \lambda \int_\Omega tr(\boldsymbol{\tau})\, d\Omega \\ &= \; -\int_\Gamma (\boldsymbol{\tau} \cdot \mathbf{n}) \cdot \mathbf{u}_\Gamma\, d\Gamma \;, \forall (\boldsymbol{\tau}, q) \in T^{'}_{div} \times P\,, \end{aligned} \tag{2.14}$$

$$-\int_\Omega \mathbf{v} \cdot div\, \boldsymbol{\psi}\, d\Omega \;+\; \eta \int_\Omega tr(\boldsymbol{\psi})\, d\Omega \;=\; \int_\Omega \mathbf{v} \cdot \mathbf{f}\, d\Omega\,, \forall (\mathbf{v}, \eta) \in U \times \mathbb{R}\,. \tag{2.15}$$

Remark 2.2 *As commented in [14], the value of the Lagrange multiplier* λ *is* 0*, as can be seen from the choice of* $\boldsymbol{\tau} = \mathbf{I}$ *and* $q = -1$*. However, it is included in the variational formulation so that the formulation has a twofold saddle point structure.*

To formally rewrite (2.13)-(2.15) as a twofold saddle point problem define the following operators:

$$\mathbf{A} : T \longrightarrow T^{'}\,, \qquad \mathbf{B} : T \longrightarrow (T^{'}_{div} \times P)^*\,, \qquad \mathbf{C} : T^{'}_{div} \times P \longrightarrow (U \times \mathbb{R})^*\,.$$

$$[\mathbf{A}(\boldsymbol{\phi})\,,\, \boldsymbol{\varsigma}] \;:=\; \int_\Omega \mathbf{g}(\boldsymbol{\phi}) : \boldsymbol{\varsigma}\, d\Omega\,, \tag{2.16}$$

$$[\mathbf{B}(\boldsymbol{\phi})\,,\, (\boldsymbol{\tau}, q)] \;:=\; -\int_\Omega \boldsymbol{\tau} : \boldsymbol{\phi}\, d\Omega \;-\; \int_\Omega q\, tr(\boldsymbol{\phi})\, d\Omega\,, \tag{2.17}$$

$$[\mathbf{C}(\boldsymbol{\psi}, p)\,,\, (\mathbf{v}, \eta)] \;:=\; -\int_\Omega \mathbf{v} \cdot div\, \boldsymbol{\psi}\, d\Omega \;+\; \eta \int_\Omega tr(\boldsymbol{\psi})\, d\Omega\,. \tag{2.18}$$

The modeling equations can then be written in the form

$$[\mathbf{A}(\boldsymbol{\phi})\,,\, \boldsymbol{\varsigma}] + [\mathbf{B}(\boldsymbol{\varsigma})\,,\, (\boldsymbol{\psi}, p)] \;=\; 0 \;, \forall \boldsymbol{\varsigma} \in T\,, \tag{2.19}$$

$$[\mathbf{B}(\boldsymbol{\phi})\,,\, (\boldsymbol{\tau}, q)] + [\mathbf{C}(\boldsymbol{\tau}, q)\,,\, (\mathbf{u}, \lambda)] \;=\; -\int_\Gamma (\boldsymbol{\tau} \cdot \mathbf{n}) \cdot \mathbf{u}_\Gamma\, d\Gamma \;, \forall (\boldsymbol{\tau}, q) \in T^{'}_{div} \times P\,, \tag{2.20}$$

$$[\mathbf{C}(\boldsymbol{\psi}, p)\,,\, (\mathbf{v}, \eta)] \;=\; \int_\Omega \mathbf{v} \cdot \mathbf{f}\, d\Omega\,, \forall (\mathbf{v}, \eta) \in U \times \mathbb{R}\,, \tag{2.21}$$

or equivalent, in the form of a twofold saddle point equation,

$$[\mathbf{A}(\boldsymbol{\phi})\,,\, \boldsymbol{\varsigma}] + [\boldsymbol{\varsigma}\,,\, \mathbf{B}^*(\boldsymbol{\psi}, p)] \;=\; 0 \;, \forall \boldsymbol{\varsigma} \in T\,, \tag{2.22}$$

$$[\mathbf{B}(\boldsymbol{\phi})\,,\, (\boldsymbol{\tau}, q)] + [(\boldsymbol{\tau}, q)\,,\, \mathbf{C}^*(\mathbf{u}, \lambda)] \;=\; -\int_\Gamma (\boldsymbol{\tau} \cdot \mathbf{n}) \cdot \mathbf{u}_\Gamma\, d\Gamma \;, \forall (\boldsymbol{\tau}, q) \in T^{'}_{div} \times P\,, \tag{2.23}$$

$$[\mathbf{C}(\boldsymbol{\psi}, p)\,,\, (\mathbf{v}, \eta)] \;=\; \int_\Omega \mathbf{v} \cdot \mathbf{f}\, d\Omega\,, \forall (\mathbf{v}, \eta) \in U \times \mathbb{R}\,, \tag{2.24}$$

where $\mathbf{B}^*$ and $\mathbf{C}^*$ denote the respective adjoint operators of $\mathbf{B}$ and $\mathbf{C}$, respectively.

3 Existence and Uniqueness

Intuitively, the solution of (2.22)-(2.24) can be found using the following steps.

1. Find a particular solution to (2.24), i.e. $(\boldsymbol{\psi}_0, p_0)$ such that

$$[\mathbf{C}(\boldsymbol{\psi}_0, p_0)\,,\,(\mathbf{v},\eta)] \;=\; \int_\Omega \mathbf{v}\cdot\mathbf{f}\,d\Omega\,,\forall(\mathbf{v},\eta)\in U\times\mathbb{R}\,. \tag{3.1}$$

2. Let $\boldsymbol{\psi} \;=\; \tilde{\boldsymbol{\psi}} + \boldsymbol{\psi}_0$ and $p \;=\; \tilde{p} + p_0$. Rewriting (2.22)-(2.24) we have

$$\begin{aligned}
[\mathbf{A}(\boldsymbol{\phi})\,,\,\boldsymbol{\varsigma}] + [\boldsymbol{\varsigma}\,,\,\mathbf{B}^*(\tilde{\boldsymbol{\psi}},\tilde{p})] \;&=\; -[\boldsymbol{\varsigma}\,,\,\mathbf{B}^*(\boldsymbol{\psi}_0,p_0)]\;,\forall\boldsymbol{\varsigma}\in T\,, && (3.2)\\
[\mathbf{B}(\boldsymbol{\phi}),(\boldsymbol{\tau},q)] + [(\boldsymbol{\tau},q),\mathbf{C}^*(\mathbf{u},\lambda)] \;&=\; -\int_\Gamma(\boldsymbol{\tau}\cdot\mathbf{n})\cdot\mathbf{u}_\Gamma\,d\Gamma\;,\forall(\boldsymbol{\tau},q)\in T^{'}_{div}\times P, && (3.3)\\
[\mathbf{C}(\tilde{\boldsymbol{\psi}},\tilde{p})\,,\,(\mathbf{v},\eta)] \;&=\; 0\,,\forall(\mathbf{v},\eta)\in U\times\mathbb{R}\,, && (3.4)
\end{aligned}$$

3. Introduce a subspace of $T^{'}_{div}\times P$ defined by

$$Z_1 \;:=\; \left\{(\boldsymbol{\tau},q)\in T^{'}_{div}\times P\;:\;[(\boldsymbol{\tau},q)\,,\,\mathbf{C}^*(\mathbf{v},\eta)] \;=\; 0\,,\,\forall(\mathbf{v},\eta)\in U\times\mathbb{R}\right\}\,.$$

Note that Z_1 can equivalently be defined as

$$Z_1 \;:=\; \left\{(\boldsymbol{\tau},q)\in T^{'}_{div}\times P\;:\;[\mathbf{C}(\boldsymbol{\tau},q)\,,\,(\mathbf{v},\eta)] \;=\; 0\,,\,\forall(\mathbf{v},\eta)\in U\times\mathbb{R}\right\}\,.$$

Equations (3.2)-(3.4) are then replaced by

$$\begin{aligned}
[\mathbf{A}(\boldsymbol{\phi})\,,\,\boldsymbol{\varsigma}] + [\boldsymbol{\varsigma}\,,\,\mathbf{B}^*(\tilde{\boldsymbol{\psi}},\tilde{p})] \;&=\; -[\boldsymbol{\varsigma}\,,\,\mathbf{B}^*(\boldsymbol{\psi}_0,p_0)]\;,\forall\boldsymbol{\varsigma}\in T\,, && (3.5)\\
[\mathbf{B}(\boldsymbol{\phi})\,,\,(\boldsymbol{\tau},q)] \;&=\; -\int_\Gamma(\boldsymbol{\tau}\cdot\mathbf{n})\cdot\mathbf{u}_\Gamma\,d\Gamma\;,\forall(\boldsymbol{\tau},q)\in Z_1\,. && (3.6)
\end{aligned}$$

4. Find a particular solution to (3.6), $\boldsymbol{\phi}_0$, i.e.

$$[\mathbf{B}(\boldsymbol{\phi}_0)\,,\,(\boldsymbol{\tau},q)] \;=\; -\int_\Gamma(\boldsymbol{\tau}\cdot\mathbf{n})\cdot\mathbf{u}_\Gamma\,d\Gamma\;,\forall(\boldsymbol{\tau},q)\in Z_1\,. \tag{3.7}$$

5. Let $\boldsymbol{\phi} \;=\; \tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0$. Rewriting (3.5)-(3.6) we have

$$\begin{aligned}
[\mathbf{A}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0)\,,\,\boldsymbol{\varsigma}] + [\boldsymbol{\varsigma}\,,\,\mathbf{B}^*(\tilde{\boldsymbol{\psi}},\tilde{p})] \;&=\; -[\boldsymbol{\varsigma}\,,\,\mathbf{B}^*(\boldsymbol{\psi}_0,p_0)]\;,\forall\boldsymbol{\varsigma}\in T\,, && (3.8)\\
[\mathbf{B}(\tilde{\boldsymbol{\phi}})\,,\,(\boldsymbol{\tau},q)] \;&=\; 0\;,\forall(\boldsymbol{\tau},q)\in Z_1\,. && (3.9)
\end{aligned}$$

6. Introduce a subspace of T defined by

$$Z_2 \;:=\; \{\boldsymbol{\varsigma}\in T\;:\;[\boldsymbol{\varsigma},\,\mathbf{B}^*(\boldsymbol{\tau},q)] \;=\; 0\,,\,\forall(\boldsymbol{\tau},q)\in Z_1\}\;.$$

Note that Z_2 can equivalently be defined as

$$Z_2 \;:=\; \{\boldsymbol{\varsigma}\in T\;:\;[\mathbf{B}(\boldsymbol{\varsigma}),\,(\boldsymbol{\tau},q)] \;=\; 0\,,\,\forall(\boldsymbol{\tau},q)\in Z_1\}\;.$$

Equations (3.8)-(3.9) are then replaced by

$$[\mathbf{A}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0)\,,\,\boldsymbol{\varsigma}] \;=\; -[\boldsymbol{\varsigma}\,,\,\mathbf{B}^*(\boldsymbol{\psi}_0,p_0)]\;,\forall\boldsymbol{\varsigma}\in Z_2\,. \tag{3.10}$$

7. Solve (3.10) for $\tilde{\boldsymbol{\phi}}$, from which we then get our solution for $\boldsymbol{\phi}$, $\boldsymbol{\phi} = \tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0$.
8. From (3.8) we solve for $(\tilde{\boldsymbol{\psi}}, \tilde{p})$ satisfying

$$[\boldsymbol{\varsigma}\,,\, \mathbf{B}^*(\tilde{\boldsymbol{\psi}}, \tilde{p})] \;=\; -[\boldsymbol{\varsigma}\,,\, \mathbf{B}^*(\boldsymbol{\psi}_0, p_0)] \;-\; [\mathbf{A}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0)\,,\, \boldsymbol{\varsigma}]\ , \forall \boldsymbol{\varsigma} \in T\,, \tag{3.11}$$

   which we use to determine $\boldsymbol{\psi} = \tilde{\boldsymbol{\psi}} + \boldsymbol{\psi}_0$ and $p = \tilde{p} + p_0$.
9. Finally, we solve for $(\mathbf{u}, \lambda)$ using (3.3)

$$[(\boldsymbol{\tau}, q)\,,\, \mathbf{C}^*(\mathbf{u}, \lambda)] \;=\; -\int_\Gamma (\boldsymbol{\tau}\cdot\mathbf{n})\cdot\mathbf{u}_\Gamma \, d\Gamma \;-\; [\mathbf{B}(\boldsymbol{\phi})\,,\, (\boldsymbol{\tau}, q)]\ , \forall (\boldsymbol{\tau}, q) \in T'_{div} \times P\,. \tag{3.12}$$

Given the above, there are five issues to address.

(i) The solvability of $\tilde{\boldsymbol{\phi}}$ in (3.10), Step 7.

(ii) The solvability of $(\tilde{\boldsymbol{\psi}}, \tilde{p})$ in (3.11), Step 8.

(iii) The solvability of $(\mathbf{u}, \lambda)$ in (3.12), Step 9.

(iv) The existence of a particular solution $(\boldsymbol{\psi}_0, p_0)$ to (3.1), Step 1.

(v) The existence of a particular solution $\boldsymbol{\phi}_0$ to (3.7), Step 4.

Issue (i):
Given the stated assumptions on $\mathbf{g}$, $\mathbf{A}$ is a bounded, continuous, strictly monotone operator on a reflexive Banach space. The solvability of $\tilde{\boldsymbol{\phi}}$ follows from monotone operator theory [25].

Issue (ii):
For the solvability of $(\tilde{\boldsymbol{\psi}}, \tilde{p})$ we need to show that $\mathbf{B}^*$ satisfies the following inf-sup condition:

$$\inf_{(\boldsymbol{\tau}, q)\in Z_1} \sup_{\boldsymbol{\varsigma}\in T} \frac{[\boldsymbol{\varsigma}\,,\, \mathbf{B}^*(\boldsymbol{\tau}, q)]}{\|(\boldsymbol{\tau}, q)\|_{T'_{div}\times P}\, \|\boldsymbol{\varsigma}\|_T} \;\geq\; c_1\ , \tag{3.13}$$

which is equivalent to

$$\inf_{(\boldsymbol{\tau}, q)\in Z_1} \sup_{\boldsymbol{\varsigma}\in T} \frac{[\mathbf{B}(\boldsymbol{\varsigma})\,,\, (\boldsymbol{\tau}, q)]}{\|\boldsymbol{\varsigma}\|_T\, \|(\boldsymbol{\tau}, q)\|_{T'_{div}\times P}} \;\geq\; c_1\ . \tag{3.14}$$

Issue (iii):
For the solvability of $(\mathbf{u}, \lambda)$ we need to show that $\mathbf{C}^*$ satisfies the following inf-sup condition:

$$\inf_{(\mathbf{u}, \lambda)\in U\times\mathbb{R}} \sup_{(\boldsymbol{\tau}, q)\in T'_{div}\times P} \frac{[(\boldsymbol{\tau}, q)\,,\, \mathbf{C}^*(\mathbf{u}, \lambda)]}{\|(\boldsymbol{\tau}, q)\|_{T'_{div}\times P}\, \|(\mathbf{u}, \lambda)\|_{U\times\mathbb{R}}} \;\geq\; c_2\ , \tag{3.15}$$

which is equivalent to

$$\inf_{(\mathbf{u}, \lambda)\in U\times\mathbb{R}} \sup_{(\boldsymbol{\tau}, q)\in T'_{div}\times P} \frac{[\mathbf{C}(\boldsymbol{\tau}, q)\,,\, (\mathbf{u}, \lambda)]}{\|(\boldsymbol{\tau}, q)\|_{T'_{div}\times P}\, \|(\mathbf{u}, \lambda)\|_{U\times\mathbb{R}}} \;\geq\; c_2\,. \tag{3.16}$$

For **issues (iv) and (v)** note the following:

Lemma 3.1 *([16], Remark 4.2, pg. 61) Let* $(X, \|\cdot\|_X)$ *and* $(M, \|\cdot\|_M)$ *be two reflexive Banach spaces. Let* $(X^*, \|\cdot\|_{X^*})$ *and* $(M^*, \|\cdot\|_{M^*})$ *be their corresponding dual spaces. Let* $B : X \to M^*$ *be a linear continuous operator and* $B^* : M^{**} \to X$ *the dual operator of* B*. Let* $V = ker(B)$ *be the kernel of* B*; we denote by* $V^o \subset X^*$ *the polar set of* $V : V^o = \{x^* \in X^*, [x^*, v] = 0, \forall v \in V\}$ *and* $\dot{B} : (X/V) \to M^*$ *the quotient operator associated with* B*. The following three properties are equivalent:*
(i) $\exists \beta > 0$*, such that*

$$\inf_{q \in M} \sup_{v \in X} \frac{[Bv, q]}{\|q\|_M \, \|v\|_X} \geq \beta \,,$$

(ii) B^* *is an isomorphism from* M^{**} *onto* V^o *and*

$$\|B^* q\| \geq C_B \|q\|_{M^{**}} \ \forall q \in M^{**} \,,$$

(iii) $\dot{B}$ *is an isomorphism from* (X/V) *onto* M^* *and*

$$\|\dot{B}\dot{v}\| \geq C_B \|\dot{v}\|_{(X/V)} \ \forall \dot{v} \in (X/V) \,.$$

■

Part (iii) of Lemma 3.1 will guarantee the existence of particular solutions, i.e. $\boldsymbol{\psi}_0, p_0, \boldsymbol{\phi}_0$, and part (i) will guarantee existence and uniqueness of $\tilde{\boldsymbol{\psi}}, \tilde{p}, \mathbf{u}$, and λ. Thus, we have two inf-sup conditions which we need to establish.

3.1 Inf-sup Condition for B

Define the null space for the operator $\mathbf{C}$, Z_1, as

$$\begin{aligned} Z_1 \ &:= \ \left\{ (\boldsymbol{\tau}, q) \in T'_{div} \times P \, : \, [\mathbf{C}(\boldsymbol{\tau}, q) \, , \, (\mathbf{v}, \eta)] \ = \ 0 \, , \, \forall (\mathbf{v}, \eta) \in U \times \mathbb{R} \right\} \, , \\ &= \ \left\{ (\boldsymbol{\tau}, q) \in T'_{div} \times P \, : \, div \, \boldsymbol{\tau} = \mathbf{0} \text{ in } \Omega, \text{ and } \int_\Omega tr(\boldsymbol{\tau}) \, d\Omega = 0 \right\} \, . \end{aligned} \tag{3.17}$$

Note that for $(\boldsymbol{\tau}, q) \in Z_1$, $\|\boldsymbol{\tau}\|_{T'_{div}} = \|\boldsymbol{\tau}\|_{T'}$. Helpful in establishing the inf-sup condition for $\mathbf{B}$ is the following lemma.

Lemma 3.2 *(See Lemma 3.1 in [2] for Hilbert space setting.)*
For $\boldsymbol{\tau} \in T'_{div}$ *satisfying* $\int_\Omega tr(\boldsymbol{\tau}) \, d\Omega = 0$*, let* $\boldsymbol{\tau}^0 = \boldsymbol{\tau} - \frac{1}{n} tr(\boldsymbol{\tau}) \mathbf{I}$*. Then, there exists* C*, depending only* Ω*, such that*

$$\|\boldsymbol{\tau}\|_{L^{r'}} \leq C \left(\|\boldsymbol{\tau}^0\|_{L^{r'}} \, + \, \|div \, \boldsymbol{\tau}\|_{W^{-1,r'}} \right) \, . \tag{3.18}$$

Proof: Now, there exists a non-zero function $\varphi \in L^r(\Omega)$ such that

$$\|tr(\boldsymbol{\tau})\|_{L^{r'}(\Omega)} \, \|\varphi\|_{L^r(\Omega)} \ = \ \int_\Omega tr(\boldsymbol{\tau}) \, \varphi \, d\Omega \, . \tag{3.19}$$

Since $\int_\Omega tr(\boldsymbol{\tau}) \, d\Omega \ = \ 0$, we can assume $\int_\Omega \varphi \, d\Omega \ = \ 0$ (shift φ by its average). From [13], pg. 116 (see also [6] pg. 220), given $\varphi \in L^r(\Omega)$, $1 < r < \infty$ with $\int_\Omega \varphi \, d\Omega \ = \ 0$, then there exists $\mathbf{v} \in W_0^{1,r}(\Omega)$ and a constant C such that

$$div \, \mathbf{v} \ = \ \varphi \ \text{ in } \Omega \qquad \text{and} \qquad \|\mathbf{v}\|_{W^{1,r}(\Omega)} \ \leq \ C \|\varphi\|_{L^r(\Omega)} \, . \tag{3.20}$$

From (3.19) and (3.20),

$$
\begin{aligned}
\frac{1}{n\,C}\|tr(\boldsymbol{\tau})\|_{L^{r'}(\Omega)}\,\|\mathbf{v}\|_{W^{1,r}(\Omega)} &\leq \frac{1}{n}\int_\Omega tr(\boldsymbol{\tau})\,div\,\mathbf{v}\,d\Omega \\
&= \frac{1}{n}\int_\Omega tr(\boldsymbol{\tau})\,\mathbf{I} : \nabla\mathbf{v}\,d\Omega \\
&= \int_\Omega (\boldsymbol{\tau} - \boldsymbol{\tau}^0) : \nabla\mathbf{v}\,d\Omega \qquad \text{(using the defn. of } \boldsymbol{\tau}^0) \\
&= -\int_\Omega (\boldsymbol{\tau}^0 : \nabla\mathbf{v} \,+\, div\,\boldsymbol{\tau}\cdot\mathbf{v})\,d\Omega \\
&\leq \left(\|\boldsymbol{\tau}^0\|_{L^{r'}(\Omega)} \,+\, \|div\,\boldsymbol{\tau}\|_{W^{-1,r'}(\Omega)}\right)\|\mathbf{v}\|_{W^{1,r}(\Omega)}\,.
\end{aligned}
$$

■

Lemma 3.3 *There exists a constant* $c_1 > 0$ *such that*

$$\inf_{(\boldsymbol{\tau},q)\in Z_1}\sup_{\boldsymbol{\phi}\in T}\frac{[\mathbf{B}(\boldsymbol{\phi})\,,\,(\boldsymbol{\tau},q)]}{\|\boldsymbol{\phi}\|_T\,\|(\boldsymbol{\tau},q)\|_{T'_{div}\times P}} \geq \; c_1\;.$$

Proof: We establish the inf-sup condition by considering two cases.
Case 1.: $\|q\|_P \leq \|\boldsymbol{\tau}\|_{T'_{div}}$.
Let

$$\boldsymbol{\tau}^0 = \boldsymbol{\tau} - \frac{1}{n}tr(\boldsymbol{\tau})\mathbf{I}, \qquad \text{and} \qquad \boldsymbol{\phi} = -|\boldsymbol{\tau}^0|^{r'/r-1}\,\boldsymbol{\tau}^0/\|\boldsymbol{\tau}^0\|_{T'}^{r'-1}\,. \tag{3.21}$$

Note that $\boldsymbol{\phi}\in T$, and $\|\boldsymbol{\phi}\|_T = 1$. Then,

$$
\begin{aligned}
\frac{[B(\boldsymbol{\phi}),(\boldsymbol{\tau},q)]}{\|\boldsymbol{\phi}\|_T} &= \int_\Omega \frac{|\boldsymbol{\tau}^0|^{r'/r-1}}{\|\boldsymbol{\tau}^0\|_{T'}^{r'-1}}\,\boldsymbol{\tau} : \boldsymbol{\tau}^0\,d\Omega \;+\; \int_\Omega q\frac{|\boldsymbol{\tau}|^{r'/r-1}\,tr(\boldsymbol{\tau}^0)}{\|\boldsymbol{\tau}^0\|_{T'}^{r'-1}}\,d\Omega \\
&= \frac{1}{\|\boldsymbol{\tau}^0\|_{T'}^{r'-1}}\int_\Omega |\boldsymbol{\tau}^0|^{r'/r-1}\,\boldsymbol{\tau}^0 : \boldsymbol{\tau}^0\,d\Omega\,, (\text{as } tr(\boldsymbol{\tau}^0) = 0\,,\ \text{and } \boldsymbol{\tau} : \boldsymbol{\tau}^0 = \boldsymbol{\tau}^0 : \boldsymbol{\tau}^0) \\
&= \frac{1}{\|\boldsymbol{\tau}^0\|_{T'}^{r'-1}}\|\boldsymbol{\tau}^0\|_{T'}^{r'} \\
&= \|\boldsymbol{\tau}^0\|_{T'} \;\geq\; \frac{1}{C}\|\boldsymbol{\tau}\|_{T'} \;=\; \frac{1}{C}\|\boldsymbol{\tau}\|_{T'_{div}} \qquad (\text{ as } (\boldsymbol{\tau},q)\in Z_1\,,\ \text{ see (3.18)}) \\
&\geq \frac{1}{2C}\left(\|\boldsymbol{\tau}\|_{T'_{div}} \,+\, \|q\|_P\right) \;=\; \frac{1}{2C}\|(\boldsymbol{\tau},q)\|_{T'_{div}\times P}\,.
\end{aligned}
\tag{3.22}
$$

Case 2.: $\|q\|_P \geq \|\boldsymbol{\tau}\|_{T'_{div}}$.
Let

$$\boldsymbol{\phi} \;=\; \frac{-|q\mathbf{I} \,+\, \boldsymbol{\tau}|^{r'/r-1}\,(q\mathbf{I} \,+\, \boldsymbol{\tau})}{\|q\mathbf{I} \,+\, \boldsymbol{\tau}\|_{T'}^{r'-1}}\,. \tag{3.23}$$

Again, $\boldsymbol{\phi} \in T$, and $\|\boldsymbol{\phi}\|_T = 1$. For this choice of $\boldsymbol{\phi}$,

$$\begin{aligned}
\frac{[B(\boldsymbol{\phi}), (\boldsymbol{\tau}, q)]}{\|\boldsymbol{\phi}\|_T} &= \int_\Omega \frac{|q\mathbf{I} \,+\, \boldsymbol{\tau}|^{r'/r-1}}{\|q\mathbf{I} \,+\, \boldsymbol{\tau}\|_{T'}^{r'-1}} \left(\boldsymbol{\tau} : (q\mathbf{I} \,+\, \boldsymbol{\tau}) \,+\, q\, tr(q\mathbf{I} \,+\, \boldsymbol{\tau})\right) d\Omega \\
&= \int_\Omega \frac{|q\mathbf{I} \,+\, \boldsymbol{\tau}|^{r'/r-1}}{\|q\mathbf{I} \,+\, \boldsymbol{\tau}\|_{T'}^{r'-1}} (q\mathbf{I} \,+\, \boldsymbol{\tau}) : (q\mathbf{I} \,+\, \boldsymbol{\tau})\, d\Omega \\
&= \|q\mathbf{I} \,+\, \boldsymbol{\tau}\|_{T'} \\
&\geq \|q\mathbf{I}\|_{T'} \,-\, \|\boldsymbol{\tau}\|_{T'} \\
&= n^{1/r'}\|q\|_P \,-\, \|\boldsymbol{\tau}\|_{T'} \\
&\geq (n^{1/r'} - 1)\|q\|_P \\
&\geq (n^{1/r'} - 1)/2\, \left(\|q\|_P \,+\, \|\boldsymbol{\tau}\|_{T'_{div}}\right) \\
&= C\|(\boldsymbol{\tau}, q)\|_{T'_{div} \times P}\,. \qquad (3.24)
\end{aligned}$$

■

3.2 Inf-sup Condition for C

The following two lemmas are helpful in establishing the inf-sup condition for **C**.

Lemma 3.4 *Let* ${}_0T'_{div} \;:=\; \left\{\boldsymbol{\tau} \in T'_{div} \,:\, \int_\Omega tr(\boldsymbol{\tau})\, d\Omega \,=\, 0\right\}$. *Then, there exists* $C > 0$ *such that for any* $\mathbf{u} \in U$

$$\sup_{\substack{\hat{\boldsymbol{\tau}} \in\, {}_0T'_{div} \\ \hat{\boldsymbol{\tau}} \neq 0}} \frac{\int_\Omega \mathbf{u} \cdot div\hat{\boldsymbol{\tau}}\, d\Omega}{\|\hat{\boldsymbol{\tau}}\|_{T'_{div}}} \;\geq\; C \sup_{\substack{\boldsymbol{\tau} \in T'_{div} \\ \boldsymbol{\tau} \neq 0}} \frac{\int_\Omega \mathbf{u} \cdot div\boldsymbol{\tau}\, d\Omega}{\|\boldsymbol{\tau}\|_{T'_{div}}}\,. \qquad (3.25)$$

Proof: For $\boldsymbol{\tau} \in T'_{div}$, let $\boldsymbol{\tau}_0 \,=\, \boldsymbol{\tau} \,-\, \frac{1}{n|\Omega|}\left(\int_\Omega tr(\boldsymbol{\tau}))\, d\Omega\right)\mathbf{I}$. Then, $\boldsymbol{\tau}_0 \in {}_0T'_{div}$, and $div\boldsymbol{\tau} \,=\, div\boldsymbol{\tau}_0$. Let

$$\varsigma \;:=\; |\boldsymbol{\tau}_0|^{r'/r\,-\,1}\boldsymbol{\tau}_0 \;+\; \frac{sgn\left(\left(\int_\Omega tr(\boldsymbol{\tau})\, d\Omega\right)\left(\int_\Omega |\boldsymbol{\tau}_0|^{r'/r\,-\,1}\boldsymbol{\tau}_0\, d\Omega\right)\right)}{n\,|\Omega|}\left(\int_\Omega |\boldsymbol{\tau}_0|^{r'/r\,-\,1}\boldsymbol{\tau}_0\, d\Omega\right)\mathbf{I}\,.$$

Note that as

$$\||\boldsymbol{\tau}_0|^{r'/r\,-\,1}\boldsymbol{\tau}_0\|_{L^r} \;=\; \left(\int_\Omega |\boldsymbol{\tau}_0|^{r'}\, d\Omega\right)^{1/r} \;=\; \|\boldsymbol{\tau}_0\|_{L^{r'}}^{r'/r}\,,$$

and

$$\begin{aligned}
\left|\int_\Omega |\boldsymbol{\tau}_0|^{r'/r\,-\,1} tr(\boldsymbol{\tau}_0)\, d\Omega\right| \;&\leq\; \sqrt{n}\int_\Omega |\boldsymbol{\tau}_0|^{r'/r}\, 1\, d\Omega \\
&\leq\; C\,\|\boldsymbol{\tau}_0\|_{L^{r'}}^{r'/r}\left(\int_\Omega 1^{r'}\, d\Omega\right)^{1/r'} \\
&\leq\; C\,\|\boldsymbol{\tau}_0\|_{L^{r'}}^{r'/r}\,.
\end{aligned}$$

Thus

$$\|\varsigma\|_{L^r} \leq C\|\boldsymbol{\tau}_0\|_{L^{r'}}^{r'/r}. \tag{3.26}$$

We have that

$$\|\boldsymbol{\tau}\|_{L^{r'}} = \sup_{\boldsymbol{\sigma}\in L^r} \frac{(\boldsymbol{\tau}\,,\,\boldsymbol{\sigma})}{\|\boldsymbol{\sigma}\|_{L^r}}\,. \tag{3.27}$$

Now, using $\boldsymbol{\tau}_0 \in\ {}_0T^{'}_{div}$,

$$\begin{aligned}(\boldsymbol{\tau}\,,\,\varsigma) &= \int_\Omega |\boldsymbol{\tau}_0|^{r'}\,d\Omega + \frac{1}{n|\Omega|}\left(\int_\Omega tr(\boldsymbol{\tau})\,d\Omega\right)\left(\int_\Omega |\boldsymbol{\tau}_0|^{r'/r\,-\,1}\,tr(\boldsymbol{\tau}_0)\,d\Omega\right)\\ &\quad + \left|\frac{1}{n|\Omega|}\left(\int_\Omega tr(\boldsymbol{\tau})\,d\Omega\right)\left(\int_\Omega |\boldsymbol{\tau}_0|^{r'/r\,-\,1}\,tr(\boldsymbol{\tau}_0)\,d\Omega\right)\right|\\ &\geq \|\boldsymbol{\tau}_0\|_{L^{r'}}^{r'}\,.\end{aligned} \tag{3.28}$$

Therefore, from (3.26), (3.27), and (3.28) we have that $\|\boldsymbol{\tau}\|_{L^{r'}} \geq C\,\|\boldsymbol{\tau}_0\|_{L^{r'}}$. Combining the above we obtain

$$\frac{\int_\Omega \mathbf{u}\cdot div\boldsymbol{\tau}\,d\Omega}{\|\boldsymbol{\tau}\|_{T^{'}_{div}}} = \frac{\int_\Omega \mathbf{u}\cdot div\boldsymbol{\tau}_0\,d\Omega}{\|\boldsymbol{\tau}\|_{T^{'}_{div}}} \leq C\frac{\int_\Omega \mathbf{u}\cdot div\boldsymbol{\tau}_0\,d\Omega}{\|\boldsymbol{\tau}_0\|_{T^{'}_{div}}}\,,$$

from which (3.25) then follows.

■

Lemma 3.5 *Given* $\mathbf{w} \in (L^{r'}(\Omega))^2$*, there exists* $\boldsymbol{\tau} \in T^{'}_{div}$ *such that*

$$div(\boldsymbol{\tau}) = \mathbf{w} \ \ in\,\Omega\,, \quad and \quad \|\boldsymbol{\tau}\|_{T^{'}_{div}} \leq C\,\|\mathbf{w}\|_{L^{r'}(\Omega)}. \tag{3.29}$$

Proof: We show the construction of a suitable $\boldsymbol{\tau}$ for the case $n = 2$. The method extends in a straight forward manner for $n = 3$. For $\mathbf{w} := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in (L^{r'}(\Omega))^2$, let $w_1^0 := w_1 - \frac{1}{|\Omega|}\int_\Omega w_1\,d\Omega$. Note that $w_1^0 \in L^{r'}(\Omega)$ and $\int_\Omega w_1^0 d\Omega = 0$. Then, from [13], pg. 116, there exist $\mathbf{v} := \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and a constant C such that

$$div\mathbf{v} = w_1^0 \ \text{ in }\Omega\,, \quad \text{and} \quad \|\mathbf{v}\|_{W^{1,r'}(\Omega)} \leq C\|w_1^0\|_{L^{r'}(\Omega)}.$$

Next consider the problem: *Find* $z \in \{f : f \in L^{r'}(\Omega)$ *and* $f_y \in L^{r'}(\Omega)\}$ *such that*

$$div\begin{bmatrix} v_1 \\ z \end{bmatrix} = w_2 \ \text{ in }\Omega.$$

This is equivalent to finding $z_y \in L^{r'}(\Omega)$ such that

$$z_y = w_2 - v_{2x} \ \text{ in }\Omega.$$

As Ω is bounded, without loss of generality, assume $\Omega \subset \{(x,y) : 0 < x < a\,, 0 < y < b\}$. For $g \in L^{r'}(\Omega)$, let $\tilde{g}$ denote the extension by zero of g to $\mathbb{R}^2\backslash\Omega$. For $(x,y) \in \Omega$, let

$$z(x,y) = \int_0^y \widetilde{(w_2(x,s) - v_{2x}(x,s))}\,ds\,.$$

Now

$$\begin{aligned}
\|z\|^{r'}_{L^{r'}(\Omega)} &= \int_0^b \int_0^a \left(\int_0^y (\widetilde{w_2 - v_{2x}})\, ds \right)^{r'} dx\, dy \\
&\le \int_0^b \int_0^a \left(\int_0^b 1^r ds \right)^{r'/r} \left(\int_0^b ((\widetilde{w_2 - v_{2x}}))^{r'}\, ds \right) dx\, dy \\
&\le b^{r'}\, \|w_2 - v_{2x}\|^{r'}_{L^{r'}(\Omega)} \\
&\le C \left(\|w_2\|^{r'}_{L^{r'}(\Omega)} + \|v_{2x}\|^{r'}_{L^{r'}(\Omega)} \right) \le C \left(\|w_2\|^{r'}_{L^{r'}(\Omega)} + \|w_1^0\|^{r'}_{L^{r'}(\Omega)} \right) .
\end{aligned}$$

Also, note that

$$\|w_1^0\|_{L^{r'}(\Omega)} \le \|w_1\|_{L^{r'}(\Omega)} + \left\| \frac{1}{|\Omega|} \int_\Omega w_1\, d\Omega \right\|_{L^{r'}(\Omega)} ,$$

and

$$\begin{aligned}
\left\| \frac{1}{|\Omega|} \int_\Omega w_1\, d\Omega \right\|^{r'}_{L^{r'}(\Omega)} &= \frac{1}{|\Omega|^{r'}} \left(\int_\Omega \left(\int_\Omega w_1\, d\Omega \right)^{r'} d\Omega \right) \\
&\le \frac{1}{|\Omega|^{r'}} \left(\int_\Omega \left(\int_\Omega 1^r\, d\Omega \right)^{\frac{r'}{r}} \left(\int_\Omega |w_1|^{r'}\, d\Omega \right) d\Omega \right) \\
&\le \frac{1}{|\Omega|^{r'}} \left(\int_\Omega |\Omega|^{r'/r}\, \|w_1\|^{r'}_{L^{r'}(\Omega)}\, d\Omega \right) \\
&= \|w_1\|^{r'}_{L^{r'}(\Omega)} .
\end{aligned}$$

Thus

$$\|w_1^0\|_{L^{r'}(\Omega)} \le 2\, \|w_1\|_{L^{r'}(\Omega)} .$$

Let $\boldsymbol{\tau}(x, y) := \begin{bmatrix} v_1 + x \frac{1}{|\Omega|} \int_\Omega w_1\, d\Omega & v_2 \\ v_2 & z \end{bmatrix}$. Then

$$div \boldsymbol{\tau} = div \begin{bmatrix} v_1 & v_2 \\ v_2 & z \end{bmatrix} + div \begin{bmatrix} x \frac{1}{|\Omega|} \int_\Omega w_1\, d\Omega & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} w_1^0 \\ w_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{|\Omega|} \int_\Omega w_1\, d\Omega \\ 0 \end{bmatrix} = \mathbf{w}.$$

Moreover,

$$\|\boldsymbol{\tau}\|_{T'_{div}} \le C\, \|\mathbf{w}\|_{L^{r'}(\Omega)}.$$

■

Lemma 3.6 *There exists a constant* $c_2 > 0$ *such that*

$$\inf_{(\mathbf{u},\lambda) \in U \times \mathbb{R}} \ \sup_{(\boldsymbol{\tau}, q) \in T'_{div} \times P} \frac{[\mathbf{C}(\boldsymbol{\tau}, q)\, ,\, (\mathbf{u}, \lambda)]}{\|(\boldsymbol{\tau}, q)\|_{T'_{div} \times P}\, \|(\mathbf{u}, \lambda)\|_{U \times \mathbb{R}}} \ge c_2 .$$

Proof: We establish the inf-sup condition by considering two cases.
<u>Case 1.</u>: $|\lambda| \geq \|\mathbf{u}\|_U$.
For this case we have

$$\begin{aligned} \sup_{(\boldsymbol{\tau},q)\in T'_{div}\times P} \frac{[\mathbf{C}(\boldsymbol{\tau},q)\,,\,(\mathbf{u},\lambda)]}{\|(\boldsymbol{\tau},q)\|_{T'_{div}\times P}} \;&\geq\; \frac{[\mathbf{C}(\lambda\mathbf{I},0)\,,\,(\mathbf{u},\lambda)]}{\|\lambda\mathbf{I}\|_{T'_{div}}} \;=\; \frac{n\,\lambda^2\,|\Omega|}{|\lambda| n^{r'/2}\,|\Omega|^{1/r'}} \\ &\geq\; C\|(\mathbf{u},\lambda)\|_{U\times\mathbb{R}}\,. \end{aligned} \tag{3.30}$$

<u>Case 2.</u>: $|\lambda| \leq \|\mathbf{u}\|_U$.
Using Lemma 3.4,

$$\begin{aligned} \sup_{(\boldsymbol{\tau},q)\in T'_{div}\times P} \frac{[\mathbf{C}(\boldsymbol{\tau},q)\,,\,(\mathbf{u},\lambda)]}{\|(\boldsymbol{\tau},q)\|_{T'_{div}\times P}} \;&\geq\; \sup_{\boldsymbol{\tau}_0\in\, {}_0T'_{div}} \frac{[\mathbf{C}(\boldsymbol{\tau}_0,0)\,,\,(\mathbf{u},\lambda)]}{\|\boldsymbol{\tau}_0\|_{T'_{div}}} \;=\; \sup_{\boldsymbol{\tau}_0\in\, {}_0T'_{div}} \frac{-\int_\Omega \mathbf{u}\cdot div\boldsymbol{\tau}_0\,d\Omega}{\|\boldsymbol{\tau}_0\|_{T'_{div}}} \\ &\geq\; C \sup_{\boldsymbol{\tau}\in T'_{div}} \frac{-\int_\Omega \mathbf{u}\cdot div\boldsymbol{\tau}\,d\Omega}{\|\boldsymbol{\tau}\|_{T'_{div}}}\,. \end{aligned} \tag{3.31}$$

Choose $\mathbf{w} \in (L^{r'}(\Omega))^n$ such that $\|\mathbf{u}\|_{L^r}\,\|\mathbf{w}\|_{L^{r'}} \;=\; \int_\Omega \mathbf{u}\cdot\mathbf{w}\,d\Omega$, and let $\boldsymbol{\tau}$ be as given in (3.29).

Then,

$$\begin{aligned} \sup_{(\boldsymbol{\tau},q)\in T'_{div}\times P} \frac{[\mathbf{C}(\boldsymbol{\tau},q)\,,\,(\mathbf{u},\lambda)]}{\|(\boldsymbol{\tau},q)\|_{T'_{div}\times P}} \;&\geq\; C\,\frac{-\int_\Omega \mathbf{u}\cdot div(-\boldsymbol{\tau})\,d\Omega}{\|-\boldsymbol{\tau}\|_{T'_{div}}} \;\geq\; C\,\frac{\int_\Omega \mathbf{u}\cdot\mathbf{w}\,d\Omega}{\|\mathbf{w}\|_{L^{r'}}} \\ &\geq\; C\|\mathbf{u}\|_U \\ &\geq\; C\|(\mathbf{u},\lambda)\|_{U\times\mathbb{R}}\,. \end{aligned} \tag{3.32}$$

■

We now summarize the above results in the following theorem.

Theorem 3.1 *There exists a unique solution* $(\boldsymbol{\phi},\boldsymbol{\psi},p,\mathbf{u},\lambda) \in T\times T'_{div}\times P\times U\times\mathbb{R}$ *satisfying (2.19)–(2.21). In addition, we have that*

$$\|\boldsymbol{\phi}\|_T \leq C\left(\|\mathbf{u}_\Gamma\|_{1-1/r,r,\Gamma} + \|\mathbf{f}\|_{0,r',\Omega}^{r'/r}\right)\,. \tag{3.33}$$

Proof: Existence and uniqueness of the solution $(\boldsymbol{\phi},\boldsymbol{\psi},p,\mathbf{u},\lambda) \in T\times T'_{div}\times P\times U\times\mathbb{R}$ follows directly from the continuity and monotonicity of $\mathbf{g}$ and the inf-sup conditions (3.14) and (3.16).

To show (3.33), we begin with some preliminary bounds. Let $\boldsymbol{\phi} = \tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0$, $\boldsymbol{\psi} = \tilde{\boldsymbol{\psi}} + \boldsymbol{\psi}_0$, and $p = \tilde{p} + p_0$ be as given in (3.1)-(3.12). Note that from (3.1) and the inf-sup condition (3.16), we have that there exists $(\mathbf{v},\eta) \in U\times\mathbb{R}$ such that

$$\begin{aligned} \frac{c_2}{2} \;&\leq\; \frac{[\mathbf{C}(\boldsymbol{\psi}_0,p_0),(\mathbf{v},\eta)]}{\|(\mathbf{v},\eta)\|_{U\times\mathbb{R}}\|(\boldsymbol{\psi}_0,p_0)\|_{T'_{div}\times P}} \\ &=\; \frac{\int_\Omega \mathbf{v}\cdot\mathbf{f}\,d\Omega}{\|(\mathbf{v},\eta)\|_{U\times\mathbb{R}}\|(\boldsymbol{\psi}_0,p_0)\|_{T'_{div}\times P}} \\ &\leq\; \frac{\|\mathbf{v}\|_U\|\mathbf{f}\|_{(L^{r'}(\Omega))^n}}{\|\mathbf{v}\|_U\left(\|\boldsymbol{\psi}_0\|_{T'} + \|p_0\|_P\right)} = \frac{\|\mathbf{f}\|_{0,r',\Omega}}{\left(\|\boldsymbol{\psi}_0\|_{T'} + \|p_0\|_P\right)}\,, \end{aligned} \tag{3.34}$$

or

$$\|\boldsymbol{\psi}_0\|_{T'} + \|p_0\|_P \leq \frac{2}{c_2}\|\mathbf{f}\|_{0,r',\Omega}\,. \tag{3.35}$$

From Lemma (3.1) (i) and (iii) with the associations $B = \mathbf{B}$, $M = Z_1 \subset T^{'}_{div} \times P$, $X = T$, and $V = ker(\mathbf{B})$, we have that there exists a $\dot{\boldsymbol{\phi}} \in T/V$ such that

$$[\mathbf{B}(\dot{\boldsymbol{\phi}}), (\boldsymbol{\tau}, q)] = -\int_\Gamma (\boldsymbol{\tau}\cdot\mathbf{n})\cdot\mathbf{u}_\Gamma\, d\Gamma\,, \quad \forall(\boldsymbol{\tau}, q) \in Z_1\,,$$

with $\|\dot{\boldsymbol{\phi}}\|_{T/V} \leq 1/C_{\mathbf{B}}\|\mathbf{u}_\Gamma\|_{1-1/r,r,\Gamma}$. Note that $\|\dot{\boldsymbol{\phi}}\|_{T/V} := \inf_{\boldsymbol{\varsigma}\in\dot{\boldsymbol{\phi}}}\|\boldsymbol{\varsigma}\|_T$. As the cosets in T/V are closed, we can choose $\boldsymbol{\phi}_0 \in \dot{\boldsymbol{\phi}}$ such that

$$\|\boldsymbol{\phi}_0\|_T = \|\dot{\boldsymbol{\phi}}\|_{T/V} \leq \frac{1}{C_{\mathbf{B}}}\|\mathbf{u}_\Gamma\|_{1-1/r,r,\Gamma}\,. \tag{3.36}$$

From (2.2) we have that

$$\hat{C}_1\left(\|\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0\|_T^r + \int_\Omega |\mathbf{g}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0)||\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0|\, d\Omega\right) \leq \int_\Omega \mathbf{g}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0) : (\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0)\, d\Omega\,. \tag{3.37}$$

Using (2.16) with $\boldsymbol{\varsigma} = \tilde{\boldsymbol{\phi}}$ we have

$$\begin{aligned}
[\mathbf{A}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0), \tilde{\boldsymbol{\phi}}] &= \int_\Omega \mathbf{g}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0) : \tilde{\boldsymbol{\phi}}\, d\Omega \\
&= \int_\Omega \mathbf{g}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0) : (\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0)\, d\Omega + \int_\Omega \mathbf{g}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0) : \boldsymbol{\phi}_0\, d\Omega \\
&\geq \hat{C}_1\left(\|\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0\|_T^r + \int_\Omega |\mathbf{g}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0)||\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0|\, d\Omega\right) \\
&\quad - \hat{C}_2\left(\int_\Omega |\mathbf{g}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0)||\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0|\, d\Omega\right)^{1/r'}\|\boldsymbol{\phi}_0\|_T \\
&\geq \hat{C}_1\|\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0\|_T^r + \left(\hat{C}_1 - \frac{\epsilon_1\hat{C}_2}{r'}\right)\int_\Omega |\mathbf{g}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0)||\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0|\, d\Omega \\
&\qquad - \frac{\hat{C}_2}{r\epsilon_1}\|\boldsymbol{\phi}_0\|_T^r\,.
\end{aligned} \tag{3.38}$$

Now we also have from (3.10), using Young's inequality and the triangle inequality,

$$\begin{aligned}
[\mathbf{A}(\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0), \tilde{\boldsymbol{\phi}}] &= -[\mathbf{B}(\tilde{\boldsymbol{\phi}}), (\boldsymbol{\psi}_0, p_0)] \\
&= \int_\Omega \boldsymbol{\psi}_0 : \tilde{\boldsymbol{\phi}}\, d\Omega + \int_\Omega p_0\, tr(\tilde{\boldsymbol{\phi}})\, d\Omega \\
&\leq \|\boldsymbol{\psi}_0\|_{T'}\|\tilde{\boldsymbol{\phi}}\|_T + \sqrt{n}\,\|p_0\|_P\|\tilde{\boldsymbol{\phi}}\|_T \\
&\leq \frac{2\epsilon_2}{r}\|\tilde{\boldsymbol{\phi}}\|_T^r + \frac{1}{r'\epsilon_2}\left(\|\boldsymbol{\psi}_0\|_{T'}^{r'} + \sqrt{n}\,\|p_o\|_P^{r'}\right) \\
&\leq \frac{2\epsilon_2}{r}\|\tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0\|_T^r + \frac{2\epsilon_2}{r}\|\boldsymbol{\phi}_0\|_T^r \\
&\qquad + \frac{1}{r'\epsilon_2}\left(\|\boldsymbol{\psi}_0\|_{T'}^{r'}\sqrt{n}\,\|p_o\|_P^{r'}\right)\,.
\end{aligned} \tag{3.39}$$

Combining (3.38), (3.39), and $\boldsymbol{\phi} = \tilde{\boldsymbol{\phi}} + \boldsymbol{\phi}_0$, we have

$$\left(\hat{C}_1 - \frac{2\epsilon_2}{r}\right) \|\boldsymbol{\phi}\|_T^r + \left(\hat{C}_1 - \frac{\epsilon_1 \hat{C}_2}{r'}\right) \int_\Omega |\mathbf{g}(\boldsymbol{\phi})||\boldsymbol{\phi}| \, d\Omega$$
$$\leq \left(\frac{\hat{C}_2}{r\epsilon_1} + \frac{2\epsilon_2}{r}\right) \|\boldsymbol{\phi}_0\|_T^r + \frac{1}{r'\epsilon_2} \left(\|\boldsymbol{\psi}_0\|_{T'}^{r'} + \sqrt{n}\, \|p_o\|_P^{r'}\right) . \quad (3.40)$$

Together with (3.35), (3.36) and choices for ϵ_1, ϵ_2 that ensure

$$\hat{C}_1 - \frac{2\epsilon_2}{r} > 0\,, \quad \text{and} \quad \hat{C}_1 - \frac{\epsilon_1 \hat{C}_2}{r'} > 0\,,$$

the result (3.33) is shown. ■

4 Finite Element Approximation

Let $\Omega \subset \mathbb{R}^n$ be a polygonal domain and let $\mathcal{T}_h$ be a triangulation of Ω into triangles ($n = 2$) or tetrahedrals ($n = 3$). Thus

$$\Omega = \cup K\,, \quad K \in \mathcal{T}_h\,,$$

and assume that there exist constants γ_1, γ_2 such that

$$\gamma_1 h \leq h_K \leq \gamma_2 \rho_K \quad (4.1)$$

where h_K is the diameter of triangle (tetrahedral) K, ρ_K is the diameter of the greatest ball (sphere) included in K, and $h = \max_{K\in\mathcal{T}_h} h_K$. Define the finite-dimensional subspaces $T_h \subseteq T$, $T'_{div,h} \subseteq T'_{div}$, $P_h \subseteq P$, and $U_h \subseteq U$. Then the discrete formulation of (2.13)-(2.15) is defined as: *Given* $\mathbf{f} \in \left(L^{r'}(\Omega)\right)^n$, $\mathbf{u}_\Gamma \in \left(W^{1-1/r,r}(\Gamma)\right)^n$, *determine* $(\boldsymbol{\phi}_h, \boldsymbol{\psi}_h, p_h, \mathbf{u}_h, \lambda_h) \in T_h \times T'_{div,h} \times P_h \times U_h \times \mathbb{R}$ *such that*

$$\int_\Omega \mathbf{g}(\boldsymbol{\phi}_h) : \boldsymbol{\varsigma}_h \, d\Omega \;-\; \int_\Omega \boldsymbol{\psi} : \boldsymbol{\varsigma}_h \, d\Omega \;-\; \int_\Omega p_h \, tr(\boldsymbol{\varsigma}_h) \, d\Omega = 0 \;, \forall \boldsymbol{\varsigma}_h \in T_h\,, \quad (4.2)$$

$$-\int_\Omega \boldsymbol{\tau}_h : \boldsymbol{\phi}_h \, d\Omega \;-\; \int_\Omega q_h \, tr(\boldsymbol{\phi}_h) \, d\Omega \;-\; \int_\Omega \mathbf{u}_h \cdot div\, \boldsymbol{\tau}_h \, d\Omega + \lambda_h \int_\Omega tr(\boldsymbol{\tau}_h) \, d\Omega$$
$$= -\int_\Gamma (\boldsymbol{\tau}_h \cdot \mathbf{n}) \cdot \mathbf{u}_\Gamma \, d\Gamma \;, \forall (\boldsymbol{\tau}_h, q_h) \in T'_{div,h} \times P_h\,, \quad (4.3)$$

$$-\int_\Omega \mathbf{v}_h \cdot div\, \boldsymbol{\psi}_h \, d\Omega \;+\; \eta_h \int_\Omega tr(\boldsymbol{\psi}_h) \, d\Omega = \int_\Omega \mathbf{v}_h \cdot \mathbf{f} \, d\Omega \,, \forall (\mathbf{v}_h, \eta_h) \in U_h \times \mathbb{R}\,. \quad (4.4)$$

Restricting the domain of the operators $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ to the appropriate finite-dimensional subspaces we can write (4.2)-(4.4) as a twofold saddle point system:

$$[\mathbf{A}(\boldsymbol{\phi}_h)\,,\, \boldsymbol{\varsigma}_h] \;+\; [\boldsymbol{\varsigma}_h\,,\, \mathbf{B}^*(\boldsymbol{\psi}_h, p_h)] \;=\; 0 \;, \forall \boldsymbol{\varsigma}_h \in T_h\,, \quad (4.5)$$

$$[\mathbf{B}(\boldsymbol{\phi}_h), (\boldsymbol{\tau}_h, q_h)] + [(\boldsymbol{\tau}_h, q_h), \mathbf{C}^*(\mathbf{u}_h, \lambda_h)] \;=\; -\int_\Gamma (\boldsymbol{\tau}_h \cdot \mathbf{n}) \cdot \mathbf{u}_\Gamma \, d\Gamma \;,$$
$$\forall (\boldsymbol{\tau}_h, q_h) \in T'_{div,h} \times P_h\,, \quad (4.6)$$

$$[\mathbf{C}(\boldsymbol{\psi}_h, p_h)\,,\, (\mathbf{v}_h, \eta_h)] \;=\; \int_\Omega \mathbf{v}_h \cdot \mathbf{f} \, d\Omega \,, \forall (\mathbf{v}_h, \eta_h) \in U_h \times \mathbb{R}\,. \quad (4.7)$$

The corresponding discrete kernels of $\mathbf{B}$ and $\mathbf{C}$ are defined similarly. We have

$$Z_{1h} \ := \ \Big\{ (\boldsymbol{\tau}_h, q_h) \in T^{'}_{div,h} \times P_h \ : \ [\mathbf{C}(\boldsymbol{\tau}_h, q_h) \, , \, (\mathbf{v}_h, \eta_h)] \ = \ 0 \, , \, \forall (\mathbf{v}_h, \eta_h) \in U_h \times \mathbb{R} \Big\} \ ,$$

and

$$Z_{2h} \ := \ \{ \boldsymbol{\varsigma}_h \in T_h \ : \ [\mathbf{B}(\boldsymbol{\varsigma}_h), \, (\boldsymbol{\tau}_h, q_h)] \ = \ 0 \, , \, \forall (\boldsymbol{\tau}_h, q_h) \in Z_{1h} \} \ .$$

4.1 A Priori Estimates

Theorem 4.1 *Let* $1 < r < 2$ *and* $\mathbf{g}$ *satisfy (2.2) and (2.3). Let* $(\boldsymbol{\phi}, \boldsymbol{\psi}, p, \mathbf{u}, \lambda) \in T \times T^{'}_{div} \times P \times U \times \mathbb{R}$ *solve (2.13)-(2.15). Assume that*
(1) There exists a positive constant c_1 *such that*

$$\inf_{(\boldsymbol{\tau}_h, q_h) \in Z_{1h}} \ \sup_{\boldsymbol{\varsigma}_h \in T_h} \frac{[\mathbf{B}(\boldsymbol{\varsigma}_h) \, , \, (\boldsymbol{\tau}_h, q_h)]}{\|\boldsymbol{\varsigma}_h\|_T \, \|(\boldsymbol{\tau}_h, q_h)\|_{T^{'}_{div} \times P}} \ \geq \ c_1 \ . \tag{4.8}$$

(2) There exists a positive constant c_2 *such that*

$$\inf_{(\mathbf{u}_h, \lambda_h) \in U_h \times \mathbb{R}} \ \sup_{(\boldsymbol{\tau}_h, q_h) \in T^{'}_{div,h} \times P_h} \frac{[\mathbf{C}(\boldsymbol{\tau}_h, q_h) \, , \, (\mathbf{u}_h, \lambda_h)]}{\|(\boldsymbol{\tau}_h, q_h)\|_{T^{'}_{div} \times P} \, \|(\mathbf{u}_h, \lambda_h)\|_{U \times \mathbb{R}}} \ \geq \ c_2 \, . \tag{4.9}$$

Then, for $\mathbf{f} \in \Big(L^{r'}(\Omega) \Big)^n$ *and* $\mathbf{u}_\Gamma \in \big(W^{1-1/r \, , \, r}(\Gamma) \big)^n$*, there exists a unique solution* $(\boldsymbol{\phi}_h, \boldsymbol{\psi}_h, p_h, \mathbf{u}_h, \lambda_h) \in T_h \times T^{'}_{div, \, h} \times P_h \times U_h \times \mathbb{R}$ *to the problem (4.5)-(4.7). In addition, we have*

$$\|\boldsymbol{\phi}_h\|_T \leq C \left(\|\mathbf{u}_\Gamma\|_{1-1/r, r, \Gamma} + \|\mathbf{f}\|^{r'/r}_{0, r', \Omega} \right) \ , \tag{4.10}$$

for some constant $C > 0$.

Proof: With the assumptions as stated above, existence and uniqueness of $(\boldsymbol{\phi}_h, \boldsymbol{\psi}_h, p_h, \mathbf{u}_h, \lambda_h) \in T_h \times T^{'}_{div, \, h} \times P_h \times U_h \times \mathbb{R}$ solving (4.5)-(4.7) follows directly from the continuous solution approach outlined in Section 3 and summarized in Theorem 3.1. The proof of (4.10) mirrors the approach for showing (3.33). ■

Theorem 4.2 *Let*

$$\mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h) = \left\| \frac{|\boldsymbol{\phi} - \boldsymbol{\phi}_h|}{|\boldsymbol{\phi}| + |\boldsymbol{\phi}_h|} \right\|_\infty^{(2-r)/r} \ . \tag{4.11}$$

Assume the hypotheses of Theorem 4.1 are satisfied. Also assume that for h sufficiently small, there is a constant $c_3 > 0$ *such that*

$$\inf_{(\boldsymbol{\tau}_h, q_h) \in T^{'}_{div,h} \times P_h} \ \sup_{(\boldsymbol{\varsigma}_h, \mathbf{v}_h, \eta_h) \in T_h \times U_h \times \mathbb{R}} \frac{[\mathbf{B}(\boldsymbol{\varsigma}_h), (\boldsymbol{\tau}_h, q_h)] + [\mathbf{C}(\boldsymbol{\tau}_h, q_h) \, , \, (\mathbf{v}_h, \eta_h)]}{\|(\boldsymbol{\varsigma}_h, \mathbf{v}_h, \eta_h)\|_{T \times U \times \mathbb{R}} \, \|(\boldsymbol{\tau}_h, q_h)\|_{T^{'}_{div} \times P}} \ \geq \ c_3 \, . \tag{4.12}$$

where $\|(\boldsymbol{\varsigma}_h, \mathbf{v}_h, \eta_h)\|_{T\times U\times\mathbb{R}} = \|\boldsymbol{\varsigma}_h\|_T + \|\mathbf{v}_h\|_U + \|\lambda_h\|_{\mathbb{R}}$. *Then*

$$
\begin{aligned}
\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T^2 &+ \int_\Omega |\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)|\, |\boldsymbol{\phi} - \boldsymbol{\phi}_h|\, d\Omega \\
&\le C \Big\{ \inf_{\boldsymbol{\varsigma}_h \in T_h} \left(\|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T^2 + \mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h)^r \, \|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T^r \right) + \inf_{\mathbf{v}_h \in U_h} \|\mathbf{u} - \mathbf{v}_h\|_U^2 \\
&\qquad + \inf_{\boldsymbol{\tau}_h \in T'_{div,\,h}} \|\boldsymbol{\psi} - \boldsymbol{\tau}_h\|_{T'_{div}}^2 + \inf_{q_h \in P_h} \|p - q_h\|_P^2 \Big\}, \qquad (4.13)
\end{aligned}
$$

$$
\begin{aligned}
\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{T'_{div}} + \|p - p_h\|_P &\le C \Big\{ \inf_{\boldsymbol{\tau}_h \in T'_{div,\,h}} \|\boldsymbol{\psi} - \boldsymbol{\tau}_h\|_{T'_{div}} + \inf_{q_h \in P_h} \|p - q_h\|_P \Big\} \\
&\qquad + \mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h) \left(\int_\Omega |\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)|\, |\boldsymbol{\phi} - \boldsymbol{\phi}_h|\, d\Omega \right)^{1/r'}, \qquad (4.14)
\end{aligned}
$$

and

$$
\|\mathbf{u} - \mathbf{u}_h\|_U + |\lambda - \lambda_h| \le C\, \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T + \inf_{\mathbf{v}_h \in U_h} \|\mathbf{u} - \mathbf{v}_h\|_U\,, \qquad (4.15)
$$

for some constant $C > 0$.

Proof: From the discrete formulation (4.2)-(4.4) we have that the approximation $(\boldsymbol{\phi}_h, \boldsymbol{\psi}_h, p_h, \mathbf{u}_h, \lambda_h)$ satisfies

$$
\begin{aligned}
[\mathbf{A}(\boldsymbol{\phi}_h)\,,\, \boldsymbol{\varsigma}_h] \,+\, [\mathbf{B}(\boldsymbol{\varsigma}_h)\,,\, (\boldsymbol{\psi}_h, p_h)] &= 0\ , \forall \boldsymbol{\varsigma}_h \in T_h\,, \\
[\mathbf{B}(\boldsymbol{\phi}_h), (\boldsymbol{\tau}_h, q_h)] + [\mathbf{C}(\boldsymbol{\tau}_h, q_h), (\mathbf{u}_h, \lambda_h)] &= -\int_\Gamma (\boldsymbol{\tau}_h \cdot \mathbf{n}) \cdot \mathbf{u}_\Gamma \, d\Gamma\ , \forall (\boldsymbol{\tau}_h, q_h) \in T'_{div,h} \times P_h\,, \\
[\mathbf{C}(\boldsymbol{\psi}_h, p_h)\,,\, (\mathbf{v}_h, \eta_h)] &= \int_\Omega \mathbf{v}_h \cdot \mathbf{f}\, d\Omega\,, \forall (\mathbf{v}_h, \eta_h) \in U_h \times \mathbb{R}\,.
\end{aligned}
$$

Define the following subspaces:

$$
\begin{aligned}
\tilde{Z}_{1h} \ :=\ \Big\{ (\boldsymbol{\tau}_h, q_h) \in T'_{div,h} \times P_h \,:\, [\mathbf{C}(\boldsymbol{\tau}_h, q_h)\,,\, (\mathbf{v}_h, \eta_h)] \,=\, \int_\Omega \mathbf{v}_h \cdot \mathbf{f}\, d\Omega\,, \\
\forall (\mathbf{v}_h, \eta_h) \in U_h \times \mathbb{R} \Big\}, \qquad (4.16)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{Z}_{2h} \ :=\ \Big\{ \boldsymbol{\varsigma}_h \in T_h \,:\, [\mathbf{B}(\boldsymbol{\varsigma}_h),\, (\boldsymbol{\tau}_h, q_h)] + [(\boldsymbol{\tau}_h, q_h),\, \mathbf{C}^*(\mathbf{u}_h, \lambda_h)] \,=\, -\int_\Gamma (\boldsymbol{\tau}_h \cdot \mathbf{n}) \cdot \mathbf{u}_\Gamma \, d\Gamma\,, \\
\forall (\boldsymbol{\tau}_h, q_h) \in \tilde{Z}_{1h} \Big\}. \qquad (4.17)
\end{aligned}
$$

From (2.2) and the definition of $\mathbf{A}$ (2.16), we have ,

$$
\begin{aligned}
\hat{C}_1 \frac{\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T^2}{\|\boldsymbol{\phi}\|_T^{2-r} + \|\boldsymbol{\phi}_h\|_T^{2-r}} + \hat{C}_1 \int_\Omega |\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)|\, |\boldsymbol{\phi} - \boldsymbol{\phi}_h| \; d\Omega \\
\le \int_\Omega (\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)) : (\boldsymbol{\phi} - \boldsymbol{\phi}_h)\, d\Omega\,, \qquad (4.18)
\end{aligned}
$$

and

$$\begin{aligned}\int_\Omega (\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)) : (\boldsymbol{\phi} - \boldsymbol{\phi}_h)\, d\Omega &= [\mathbf{A}(\boldsymbol{\phi}) - \mathbf{A}(\boldsymbol{\phi}_h),\, \boldsymbol{\phi} - \boldsymbol{\phi}_h] \\ &= [\mathbf{A}(\boldsymbol{\phi}) - \mathbf{A}(\boldsymbol{\phi}_h),\, \boldsymbol{\phi} - \boldsymbol{\varsigma}_h] \\ &\qquad + [\mathbf{A}(\boldsymbol{\phi}) - \mathbf{A}(\boldsymbol{\phi}_h),\, \boldsymbol{\varsigma}_h - \boldsymbol{\phi}_h] \,. \end{aligned} \tag{4.19}$$

We examine the first term on the RHS of (4.19). For $\mathcal{E}$ given by (4.11), note that $\mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h) \leq 1$. From (2.3) and Young's inequality, we have

$$\begin{aligned}[\mathbf{A}(\boldsymbol{\phi}) - \mathbf{A}(\boldsymbol{\phi}_h),\, \boldsymbol{\phi} - \boldsymbol{\varsigma}_h] &= \int_\Omega (\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)) : (\boldsymbol{\phi} - \boldsymbol{\varsigma}_h)\, d\Omega \\ &\leq \hat{C}_2\, \mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h) \left(\int_\Omega |\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)|\, |\boldsymbol{\phi} - \boldsymbol{\phi}_h|\, d\Omega \right)^{1/r'} \|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T \\ &\leq \frac{\hat{C}_2^{r'} \epsilon_1}{r'} \int_\Omega |\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)|\, |\boldsymbol{\phi} - \boldsymbol{\phi}_h|\, d\Omega + \frac{1}{r\epsilon_1}\, \mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h)^r\, \|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T^r \,. \end{aligned} \tag{4.20}$$

For the second term on the RHS of (4.19), if $\boldsymbol{\varsigma}_h \in \tilde{Z}_{2h}$, we have

$$\begin{aligned}[\mathbf{A}(\boldsymbol{\phi}) - \mathbf{A}(\boldsymbol{\phi}_h),\, \boldsymbol{\varsigma}_h - \boldsymbol{\phi}_h] &= [\mathbf{A}(\boldsymbol{\phi}),\, \boldsymbol{\varsigma}_h - \boldsymbol{\phi}_h] - [\mathbf{A}(\boldsymbol{\phi}_h),\, \boldsymbol{\varsigma}_h - \boldsymbol{\phi}_h] \\ &= -[\mathbf{B}(\boldsymbol{\varsigma}_h - \boldsymbol{\phi}_h),\, (\boldsymbol{\psi}, p)] + [\mathbf{B}(\boldsymbol{\varsigma}_h - \boldsymbol{\phi}_h),\, (\boldsymbol{\psi}_h, p_h)] \\ &= [\mathbf{B}(\boldsymbol{\phi}_h - \boldsymbol{\varsigma}_h),\, (\boldsymbol{\psi}, p)] \qquad (\text{as } \boldsymbol{\varsigma}_h, \boldsymbol{\phi}_h \in \tilde{Z}_{2h}) \\ &= [\mathbf{B}(\boldsymbol{\phi}_h - \boldsymbol{\varsigma}_h),\, (\boldsymbol{\psi}, p)] - [\mathbf{B}(\boldsymbol{\phi}_h - \boldsymbol{\varsigma}_h),\, (\boldsymbol{\tau}_h, q_h)] \qquad (\text{for } (\boldsymbol{\tau}_h, q_h) \in \tilde{Z}_{1h}) \\ &= [\mathbf{B}(\boldsymbol{\phi}_h - \boldsymbol{\varsigma}_h),\, (\boldsymbol{\psi} - \boldsymbol{\tau}_h, p - q_h)] \\ &= [\mathbf{B}(\boldsymbol{\phi}_h - \boldsymbol{\phi}),\, (\boldsymbol{\psi} - \boldsymbol{\tau}_h, p - q_h)] + [\mathbf{B}(\boldsymbol{\phi} - \boldsymbol{\varsigma}_h),\, (\boldsymbol{\psi} - \boldsymbol{\tau}_h, p - q_h)] \\ &= -\int_\Omega (\boldsymbol{\phi}_h - \boldsymbol{\phi}) : (\boldsymbol{\psi} - \boldsymbol{\tau}_h)\, d\Omega - \int_\Omega (p - q_h) tr(\boldsymbol{\phi}_h - \boldsymbol{\phi})\, d\Omega \\ &\quad - \int_\Omega (\boldsymbol{\phi} - \boldsymbol{\varsigma}_h) : (\boldsymbol{\psi} - \boldsymbol{\tau}_h)\, d\Omega - \int_\Omega (p - q_h) tr(\boldsymbol{\phi} - \boldsymbol{\varsigma}_h)\, d\Omega \\ &\leq \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T \|\boldsymbol{\psi} - \boldsymbol{\tau}_h\|_{T'} + \sqrt{n} \|p - q_h\|_P \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T \\ &\quad + \|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T \|\boldsymbol{\psi} - \boldsymbol{\tau}_h\|_{T'} + \sqrt{n} \|p - q_h\|_P \|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T \\ &\leq \frac{\epsilon_2 + \epsilon_3}{2} \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T^2 + \frac{\epsilon_4 + \epsilon_5}{2} \|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T^2 \\ &\quad + \left(\frac{1}{2\epsilon_2} + \frac{1}{2\epsilon_4} \right) \|\boldsymbol{\psi} - \boldsymbol{\tau}_h\|_{T'}^2 + \sqrt{n} \left(\frac{1}{2\epsilon_3} + \frac{1}{2\epsilon_5} \right) \|p - q_h\|_P^2 \,. \end{aligned} \tag{4.21}$$

Combining (4.18)-(4.21) with $\epsilon_4 = \epsilon_5 = 1$ we have

$$\begin{aligned} &\left(\frac{\hat{C}_1}{\|\boldsymbol{\phi}\|_T^{2-r} + \|\boldsymbol{\phi}_h\|_T^{2-r}} - \frac{\epsilon_2 + \epsilon_3}{2} \right) \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T^2 \\ &\qquad + \left(\hat{C}_1 - \frac{\hat{C}_2^{r'} \epsilon_1}{r'} \right) \int_\Omega |\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)|\, |\boldsymbol{\phi} - \boldsymbol{\phi}_h|\; d\Omega \\ &\qquad\qquad \leq \|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T^2 + \frac{1}{r\epsilon_1}\, \mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h)^r\, \|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T^r \\ &\qquad\qquad\qquad + \left(\frac{1}{2\epsilon_2} + \frac{1}{2} \right) \|\boldsymbol{\psi} - \boldsymbol{\tau}_h\|_{T'}^2 + \sqrt{n} \left(\frac{1}{2\epsilon_3} + \frac{1}{2} \right) \|p - q_h\|_P^2 \,. \end{aligned} \tag{4.22}$$

Choosing $\epsilon_1, \epsilon_2, \epsilon_3$ small enough to ensure

$$\left(\frac{\hat{C}_1}{\|\boldsymbol{\phi}\|_T^{2-r} + \|\boldsymbol{\phi}_h\|_T^{2-r}} - \frac{\epsilon_2 + \epsilon_3}{2} \right) > 0 \,,$$

and

$$\left(\hat{C}_1 - \frac{\hat{C}_2^{r'} \epsilon_1}{r'} \right) > 0 \,,$$

we have

$$\begin{aligned} \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T^2 + \int_\Omega |\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)| \, |\boldsymbol{\phi} - \boldsymbol{\phi}_h| \, d\Omega \leq C \Bigg\{ & \inf_{\boldsymbol{\varsigma}_h \in \tilde{Z}_{2h}} \left(\|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T^2 + \mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h)^r \, \|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T^r \right) \\ & + \inf_{(\boldsymbol{\tau}_h, q_h) \in \tilde{Z}_{1h}} \left(\|\boldsymbol{\psi} - \boldsymbol{\tau}_h\|_{T'}^2 + \|p - q_h\|_P^2 \right) \Bigg\} \,. \end{aligned} \tag{4.23}$$

The estimate (4.23) holds for $(\boldsymbol{\varsigma}_h, \boldsymbol{\tau}_h, q_h) \in \tilde{Z}_{2h} \times \tilde{Z}_{1h} \subseteq T_h \times T^{'}_{div,\,h} \times P_h$. In order to show that this estimate holds in all of $T_h \times T^{'}_{div,\,h} \times P_h$, we employ a lifting argument similar to that in [10]. Define the subspace

$$\begin{aligned} \tilde{W}_h := \Bigg\{ \boldsymbol{\varsigma}_h \in T_h : & [\mathbf{B}(\boldsymbol{\varsigma}_h), (\boldsymbol{\tau}_h, q_h)] + [(\boldsymbol{\tau}_h, q_h), \mathbf{C}^*(\mathbf{u}_h, \lambda_h)] \\ & = - \int_\Gamma (\boldsymbol{\tau}_h \cdot \mathbf{n}) \cdot \mathbf{u}_\Gamma \, d\Gamma \quad \forall\, (\boldsymbol{\tau}_h, q_h) \in T^{'}_{div,\,h} \times P_h \Bigg\} \,. \end{aligned}$$

We first show that (4.23) holds for all $\boldsymbol{\varsigma}_h \in T_h$. Then we show that (4.23) holds for all $(\boldsymbol{\tau}_h, q_h) \in T^{'}_{div,\,h} \times P_h$.

Note that $\boldsymbol{\varsigma}_h \in \tilde{W}_h \Rightarrow \boldsymbol{\varsigma}_h \in \tilde{Z}_{2h}$. Thus, for $\mathbf{v}_h \in U_h$,

$$\inf_{\boldsymbol{\varsigma}_h \in \tilde{Z}_{2h}} \|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T \leq \inf_{\boldsymbol{\varsigma}_h \in \tilde{W}_h} \|(\boldsymbol{\phi}, \mathbf{u}) - (\boldsymbol{\varsigma}_h, \mathbf{v}_h)\|_{T \times U} \,. \tag{4.24}$$

From the inf-sup condition (4.12), there exist operators $\Pi_T : T \to T_h$ and $\Pi_U : U \to U_h$ such that

$$[\mathbf{B}(\boldsymbol{\varsigma} - \Pi_T \boldsymbol{\varsigma}), (\boldsymbol{\tau}_h, q_h)] + [\mathbf{C}(\boldsymbol{\tau}_h, q_h), (\mathbf{v} - \Pi_U \mathbf{v}, \lambda_h)] = 0 \,, \quad \forall\, (\boldsymbol{\tau}_h, q_h) \in T^{'}_{div,\,h} \times P_h \,, \tag{4.25}$$

and

$$\|(\Pi_T \boldsymbol{\varsigma}, \Pi_U \mathbf{v})\|_{T \times U} \leq \tilde{C} \|(\boldsymbol{\varsigma}, \mathbf{v})\|_{T \times U} \,, \quad \forall\, (\boldsymbol{\varsigma}, \mathbf{v}) \in T \times U \,. \tag{4.26}$$

Now, let $(\boldsymbol{\varsigma}_h, \mathbf{v}_h) \in T_h \times U_h$ and set $\tilde{\boldsymbol{\phi}} := \boldsymbol{\varsigma}_h - \Pi_T(\boldsymbol{\varsigma}_h - \boldsymbol{\phi})$ and $\tilde{\mathbf{u}} := \mathbf{v}_h - \Pi_U(\mathbf{v}_h - \mathbf{u})$. Note that $(\tilde{\boldsymbol{\phi}}, \tilde{\mathbf{u}}) \in T_h \times U_h$. Then for all $(\boldsymbol{\tau}_h, q_h) \in T^{'}_{div,\,h} \times P_h$,

$$\begin{aligned} [\mathbf{B}(\tilde{\boldsymbol{\phi}}), (\boldsymbol{\tau}_h, q_h)] + [\mathbf{C}(\boldsymbol{\tau}_h, q_h), (\tilde{\mathbf{u}}, \lambda_h)] &= [\mathbf{B}(\boldsymbol{\phi}), (\boldsymbol{\tau}_h, q_h)] + [\mathbf{C}(\boldsymbol{\tau}_h, q_h), (\mathbf{u}, \lambda_h)] \\ &= - \int_\Gamma (\boldsymbol{\tau}_h \cdot \mathbf{n}) \cdot \mathbf{u}_\Gamma \, d\Gamma, . \end{aligned} \tag{4.27}$$

Thus $\tilde{\boldsymbol{\phi}} \in \tilde{W}_h$. Now, using (4.26), we have

$$\begin{aligned} \|(\tilde{\boldsymbol{\phi}}, \tilde{\mathbf{u}}) - (\boldsymbol{\varsigma}_h, \mathbf{v}_h)\|_{T \times U} &= \|(\Pi_T(\boldsymbol{\phi} - \boldsymbol{\varsigma}_h), \Pi_U(\mathbf{u} - \mathbf{v}_h))\|_{T \times U} \\ &\leq \tilde{C} \|(\boldsymbol{\phi} - \boldsymbol{\varsigma}_h, \mathbf{u} - \mathbf{v}_h)\|_{T \times U} \,. \end{aligned} \tag{4.28}$$

Thus we have

$$\begin{aligned}
\inf_{\boldsymbol{\varsigma}_h \in \tilde{Z}_{2h}} \|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T &\leq \inf_{(\boldsymbol{\varsigma}_h, \mathbf{v}_h) \in \tilde{W}_h \times U_h} \|(\boldsymbol{\phi}, \mathbf{u}) - (\boldsymbol{\varsigma}_h, \mathbf{v}_h)\|_{T\times U} \\
&\leq \inf_{(\boldsymbol{\varsigma}_h, \mathbf{v}_h) \in T_h \times U_h} \|(\boldsymbol{\phi}, \mathbf{u}) - (\tilde{\boldsymbol{\phi}}, \tilde{\mathbf{u}})\|_{T\times U} \\
&\leq \inf_{(\boldsymbol{\varsigma}_h, \mathbf{v}_h) \in T_h \times U_h} \Big(\|(\boldsymbol{\phi}, \mathbf{u}) - (\boldsymbol{\varsigma}_h, \mathbf{v}_h)\|_{T\times U} + \|(\tilde{\boldsymbol{\phi}}, \tilde{\mathbf{u}}) - (\boldsymbol{\varsigma}_h, \mathbf{v}_h)\|_{T\times U} \Big) \\
&\leq (1 + \tilde{C}) \inf_{(\boldsymbol{\varsigma}_h, \mathbf{v}_h) \in T_h \times U_h} \|(\boldsymbol{\phi}, \mathbf{u}) - (\boldsymbol{\varsigma}_h, \mathbf{v}_h)\|_{T\times U} \,, \qquad (4.29)
\end{aligned}$$

which lifts the best approximation of $\boldsymbol{\phi}$ from $\tilde{Z}_{2h}$ to T_h.
Now, we must also show

$$\inf_{(\boldsymbol{\tau}_h, q_h) \in \tilde{Z}_{1h}} \|(\boldsymbol{\psi}, p) - (\boldsymbol{\tau}_h, q_h)\|_{T'_{div} \times P} \leq C \inf_{(\boldsymbol{\tau}_h, q_h) \in T'_{div,\,h} \times P_h} \|(\boldsymbol{\psi}, p) - (\boldsymbol{\tau}_h, q_h)\|_{T'_{div} \times P} \,. \qquad (4.30)$$

From (4.9), we have the existence of operators $\Pi_{T'} : T'_{div} \to T'_{div,\,h}$ and $\Pi_P : P \to P_h$ such that

$$[\mathbf{C}(\boldsymbol{\tau} - \Pi_{T'}\boldsymbol{\tau}, q - \Pi_P q), (\mathbf{v}_h, \eta_h)] = 0 \,, \quad \forall\, (\mathbf{v}_h, \eta_h) \in U_h \times \mathbb{R} \,, \qquad (4.31)$$

and

$$\|(\Pi_{T'}\boldsymbol{\tau}, \Pi_P q)\|_{T'_{div} \times P} \leq \tilde{C} \|(\boldsymbol{\tau}, q)\|_{T'_{div} \times P} \,. \qquad (4.32)$$

Now for $(\boldsymbol{\tau}_h, q_h) \in T'_{div,\,h} \times P_h$, let $\tilde{\boldsymbol{\psi}} := \boldsymbol{\tau}_h - \Pi_{T'}(\boldsymbol{\tau}_h - \boldsymbol{\psi})$ and $\tilde{p} := q_h - \Pi_P(q_h - p)$. Note that $(\tilde{\boldsymbol{\psi}}, \tilde{p}) \in T'_{div,\,h} \times P_h$. Then for all $(\mathbf{v}_h, \eta_h) \in U_h \times \mathbb{R}$ we have

$$[\mathbf{C}(\tilde{\boldsymbol{\psi}}, \tilde{p}), (\mathbf{v}_h, \eta_h)] = [\mathbf{C}(\boldsymbol{\psi}, p), (\mathbf{v}_h, \eta_h)] = \int_\Omega \mathbf{v}_h \cdot \mathbf{f} \,, d\Omega \,. \qquad (4.33)$$

So $(\tilde{\boldsymbol{\psi}}, \tilde{p}) \in \tilde{Z}_{1h}$. Now, using (4.32) we have

$$\begin{aligned}
\|(\tilde{\boldsymbol{\psi}}, \tilde{p}) - (\boldsymbol{\tau}_h, q_h)\|_{T'_{div} \times P} &= \|(\Pi_{T'}(\boldsymbol{\psi} - \boldsymbol{\tau}_h), \Pi_P(p - q_h)\|_{T'_{div} \times P} \\
&\leq \tilde{C} \|(\boldsymbol{\psi} - \boldsymbol{\tau}_h, p - q_h)\|_{T'_{div} \times P} \,. \qquad (4.34)
\end{aligned}$$

Thus

$$\begin{aligned}
\inf_{(\boldsymbol{\tau}_h, q_h) \in \tilde{Z}_{1h}} \|(\boldsymbol{\psi}, p) - (\boldsymbol{\tau}_h, q_h)\|_{T'_{div} \times P} &\leq \inf_{(\boldsymbol{\tau}_h, q_h) \in T'_{div,\,h} \times P_h} \|(\boldsymbol{\psi}, p) - (\tilde{\boldsymbol{\psi}}, \tilde{p})\|_{T'_{div} \times P} \\
&\leq \inf_{(\boldsymbol{\tau}_h, q_h) \in T'_{div,\,h} \times P_h} \Big(\|(\boldsymbol{\psi}, p) - (\boldsymbol{\tau}_h, q_h)\|_{T'_{div} \times P} + \|(\tilde{\boldsymbol{\psi}}, \tilde{p}) - (\boldsymbol{\tau}_h, q_h)\|_{T'_{div} \times P} \Big) \\
&\leq (1 + \tilde{C}) \inf_{(\boldsymbol{\tau}_h, q_h) \in T'_{div,\,h} \times P_h} \|(\boldsymbol{\psi}, p) - (\boldsymbol{\tau}_h, q_h)\|_{T'_{div} \times P} \,. \qquad (4.35)
\end{aligned}$$

This lifts the best approximation of $(\boldsymbol{\psi}, p)$ from $\tilde{Z}_{1h}$ to $T'_{div} \times P$. Thus, from (4.23), (4.29), and (4.35) we have

$$\begin{aligned}
&\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T^2 + \int_\Omega |\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)| \, |\boldsymbol{\phi} - \boldsymbol{\phi}_h| \, d\Omega \\
&\qquad \leq C \bigg\{ \inf_{\boldsymbol{\varsigma}_h \in T_h} \left(\|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T^2 + \mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h)^r \, \|\boldsymbol{\phi} - \boldsymbol{\varsigma}_h\|_T^r \right) + \inf_{\mathbf{v}_h \in U_h} \|\mathbf{u} - \mathbf{v}_h\|_U^2 \\
&\qquad\qquad + \inf_{\boldsymbol{\tau}_h \in T'_{div,\,h}} \|\boldsymbol{\psi} - \boldsymbol{\tau}_h\|_{T'_{div}}^2 + \inf_{q_h \in P_h} \|p - q_h\|_P^2 \bigg\} \,. \qquad (4.36)
\end{aligned}$$

To obtain the a priori estimate for $\boldsymbol{\phi}$ and p, we begin with the discrete inf-sup condition satisfied by $\mathbf{B}$. We have

$$\begin{aligned}
c_1\bigg(\|\boldsymbol{\psi}_h \; &- \; \boldsymbol{\tau}_h\|_{T'_{div}} + \|p_h - q_h\|_P \bigg) \le \sup_{\boldsymbol{\varsigma}_h \in T_h} \frac{[\mathbf{B}(\boldsymbol{\varsigma}_h), (\boldsymbol{\psi}_h - \boldsymbol{\tau}_h, p_h - q_h)]}{\|\boldsymbol{\varsigma}_h\|_T} \\
&= \sup_{\boldsymbol{\varsigma}_h \in T_h} \frac{-\int_\Omega \boldsymbol{\varsigma}_h : (\boldsymbol{\psi}_h - \boldsymbol{\tau}_h)\, d\Omega - \int_\Omega (p_h - q_h) tr(\boldsymbol{\varsigma}_h)\, d\Omega}{\|\boldsymbol{\varsigma}_h\|_T} \\
&= \sup_{\boldsymbol{\varsigma}_h \in T_h} \bigg(\frac{-\int_\Omega \boldsymbol{\varsigma}_h : (\boldsymbol{\psi}_h - \boldsymbol{\psi})\, d\Omega - \int_\Omega \boldsymbol{\varsigma}_h : (\boldsymbol{\psi} - \boldsymbol{\tau}_h)\, d\Omega}{\|\boldsymbol{\varsigma}_h\|_T} \\
&\qquad\qquad + \frac{-\int_\Omega (p_h - p) tr(\boldsymbol{\varsigma}_h)\, d\Omega - \int_\Omega (p - q_h) tr(\boldsymbol{\varsigma}_h)\, d\Omega}{\|\boldsymbol{\varsigma}_h\|_T} \bigg) \\
&\le \sup_{\boldsymbol{\varsigma}_h \in T_h} \frac{-\int_\Omega \boldsymbol{\varsigma}_h : (\boldsymbol{\psi}_h - \boldsymbol{\psi})\, d\Omega - \int_\Omega (p_h - p) tr(\boldsymbol{\varsigma}_h)\, d\Omega}{\|\boldsymbol{\varsigma}_h\|_T} \\
&\qquad + \|\boldsymbol{\psi} - \boldsymbol{\tau}_h\|_{T'} + \sqrt{n}\|p - q_h\|_P \\
&= \sup_{\boldsymbol{\varsigma}_h \in T_h} \frac{[\mathbf{B}(\boldsymbol{\varsigma}_h), (\boldsymbol{\psi}_h - \boldsymbol{\psi}, p_h - p)]}{\|\boldsymbol{\varsigma}_h\|_T} + \|\boldsymbol{\psi} - \boldsymbol{\tau}_h\|_{T'} + \sqrt{n}\|p - q_h\|_P \\
&= \sup_{\boldsymbol{\varsigma}_h \in T_h} \frac{[\mathbf{B}(\boldsymbol{\varsigma}_h), (\boldsymbol{\psi}_h, p_h)] - [\mathbf{B}(\boldsymbol{\varsigma}_h), (\boldsymbol{\psi}, p)]}{\|\boldsymbol{\varsigma}_h\|_T} + \|\boldsymbol{\psi} - \boldsymbol{\tau}_h\|_{T'} + \sqrt{n}\|p - q_h\|_P \\
&= \sup_{\boldsymbol{\varsigma}_h \in T_h} \frac{[\mathbf{A}(\boldsymbol{\phi}_h), \boldsymbol{\varsigma}_h] - [\mathbf{A}(\boldsymbol{\phi}), \boldsymbol{\varsigma}_h]}{\|\boldsymbol{\varsigma}_h\|_T} + \|\boldsymbol{\psi} - \boldsymbol{\tau}_h\|_{T'} + \sqrt{n}\|p - q_h\|_P \\
&= \sup_{\boldsymbol{\varsigma}_h \in T_h} \frac{\int_\Omega (\mathbf{g}(\boldsymbol{\phi}_h) - \mathbf{g}(\boldsymbol{\phi})) : \boldsymbol{\varsigma}_h\, d\Omega}{\|\boldsymbol{\varsigma}_h\|_T} + \|\boldsymbol{\psi} - \boldsymbol{\tau}_h\|_{T'} + \sqrt{n}\|p - q_h\|_P \,. \qquad (4.37)
\end{aligned}$$

The first term on the RHS of (4.37) can be handled using (2.3) and the definition of $\mathcal{E}$:

$$\begin{aligned}
&\sup_{\boldsymbol{\varsigma}_h \in T_h} \frac{\int_\Omega (\mathbf{g}(\boldsymbol{\phi}_h) - \mathbf{g}(\boldsymbol{\phi})) : \boldsymbol{\varsigma}_h\, d\Omega}{\|\boldsymbol{\varsigma}_h\|_T} \\
&\qquad \le \sup_{\boldsymbol{\varsigma}_h \in T_h} \hat{C}_2\, \mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h) \frac{\left(\int_\Omega |\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)|\, |\boldsymbol{\phi} - \boldsymbol{\phi}_h|\, d\Omega \right)^{1/r'} \|\boldsymbol{\varsigma}_h\|_T}{\|\boldsymbol{\varsigma}_h\|_T} \\
&\qquad\qquad \le \hat{C}_2\, \mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h) \left(\int_\Omega |\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)|\, |\boldsymbol{\phi} - \boldsymbol{\phi}_h|\, d\Omega \right)^{1/r'} . \qquad (4.38)
\end{aligned}$$

Combining (4.37), (4.38), and an application of the triangle inequality imply

$$\begin{aligned}
&\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{T'_{div}} + \|p - p_h\|_P \\
&\qquad \le C \bigg\{ \inf_{(\boldsymbol{\tau}_h, q_h) \in \tilde{Z}_{1h}} \Big(\|\boldsymbol{\psi} - \boldsymbol{\tau}_h\|_{T'} + \|p - q_h\|_P \Big) \bigg\} \\
&\qquad\qquad + \hat{C}_2\, \mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h) \left(\int_\Omega |\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)|\, |\boldsymbol{\phi} - \boldsymbol{\phi}_h|\, d\Omega \right)^{1/r'} . \qquad (4.39)
\end{aligned}$$

Now the previously described argument to lift the best approximations of $(\boldsymbol{\tau}_h, q_h)$ from $\tilde{Z}_{1h}$ to $T'_{div,\,h} \times P_h$ can be applied here. Thus we have, from (4.39)

$$\begin{aligned}
&\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{T'_{div}} + \|p - p_h\|_P \\
&\qquad \le C\left\{ \inf_{\boldsymbol{\tau}_h \in T'_{div,\,h}} \|\boldsymbol{\psi} - \boldsymbol{\tau}_h\|_{T'_{div}} + \inf_{q_h \in P_h} \|p - q_h\|_P \right\} \\
&\qquad\qquad + \hat{C}_2\, \mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h) \left(\int_\Omega |\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)|\, |\boldsymbol{\phi} - \boldsymbol{\phi}_h|\, d\Omega \right)^{1/r'} . \qquad (4.40)
\end{aligned}$$

From the discrete inf-sup condition for $\mathbf{C}$ we have,

$$\begin{aligned}
c_2(\|\mathbf{u}_h - \mathbf{v}_h\|_U + |\lambda_h - \lambda|) &\le \sup_{(\boldsymbol{\tau}_h, q_h) \in T'_{div,\,h} \times P_h} \frac{[\mathbf{C}(\boldsymbol{\tau}_h, q_h), (\mathbf{u}_h - \mathbf{v}_h, \lambda_h - \lambda)]}{\|(\boldsymbol{\tau}_h, q_h)\|_{T'_{div,\,h} \times P_h}} \\
&\le \sup_{(\boldsymbol{\tau}_h, q_h) \in T'_{div,\,h} \times P_h} \frac{-\int_\Omega (\mathbf{u}_h - \mathbf{u})\, div(\boldsymbol{\tau}_h)\, d\Omega - (\lambda_h - \lambda) \int_\Omega tr(\boldsymbol{\tau}_h)\, d\Omega}{\|(\boldsymbol{\tau}_h, q_h)\|_{T'_{div,\,h} \times P_h}} \\
&\qquad + \frac{\|\mathbf{u} - \mathbf{v}_h\|_U \|\boldsymbol{\tau}_h\|_{T'_{div}}}{\|\boldsymbol{\tau}_h\|_{T'_{div}} + \|q_h\|_P} . \qquad (4.41)
\end{aligned}$$

Now the first term on the RHS of (4.41) gives

$$\begin{aligned}
&\sup_{(\boldsymbol{\tau}_h, q_h) \in T'_{div,\,h} \times P_h} \frac{-\int_\Omega (\mathbf{u}_h - \mathbf{u})\, div(\boldsymbol{\tau}_h)\, d\Omega - (\lambda_h - \lambda) \int_\Omega tr(\boldsymbol{\tau}_h)\, d\Omega}{\|(\boldsymbol{\tau}_h, q_h)\|_{T'_{div,\,h} \times P_h}} \\
&\qquad = \sup_{(\boldsymbol{\tau}_h, q_h) \in T'_{div,\,h} \times P_h} \frac{[\mathbf{C}(\boldsymbol{\tau}_h, q_h), (\mathbf{u}_h - \mathbf{u}, \lambda_h - \lambda)]}{\|(\boldsymbol{\tau}_h, q_h)\|_{T'_{div,\,h} \times P_h}} \\
&\qquad = \sup_{(\boldsymbol{\tau}_h, q_h) \in T'_{div,\,h} \times P_h} \frac{[\mathbf{B}(\boldsymbol{\phi}_h - \boldsymbol{\phi}), (\boldsymbol{\tau}_h, q_h)]}{\|(\boldsymbol{\tau}_h, q_h)\|_{T'_{div,\,h} \times P_h}} \\
&\qquad \le \frac{\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T \|\boldsymbol{\tau}_h\|_{T'} + \sqrt{n} \|q_h\|_P \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T}{\|\boldsymbol{\tau}_h\|_{T'_{div}} + \|q_h\|_P} \\
&\qquad \le \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T + \sqrt{n} \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T \le C \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T . \qquad (4.42)
\end{aligned}$$

Thus, from (4.36), (4.41), (4.42), and the triangle inequality we have

$$\|\mathbf{u} - \mathbf{u}_h\|_U + |\lambda - \lambda_h| \le C\, \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T + \inf_{\mathbf{v}_h \in U_h} \|\mathbf{u} - \mathbf{v}_h\|_U . \qquad (4.43)$$

Thus the estimates (4.13)–(4.15) are proven. ■

Remark 4.1 *As previously noted,* $\mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h) \le 1$. *In addition, if* $1/(|\boldsymbol{\phi}| + |\boldsymbol{\phi}_h|) \le C$ *for some constant* $C > 0$, *then*

$$\mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h) \le min \left\{ 1, C\, \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_\infty^{(2-r)/r} \right\} .$$

Furthermore, if $\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_\infty \sim \|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_T$*, the estimates (4.13)–(4.15) may be written as*

$$\begin{aligned}
&\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_T + \|\boldsymbol{\psi}-\boldsymbol{\psi}_h\|_{T'_{div}} + \|p-p_h\|_P + \|\mathbf{u}-\mathbf{u}_h\|_U + |\lambda-\lambda_h| \\
&\qquad \le C\Bigg\{ \inf_{\boldsymbol{\varsigma}_h\in T_h} \|\boldsymbol{\phi}-\boldsymbol{\varsigma}_h\|_T + \inf_{\mathbf{v}_h\in U_h} \|\mathbf{u}-\mathbf{v}_h\|_U \\
&\qquad\qquad + \inf_{\boldsymbol{\tau}_h\in T'_{div,\,h}} \|\boldsymbol{\psi}-\boldsymbol{\tau}_h\|_{T'_{div}} + \inf_{q_h\in P_h} \|p-q_h\|_P \Bigg\}\,. \qquad (4.44)
\end{aligned}$$

4.2 Discrete Approximation Spaces

Let $n=2$. Let $K\in\mathcal{T}_h$ and let $\mathbb{P}_k(K)$ be the set of all polynomials in the variables x_1, x_2 of degree less than or equal to k defined on the triangle K. Let $\mathbb{RT}_k(K)$ be the 2-vector of Raviart-Thomas elements [24, 26] on K defined by

$$\mathbb{RT}_k(K) = (\mathbb{P}_k(K))^2 + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mathbb{P}_k(K)\,.$$

Let the following discrete spaces be defined as:

$$\begin{aligned}
T_h &:= \left\{\boldsymbol{\phi}\in T : \boldsymbol{\phi}|_K \in (\mathbb{P}_0(K))^{2\times 2}\,, \quad \forall K\in\mathcal{T}_h\right\}\,, \\
T'_{div,\,h} &:= \Bigg\{\boldsymbol{\psi}\in T'_{div} : \boldsymbol{\psi} = (\boldsymbol{\psi}_1 \quad \boldsymbol{\psi}_2)^{\mathrm{T}}|_K \in (\mathbb{RT}_0(K))^2\,, \\
&\qquad (\psi_{i1} \quad \psi_{i2})^{\mathrm{T}}|_K \in \mathbb{RT}_0(K)\,, \quad \forall i\in\{1,2\}, \quad \forall K\in\mathcal{T}_h \Bigg\}\,, \\
P_h &:= \left\{p\in P : p|_K \in \mathbb{P}_0(K)\,, \quad \forall K\in\mathcal{T}_h\right\}\,, \\
U_h &:= \left\{\mathbf{u}\in U : \mathbf{u}|_K \in (\mathbb{P}_0(K))^2\,, \quad \forall K\in\mathcal{T}_h\right\}\,.
\end{aligned}$$

Let $s>1$ and let $\mathcal{I}_h^0 : \left(W^{1,s}(\Omega)\right)^{2\times 2} \longrightarrow T'_{div,\,h}$ be the lowest-order Raviart-Thomas interpolation operator [24, 7, 9] defined by, for row $j=1,2$ of $\boldsymbol{\tau}\in T'_{div}$,

$$\int_{e_i} (\boldsymbol{\tau}_j - \mathcal{I}_h^0\boldsymbol{\tau}_j)\cdot\mathbf{n}_{e_i}\,ds = 0, \quad \forall e_i\in\partial K, \quad i=1,2,3, \quad \forall K\in\mathcal{T}_h\,, \qquad (4.45)$$

where $\mathbf{n}_{e_i}$ denotes the outer unit normal vector to edge e_i of K. For every $K\in\mathcal{T}_h$, $\boldsymbol{\tau}|_K \in \left(W^{1,s}(K)\right)^{2\times 2}$, and thus $\boldsymbol{\tau}|_{\partial K} \in \left(W^{1-1/s,s}(\partial K)\right)^{2\times 2} \subset (L^s(\partial K))^{2\times 2} \subset \left(L^1(\partial K)\right)^{2\times 2}$. Thus we have, if $div\,\boldsymbol{\tau}\in\left(W^{m,r'}(\Omega)\right)^2$ with $0\le m\le 1$,

$$\|\boldsymbol{\tau}-\mathcal{I}_h^0\boldsymbol{\tau}\|_{0,r',\Omega} \le Ch^m|\boldsymbol{\tau}|_{m,r',\Omega}\,, \qquad (4.46)$$

$$\|div\,(\boldsymbol{\tau}-\mathcal{I}_h^0\boldsymbol{\tau})\|_{0,r',\Omega} \le Ch^m|div\,\boldsymbol{\tau}|_{m,r',\Omega}\,, \qquad (4.47)$$

and, for $\mathbf{v}\in U$,

$$\int_\Omega \mathbf{v}\cdot div(\boldsymbol{\tau}-\mathcal{I}_h^0\boldsymbol{\tau})\,d\Omega = 0, \quad \forall\boldsymbol{\tau}\in T'_{div}\,. \qquad (4.48)$$

Lemma 4.1 *For the choices of T_h, $T^{'}_{div,\,h}$, P_h, and U_h above, there exists a positive constant c_1 such that*

$$\inf_{(\boldsymbol{\tau}_h, q_h)\in Z_{1h}} \ \sup_{\boldsymbol{\phi}_h \in T_h} \ \frac{[\mathbf{B}(\boldsymbol{\phi}_h)\,,\,(\boldsymbol{\tau}_h, q_h)]}{\|\boldsymbol{\phi}_h\|_T \,\|(\boldsymbol{\tau}_h, q_h)\|_{T'_{div}\times P}} \ \geq \ c_1 \ . \tag{4.49}$$

Proof: Note that for $\boldsymbol{\tau}_h \in T^{'}_{div,\,h}$, $div\,\boldsymbol{\tau}_h = \mathbf{0}$ implies $\boldsymbol{\tau}_h \in (\mathbb{P}_0(K))^{2\times 2}$ for all $K \in \mathcal{T}_h$. We consider two cases.
<u>Case 1:</u> $\|q_h\|_P \leq \|\boldsymbol{\tau}_h\|_{T'_{div}}$.
Let $\boldsymbol{\tau}^0_h = \boldsymbol{\tau}_h - \frac{1}{n}tr(\boldsymbol{\tau}_h)\mathbf{I}$, and $\boldsymbol{\phi}_h = -|\boldsymbol{\tau}^0_h|^{r'/r-1}\,\boldsymbol{\tau}^0_h/\|\boldsymbol{\tau}^0_h\|^{r'-1}_{T'}$. Note that $\boldsymbol{\phi}_h \in T_h$ as $(\boldsymbol{\tau}_h, q_h) \in Z_{1h}$ implies $\boldsymbol{\tau}^0_h \in (\mathbb{P}_0(K))^{2\times 2}$. Proceeding as in Case 1 of the proof of Lemma 3.3, we obtain (4.49).
<u>Case 2:</u> $\|q_h\|_P \geq \|\boldsymbol{\tau}_h\|_{T'_{div}}$.
Let

$$\boldsymbol{\phi}_h \ = \ \frac{-|q_h\mathbf{I} \,+\, \boldsymbol{\tau}_h|^{r'/r-1}\,(q_h\mathbf{I} \,+\, \boldsymbol{\tau}_h)}{\|q_h\mathbf{I} \,+\, \boldsymbol{\tau}_h\|^{r'-1}_{T'}}\,.$$

Again $\boldsymbol{\phi}_h \in T_h$ as $q_h \in \mathbb{P}_0(K)$ and $\boldsymbol{\tau}_h \in (\mathbb{P}_0(K))^{2\times 2}$ for all $K \in \mathcal{T}_h$. Proceeding as in Case 2 of the proof of Lemma 3.3, we obtain (4.49). ■

Lemma 4.2 *For the choices of T_h, $T^{'}_{div,\,h}$, P_h, and U_h above, there exists a positive constant c_2 such that*

$$\inf_{(\mathbf{u}_h,\lambda_h)\in U_h\times\mathbb{R}} \ \sup_{(\boldsymbol{\tau}_h,q_h)\in T'_{div,h}\times P_h} \ \frac{[\mathbf{C}(\boldsymbol{\tau}_h, q_h)\,,\,(\mathbf{u}_h, \lambda_h)]}{\|(\boldsymbol{\tau}_h, q_h)\|_{T'_{div}\times P}\,\|(\mathbf{u}_h, \lambda_h)\|_{U\times\mathbb{R}}} \ \geq \ c_2\,. \tag{4.50}$$

Proof: As in the approach to the proof of Lemma 3.6, we consider two cases:
<u>Case 1:</u> $|\lambda_h| \geq \|\mathbf{u}_h\|_U$.
The choice $(\boldsymbol{\tau}_h, q_h) = (\lambda_h\mathbf{I}, 0) \in T^{'}_{div,\,h} \times P_h$ shows the result as in Case 1 of the proof of Lemma 3.6.
<u>Case 2:</u> $|\lambda_h| \leq \|\mathbf{u}_h\|_U$.
Note that Lemma 3.4 applies to the subspace $T^{'}_{div,\,h} \subset T^{'}_{div}$, thus we have

$$\begin{aligned}
\sup_{(\boldsymbol{\tau}_h,q_h)\in T'_{div,h}\times P_h} \frac{[\mathbf{C}(\boldsymbol{\tau}_h, q_h)\,,\,(\mathbf{u}_h, \lambda_h)]}{\|(\boldsymbol{\tau}_h, q_h)\|_{T'_{div}\times P}} \ &\geq \ \sup_{\boldsymbol{\tau}_0\in\ {}_0T'_{div,h}} \frac{[\mathbf{C}(\boldsymbol{\tau}_0, 0)\,,\,(\mathbf{u}_h, \lambda_h)]}{\|\boldsymbol{\tau}_0\|_{T'_{div}}} \\
&= \ \sup_{\boldsymbol{\tau}_0\in\ {}_0T'_{div,h}} \frac{-\int_\Omega \mathbf{u}_h\cdot div\,\boldsymbol{\tau}_0\,d\Omega}{\|\boldsymbol{\tau}_0\|_{T'_{div,h}}} \\
&\geq \ C \sup_{\boldsymbol{\tau}\in T'_{div,h}} \frac{-\int_\Omega \mathbf{u}_h\cdot div\,\boldsymbol{\tau}_h\,d\Omega}{\|\boldsymbol{\tau}_h\|_{T'_{div,h}}}\,.
\end{aligned} \tag{4.51}$$

Now we proceed in a manner similar to that of Proposition 5 of [22] (as well as Proposition 3.1 of [12]). Let $\mathbf{w}$ be the solution of the Laplacian problem

$$\begin{aligned}
-\Delta\mathbf{w} \ &= \ |\mathbf{u}_h|^{r-2}\mathbf{u}_h\,, \quad && \text{in } \Omega\,, \\
\mathbf{w} \ &= \ \mathbf{0}\,, && \text{on } \Gamma\,.
\end{aligned}$$

Note that $|\mathbf{u}_h|^{r-2}\mathbf{u}_h \in (W^{0,r'}(\Omega))^2$. Hence, from [16], this problem has a unique solution $\mathbf{w} \in (W_0^{2,r}(\Omega))^2$, and there exists a constant $C > 0$ such that

$$\begin{aligned}
\|\mathbf{w}\|_{2,r',\Omega} &\leq C \left\| |\mathbf{u}_h|^{r-2}\mathbf{u}_h \right\|_{0,r',\Omega} \\
&= C \left(\int_\Omega \left| |\mathbf{u}_h|^{r-2}\mathbf{u}_h \right|^{r'} \, d\Omega \right)^{1/r'} \\
&= C \left(\int_\Omega |\mathbf{u}_h|^{r'(r-1)} \, d\Omega \right)^{(r-1)/r} \\
&= C \left(\int_\Omega |\mathbf{u}_h|^{r} \, d\Omega \right)^{(1/r)(r-1)} = C \, \|\mathbf{u}_h\|^{r-1}_{0,r,\Omega} \,.
\end{aligned} \tag{4.52}$$

Now let $\boldsymbol{\tau}^* = \nabla \mathbf{w}$. Thus from (4.52) we have

$$\|\boldsymbol{\tau}^*\|_{1,r',\Omega} = \|\mathbf{w}\|_{2,r',\Omega} \leq C \, \|\mathbf{u}_h\|^{r-1}_{0,r,\Omega} \,, \tag{4.53}$$

and $\nabla \cdot \boldsymbol{\tau}^* = \Delta \mathbf{w} = -|\mathbf{u}_h|^{r-2}\mathbf{u}_h$. Thus $\boldsymbol{\tau}^* \in T'_{div}$ and we have

$$\|\boldsymbol{\tau}^*\|_{T'_{div}} \leq C \, \|\mathbf{u}_h\|^{r-1}_{0,r,\Omega} \,. \tag{4.54}$$

Let $\boldsymbol{\tau}_h = \mathcal{I}_h^0 \boldsymbol{\tau}^*$. Then we have, from (4.48), that

$$\begin{aligned}
-\int_\Omega \mathbf{u}_h \cdot div\, \boldsymbol{\tau}_h \, d\Omega &= -\int_\Omega \mathbf{u}_h \cdot div\, \boldsymbol{\tau}^* \, d\Omega \\
&= \int_\Omega |\mathbf{u}_h|^{r-2}\mathbf{u}_h \cdot \mathbf{u}_h \, d\Omega \\
&= \|\mathbf{u}_h\|^r_U \,.
\end{aligned} \tag{4.55}$$

The estimate (4.47) and the reverse triangle inequality gives, for $m = 0$,

$$\|\nabla \cdot \boldsymbol{\tau}_h\|_{0,r',\Omega} \leq C \|\nabla \cdot \boldsymbol{\tau}^*\|_{0,r',\Omega} \,. \tag{4.56}$$

Then, from (4.46), (4.54), (4.56), and the triangle inequality, we have

$$\begin{aligned}
\|\boldsymbol{\tau}_h\|_{T'_{div}} &\leq \|\boldsymbol{\tau}^h\|_{0,r',\Omega} + \|\nabla \cdot \boldsymbol{\tau}_h\|_{0,r',\Omega} \\
&\leq C \left(\|\boldsymbol{\tau}^*\|_{0,r',\Omega} + \|\boldsymbol{\tau}^* - \boldsymbol{\tau}_h\|_{0,r',\Omega} + \|\nabla \cdot \boldsymbol{\tau}^*\|_{0,r',\Omega} \right) \\
&\leq C \left(\|\boldsymbol{\tau}^*\|_{T'_{div}} + h \, \|\boldsymbol{\tau}^*\|_{1,r',\Omega} + \|\mathbf{u}\|^{r-1}_{0,r,\Omega} \right) \\
&\leq C \left(\|\mathbf{u}_h\|^{r-1}_{0,r,\Omega} + h \|\mathbf{u}_h\|^{r-1}_{0,r,\Omega} + \|\mathbf{u}_h\|^{r-1}_{0,r,\Omega} \right) \\
&\leq C \|\mathbf{u}_h\|^{r-1}_U \,.
\end{aligned} \tag{4.57}$$

Combining (4.51), (4.55), and (4.57), we have that

$$\begin{aligned}
\sup_{(\boldsymbol{\tau}_h, q_h) \in T'_{div,h} \times P_h} \frac{[\mathbf{C}(\boldsymbol{\tau}_h, q_h) \, , \, (\mathbf{u}_h, \lambda_h)]}{\|(\boldsymbol{\tau}_h, q_h)\|_{T'_{div} \times P}} \geq \frac{-\int_\Omega \mathbf{u}_h \cdot div\, \boldsymbol{\tau}_h \, d\Omega}{\|\boldsymbol{\tau}_h\|_{T'_{div}}} &\geq \frac{\|\mathbf{u}_h\|^r_U}{C \|\mathbf{u}_h\|^{r-1}_U} \\
&\geq C \|\mathbf{u}_h\|_U \geq c_2 \|(\mathbf{u}_h, \lambda_h)\|_{U \times \mathbb{R}} \,,
\end{aligned}$$

and thus we obtain (4.50). ■

In order to apply the abstract a priori estimate from Theorem 4.2, we must first show that the chosen approximation spaces satisfy the condition (4.12).

Lemma 4.3 *For h sufficiently small, there is a constant $c_3 > 0$ such that*

$$\inf_{(\boldsymbol{\tau}_h, q_h) \in T^{'}_{div,h} \times P_h} \ \sup_{(\boldsymbol{\varsigma}_h, \mathbf{v}_h, \eta_h) \in T_h \times U_h \times \mathbb{R}} \frac{[\mathbf{B}(\boldsymbol{\varsigma}_h), (\boldsymbol{\tau}_h, q_h)] + [\mathbf{C}(\boldsymbol{\tau}_h, q_h)\,,\, (\mathbf{v}_h, \eta_h)]}{\|(\boldsymbol{\varsigma}_h, \mathbf{v}_h, \eta_h)\|_{T \times U \times \mathbb{R}} \, \|(\boldsymbol{\tau}_h, q_h)\|_{T^{'}_{div} \times P}} \geq \ c_3 \,, \qquad (4.58)$$

where $\|(\boldsymbol{\varsigma}_h, \mathbf{v}_h, \eta_h)\|_{T \times U \times \mathbb{R}} = \|\boldsymbol{\varsigma}_h\|_T + \|\mathbf{v}_h\|_U + \|\lambda_h\|_{\mathbb{R}}$.

Proof: If $(\boldsymbol{\tau}_h, q_h) \in Z_{1h}$, then for all $(\mathbf{v}_h, \eta_h) \in U_h \times \mathbb{R}$, we have $[\mathbf{C}(\boldsymbol{\tau}_h, q_h)\,,\, (\mathbf{v}_h, \eta_h)] = 0$, thus (4.58) follows immediately from Lemma 4.1.

For $K \in \mathcal{T}_h$, let $\Pi_{0,K} : T^{'}_{div,\,h}(K) \longrightarrow T_h(K)$ be the $\mathcal{P}_0(K)$ interpolation operator defined by

$$\int_K (\boldsymbol{\tau}_h - \Pi_{0,K} \boldsymbol{\tau}_h) \, dK = 0 \,, \qquad \forall \boldsymbol{\tau}_h \in T^{'}_{div,\,h}(K) \,.$$

For $\boldsymbol{\tau}_h \in T^{'}_{div,\,h}(\Omega)$, let $\hat{\boldsymbol{\tau}} = \Pi_0 \boldsymbol{\tau}_h = \cup_{K \in \mathcal{T}_h} \Pi_{0,K} \boldsymbol{\tau}_h$. From [6, 9], we have that there exists a constant $\hat{C}$ such that

$$\|\boldsymbol{\tau}_h - \hat{\boldsymbol{\tau}}\|_{0,r',\Omega} = \left(\sum_{K \in \mathcal{T}_h} \|\boldsymbol{\tau}_h - \hat{\boldsymbol{\tau}}\|^{r'}_{0,r',K} \right)^{1/r'} \leq \hat{C}\, h \left(\sum_{K \in \mathcal{T}_h} |\boldsymbol{\tau}_h|^{r'}_{1,r',K} \right)^{1/r'}$$
$$= \hat{C}\, h \left(\sum_{K \in \mathcal{T}_h} \|\nabla \boldsymbol{\tau}_h\|^{r'}_{0,r',K} \right)^{1/r'} . \qquad (4.59)$$

Note that, since $\boldsymbol{\tau}_h|_K \in (\mathbb{RT}_0(K))^2$ for all $K \in \mathcal{T}_h$, the partial derivatives of $\nabla \boldsymbol{\tau}_h$ that are not also present in $div\, \boldsymbol{\tau}_h$ are zero, and thus

$$\left(\sum_{K \in \mathcal{T}_h} \|\nabla \boldsymbol{\tau}_h\|^{r'}_{0,r',K} \right)^{1/r'} = \left(\sum_{K \in \mathcal{T}_h} (\sqrt{n}\, \|div\, \boldsymbol{\tau}_h\|_{0,r',K})^{r'} \right)^{1/r'} = \sqrt{n}\, \|div\, \boldsymbol{\tau}_h\|_{0,r',\Omega} \,. \qquad (4.60)$$

Combining (4.59) and (4.60) we can bound the error in approximating $\boldsymbol{\tau}_h$ with $\hat{\boldsymbol{\tau}}$ by

$$\|\boldsymbol{\tau}_h - \hat{\boldsymbol{\tau}}\|_{0,r',\Omega} \leq \sqrt{n}\, \hat{C}\, h\, \|div\, \boldsymbol{\tau}_h\|_{0,r',\Omega} \,. \qquad (4.61)$$

We will assume that $div\, \boldsymbol{\tau} \neq \mathbf{0}$, for if $div\, \boldsymbol{\tau} = \mathbf{0}$ then $\boldsymbol{\tau}$ is piecewise constant and $\hat{\boldsymbol{\tau}} = \boldsymbol{\tau}$.

<u>Case 1:</u> ($\|\boldsymbol{\tau}_h\|_{T^{'}_{div}} \leq \|q_h\|_P$)

Let

$$\boldsymbol{\varsigma}_h = \frac{-1}{\|q_h \mathbf{I} + \hat{\boldsymbol{\tau}}\|^{r'-1}_{T'}} |q_h \mathbf{I} + \hat{\boldsymbol{\tau}}|^{r'/r-1} (q_h \mathbf{I} + \hat{\boldsymbol{\tau}}) \,.$$

Note that $\boldsymbol{\varsigma}_h \in T_h$ and $\|\boldsymbol{\varsigma}_h\|_T = 1$. Then we have

$$
\begin{aligned}
[\mathbf{B}(\boldsymbol{\varsigma}_h), (\boldsymbol{\tau}_h, q_h)] &= \int_\Omega \frac{|q_h\mathbf{I} + \hat{\boldsymbol{\tau}}|^{r'/r-1}}{\|q_h\mathbf{I} + \hat{\boldsymbol{\tau}}\|_{T'}^{r'-1}} \left(\boldsymbol{\tau}_h : (q_h\mathbf{I} + \hat{\boldsymbol{\tau}}) + q_h\, tr(q_h\mathbf{I} + \hat{\boldsymbol{\tau}})\right) d\Omega \\
&= \int_\Omega \frac{|q_h\mathbf{I} + \hat{\boldsymbol{\tau}}|^{r'/r-1}}{\|q_h\mathbf{I} + \hat{\boldsymbol{\tau}}\|_{T'}^{r'-1}} \left((q_h\mathbf{I} + \hat{\boldsymbol{\tau}}) : (q_h\mathbf{I} + \hat{\boldsymbol{\tau}}) + (q_h\mathbf{I} + \hat{\boldsymbol{\tau}}) : (\boldsymbol{\tau}_h - \hat{\boldsymbol{\tau}}\right) d\Omega \\
&= \|q_h\mathbf{I} + \hat{\boldsymbol{\tau}}\|_{T'} - \int_\Omega \frac{|q_h\mathbf{I} + \hat{\boldsymbol{\tau}}|^{r'/r-1}}{\|q_h\mathbf{I} + \hat{\boldsymbol{\tau}}\|_{T'}^{r'-1}} \left((q_h\mathbf{I} + \hat{\boldsymbol{\tau}}) : (\boldsymbol{\tau}_h - \hat{\boldsymbol{\tau}}\right) d\Omega \\
&\geq \|q_h\mathbf{I} + \hat{\boldsymbol{\tau}}\|_{T'} - \|\boldsymbol{\varsigma}_h\|_T \|\boldsymbol{\tau}_h - \hat{\boldsymbol{\tau}}\|_{T'} \\
&\geq n^{1/r'}\|q_h\|_P - \|\hat{\boldsymbol{\tau}}\|_{T'} - \sqrt{n}\,\hat{C}h\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega} \\
&\geq n^{1/r'}\|q_h\|_P - \|\boldsymbol{\tau}_h\|_{T'} - 2\sqrt{n}\,\hat{C}h\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega} \\
&\geq (n^{1/r'} - 1)\|q_h\|_P - 2\sqrt{n}\,\hat{C}h\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega}\,.
\end{aligned}
\tag{4.62}
$$

Let

$$
\mathbf{v}_h = \frac{-(n^{1/r'} - 1)|div\,\boldsymbol{\tau}_h|^{r'/r-1}}{\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega}^{r'-1}}\,(div\,\boldsymbol{\tau}_h)\,,
$$

and note that $\mathbf{v}_h \in U_h$ and $\|\mathbf{v}_h\|_U = n^{1/r'} - 1$. Let $\eta_h = 0$, then we have

$$
\begin{aligned}
[\mathbf{C}(\boldsymbol{\tau}_h, q_h)\,,\,(\mathbf{v}_h, \eta_h)] &= (n^{1/r'} - 1)\int_\Omega \frac{|div\,\boldsymbol{\tau}_h|^{r'/r-1}}{\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega}^{r'-1}}\,(div\,\boldsymbol{\tau}_h)\cdot(div\,\boldsymbol{\tau}_h)\,d\Omega \\
&= (n^{1/r'} - 1)\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega}\,.
\end{aligned}
\tag{4.63}
$$

Thus, from (4.62) and (4.63), we have

$$
\begin{aligned}
&\frac{[\mathbf{B}(\boldsymbol{\varsigma}_h), (\boldsymbol{\tau}_h, q_h)] + [\mathbf{C}(\boldsymbol{\tau}_h, q_h)\,,\,(\mathbf{v}_h, \eta_h)]}{\|(\boldsymbol{\varsigma}_h, \mathbf{v}_h, \eta_h)\|_{T\times U\times\mathbb{R}}} \\
&\qquad\qquad \geq \frac{1}{n^{1/r'}}\left((n^{1/r'} - 1)\|q_h\|_P + (n^{1/r'} - 1 - 2\sqrt{n}\hat{C}h)\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega}\right)\,,
\end{aligned}
\tag{4.64}
$$

and, for h small enough to satisfy $n^{1/r'} - 1 - 2\sqrt{n}\hat{C}h > 0$, we have that

$$
\frac{[\mathbf{B}(\boldsymbol{\varsigma}_h), (\boldsymbol{\tau}_h, q_h)] + [\mathbf{C}(\boldsymbol{\tau}_h, q_h)\,,\,(\mathbf{v}_h, \eta_h)]}{\|(\boldsymbol{\varsigma}_h, \mathbf{v}_h, \eta_h)\|_{T\times U\times\mathbb{R}}} \geq C\|(\boldsymbol{\tau}_h, q_h)\|_{T'_{div}\times P}\,,
$$

for some constant $C > 0$.
Case 2: ($\|\boldsymbol{\tau}_h\|_{T'_{div}} \geq \|q_h\|_P$)
First note that for $\boldsymbol{\tau} \in T'_{div}$, we have by Hölder's inequality,

$$
\left|\int_\Omega tr(\boldsymbol{\tau})\,d\Omega\right| \leq \sqrt{n}\,|\Omega|^{1/r}\|\boldsymbol{\tau}\|_{T'}\,,
$$

and thus

$$
\left|\int_\Omega tr(\hat{\boldsymbol{\tau}} - \boldsymbol{\tau}_h)\,d\Omega\right| \leq \hat{C}\,n\,|\Omega|^{1/r}h\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega}\,.
\tag{4.65}
$$

Given the piecewise constant interpolant $\hat{\boldsymbol{\tau}}$, let

$$\tilde{\boldsymbol{\tau}} = \hat{\boldsymbol{\tau}} - \frac{1}{n}\left(\int_\Omega tr(\hat{\boldsymbol{\tau}})\, d\Omega\right)\mathbf{I}\,,$$

and note that $div\,\hat{\boldsymbol{\tau}} = div\,\tilde{\boldsymbol{\tau}}$. From (4.65) we have

$$\begin{aligned}
\|\tilde{\boldsymbol{\tau}} - \hat{\boldsymbol{\tau}}\|_{T'} &= \left\|\frac{1}{n}\left(\int_\Omega tr(\hat{\boldsymbol{\tau}})\, d\Omega\right)\mathbf{I}\right\|_{T'} \\
&= \left|\int_\Omega tr(\hat{\boldsymbol{\tau}})\, d\Omega\right| \left\|\frac{1}{n}\mathbf{I}\right\|_{T'} \\
&= n^{-1/2}|\Omega|^{1/r'}\left|\int_\Omega tr(\hat{\boldsymbol{\tau}})\, d\Omega\right| \\
&\le n^{-1/2}|\Omega|^{1/r'}\left(\left|\int_\Omega tr(\hat{\boldsymbol{\tau}} - \boldsymbol{\tau}_h)\, d\Omega\right| + \left|\int_\Omega tr(\boldsymbol{\tau}_h)\, d\Omega\right|\right) \\
&\le n^{-1/2}|\Omega|^{1/r'}\left(\hat{C}\, n\, |\Omega|^{1/r} h \|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega} + \left|\int_\Omega tr(\boldsymbol{\tau}_h)\, d\Omega\right|\right) \\
&= \sqrt{n}\,\hat{C}\,|\Omega|\, h\, \|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega} + n^{-1/2}|\Omega|^{1/r'}\left|\int_\Omega tr(\boldsymbol{\tau}_h)\, d\Omega\right|\,. \qquad (4.66)
\end{aligned}$$

We have, from (4.61) and (4.66),

$$\begin{aligned}
\|\tilde{\boldsymbol{\tau}} - \boldsymbol{\tau}_h\|_{T'} &= \|\tilde{\boldsymbol{\tau}} - \hat{\boldsymbol{\tau}}\|_{T'} + \|\hat{\boldsymbol{\tau}} - \boldsymbol{\tau}_h\|_{T'} \\
&\le \sqrt{n}\,\hat{C}\,(1 + |\Omega|)\, h\, \|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega} + n^{-1/2}|\Omega|^{1/r'}\left|\int_\Omega tr(\boldsymbol{\tau}_h)\, d\Omega\right|. \qquad (4.67)
\end{aligned}$$

Lemma 3.2 applies to $\tilde{\boldsymbol{\tau}}$ and we have that, for

$$\boldsymbol{\tau}^0 = \tilde{\boldsymbol{\tau}} - \frac{1}{n}tr(\tilde{\boldsymbol{\tau}})\mathbf{I}\,,$$

there exists a constant C_0 such that

$$\|\tilde{\boldsymbol{\tau}}\|_{0,r'} \le C_0\left(\|\boldsymbol{\tau}^0\|_{0,r'} + \|div\,\tilde{\boldsymbol{\tau}}\|_{-1,r'}\right)\,, \qquad (4.68)$$

and, since $div\,\tilde{\boldsymbol{\tau}} = div\,\hat{\boldsymbol{\tau}}$, we have that

$$\|div\,\tilde{\boldsymbol{\tau}}\|_{-1,r'} \le \|div\,\boldsymbol{\tau}_h\|_{-1,r'} + \|div\,(\boldsymbol{\tau}_h - \hat{\boldsymbol{\tau}})\|_{-1,r'} \le \|div\,\boldsymbol{\tau}_h\|_{0,r'} + \|div\,(\boldsymbol{\tau}_h - \hat{\boldsymbol{\tau}})\|_{-1,r'}\,. \qquad (4.69)$$

Observe that

$$\|div\,(\boldsymbol{\tau}_h - \hat{\boldsymbol{\tau}})\|_{-1,r'} = \sup_{g\in W_0^{1,r}(\Omega)} \frac{\langle div\,(\boldsymbol{\tau}_h - \hat{\boldsymbol{\tau}}), g\rangle}{\|g\|_{1,r}} = \sup_{g\in W_0^{1,r}(\Omega)} \frac{-\langle \boldsymbol{\tau}_h - \hat{\boldsymbol{\tau}}, \nabla g\rangle}{\|g\|_{1,r}} \le \|\boldsymbol{\tau}_h - \hat{\boldsymbol{\tau}}\|_{T'}\,. \qquad (4.70)$$

Combining (4.68) - (4.70) we have

$$\begin{aligned}
\|\tilde{\boldsymbol{\tau}}\|_{0,r'} &\le C_0\left(\|\boldsymbol{\tau}^0\|_{0,r'} + \|div\,\boldsymbol{\tau}_h\|_{0,r'} + \sqrt{n}\,\hat{C}\,h\|div\,\boldsymbol{\tau}_h\|_{0,r'}\right) \\
&\le C_0\left(\|\boldsymbol{\tau}^0\|_{0,r'} + (1 + \sqrt{n}\,\hat{C}h)\|div\,\boldsymbol{\tau}_h\|_{0,r'}\right)\,. \qquad (4.71)
\end{aligned}$$

Rearranging and using (4.67),

$$\begin{aligned}
\|\boldsymbol{\tau}^0\|_{T'} &\geq \frac{1}{C_0}\|\tilde{\boldsymbol{\tau}}\|_{T'} - (1+\sqrt{n}\,\hat{C}h)\|div\,\boldsymbol{\tau}_h\|_{0,r'} \\
&\geq \frac{1}{C_0}\left(\|\boldsymbol{\tau}_h\|_{T'} - \|\tilde{\boldsymbol{\tau}}-\boldsymbol{\tau}_h\|_{T'}\right) - (1+\sqrt{n}\,\hat{C}h)\|div\,\boldsymbol{\tau}_h\|_{0,r'} \\
&\geq \frac{1}{C_0}\|\boldsymbol{\tau}_h\|_{T'} - \left(1+\frac{\sqrt{n}\,\hat{C}\,(1+C_0+|\Omega|)}{C_0}\,h\right)\|div\,\boldsymbol{\tau}_h\|_{0,r'} \\
&\qquad - \frac{n^{-1/2}|\Omega|^{1/r'}}{C_0}\left|\int_\Omega tr(\boldsymbol{\tau}_h)\,d\Omega\right| .
\end{aligned} \tag{4.72}$$

Let

$$\boldsymbol{\varsigma}_h = \frac{-|\boldsymbol{\tau}^0|^{r'/r-1}}{\|\boldsymbol{\tau}^0\|_{T'}^{r'-1}}\,\boldsymbol{\tau}^0 ,$$

and note that $\boldsymbol{\varsigma}_h \in T_h$, $tr(\boldsymbol{\varsigma}_h)=0$, and $\|\boldsymbol{\varsigma}_h\|_T = 1$. Let

$$\mathbf{v}_h = \frac{-2|div\,\boldsymbol{\tau}_h|^{r'/r-1}}{\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega}^{r'-1}}\,(div\,\boldsymbol{\tau}_h) ,$$

and note that $\mathbf{v}_h \in U_h$ and $\|\mathbf{v}_h\|_U = 2$. Let

$$\eta_h = sgn\left(\int_\Omega tr(\boldsymbol{\tau}_h)\,d\Omega\right)\left(1+\frac{1}{C_0}\right)n^{-1/2}|\Omega|^{1/r'}.$$

Then we have, from (4.67) and (4.72),

$$\begin{aligned}
[\mathbf{B}(\boldsymbol{\varsigma}_h),(\boldsymbol{\tau}_h,q_h)] &= \int_\Omega \frac{|\boldsymbol{\tau}^0|^{r'/r-1}}{\|\boldsymbol{\tau}^0\|_{T'}^{r'-1}}\,\boldsymbol{\tau}^0 : \boldsymbol{\tau}_h\,d\Omega \\
&= \int_\Omega \frac{|\boldsymbol{\tau}^0|^{r'/r-1}}{\|\boldsymbol{\tau}^0\|_{T'}^{r'-1}}\,\boldsymbol{\tau}^0 : \tilde{\boldsymbol{\tau}}\,d\Omega - \int_\Omega \frac{|\boldsymbol{\tau}^0|^{r'/r-1}}{\|\boldsymbol{\tau}^0\|_{T'}^{r'-1}}\,\boldsymbol{\tau}^0 : (\tilde{\boldsymbol{\tau}}-\boldsymbol{\tau}_h)\,d\Omega \\
&\geq \int_\Omega \frac{|\boldsymbol{\tau}^0|^{r'/r-1}}{\|\boldsymbol{\tau}^0\|_{T'}^{r'-1}}\,\boldsymbol{\tau}^0 : \boldsymbol{\tau}^0\,d\Omega - \|\boldsymbol{\varsigma}_h\|_T\|\tilde{\boldsymbol{\tau}}-\boldsymbol{\tau}_h\|_{T'} \\
&\geq \|\boldsymbol{\tau}^0\|_{T'} - \|\tilde{\boldsymbol{\tau}}-\boldsymbol{\tau}_h\|_{T'} \\
&\geq \frac{1}{C_0}\|\boldsymbol{\tau}_h\|_{T'} - \left(1+\frac{\sqrt{n}\,\hat{C}\,(1+C_0+|\Omega|)}{C_0}\,h\right)\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega} \\
&\qquad - \frac{n^{-1/2}|\Omega|^{1/r'}}{C_0}\left|\int_\Omega tr(\boldsymbol{\tau}_h)\,d\Omega\right| \\
&\qquad - \sqrt{n}\,\hat{C}\,(1+|\Omega|)\,h\,\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega} - n^{-1/2}|\Omega|^{1/r'}\left|\int_\Omega tr(\boldsymbol{\tau}_h)\,d\Omega\right| \\
&= \frac{1}{C_0}\|\boldsymbol{\tau}_h\|_{T'} - \left(1+\frac{\sqrt{n}\,\hat{C}h(1+2C_0+(1+C_0)|\Omega|)}{C_0}\right)\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega} \\
&\qquad - \left(1+\frac{1}{C_0}\right)n^{-1/2}|\Omega|^{1/r'}\left|\int_\Omega tr(\boldsymbol{\tau}_h)\,d\Omega\right| ,
\end{aligned} \tag{4.73}$$

and

$$
\begin{aligned}
[\mathbf{C}(\boldsymbol{\tau}_h, q_h), (\mathbf{v}_h, \eta_h)] &= \int_\Omega \frac{2|div\,\boldsymbol{\tau}_h|^{r'/r-1}}{\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega}^{r'-1}} (div\,\boldsymbol{\tau}_h)\cdot(div\,\boldsymbol{\tau}_h)\,d\Omega \\
&\quad + \left(1+\frac{1}{C_0}\right) n^{-1/2}|\Omega|^{1/r'} \left|\int_\Omega tr(\boldsymbol{\tau}_h)\,d\Omega\right| \\
&= 2\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega} + \left(1+\frac{1}{C_0}\right) n^{-1/2}|\Omega|^{1/r'} \left|\int_\Omega tr(\boldsymbol{\tau}_h)\,d\Omega\right| , \qquad (4.74)
\end{aligned}
$$

and

$$
\|\boldsymbol{\varsigma}_h\|_T + \|\mathbf{v}_h\|_U + |\eta_h| = 3 + \left(1+\frac{1}{C_0}\right) n^{-1/2}|\Omega|^{1/r'} = \tilde{C}\,. \qquad (4.75)
$$

Thus, (4.73)-(4.75) and h small enough to guarantee that

$$
C_0 > \sqrt{n}\,\hat{C}h(1+2C_0+(1+C_0)|\Omega|)
$$

imply

$$
\begin{aligned}
&\frac{[\mathbf{B}(\boldsymbol{\varsigma}_h), (\boldsymbol{\tau}_h, q_h)] + [\mathbf{C}(\boldsymbol{\tau}_h, q_h), (\mathbf{v}_h, \eta_h)]}{\|(\boldsymbol{\varsigma}_h, \mathbf{v}_h, \eta_h)\|_{T\times U\times\mathbb{R}}} \\
&\qquad \geq \frac{1}{\tilde{C}\,C_0}\Bigg(\|\boldsymbol{\tau}_h\|_{T'} \\
&\qquad\qquad + \left(C_0 - \sqrt{n}\,\hat{C}h(1+2C_0+(1+C_0)|\Omega|)\right)\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega}\Bigg) \\
&\qquad \geq C\|\boldsymbol{\tau}_h\|_{T'_{div}} \geq \frac{C}{2}\|(\boldsymbol{\tau}_h, q_h)\|_{T'_{div}\times P}\,. \qquad (4.76)
\end{aligned}
$$

Thus (4.58) is shown. ■

Standard approximation properties for the discrete spaces are shown in [9, 26]: For all $(\boldsymbol{\varsigma}, \boldsymbol{\tau}, q, \mathbf{v}) \in \left(W^{1,r}(\Omega)\right)^{2\times 2} \times \left(W^{1,r'}(\Omega)\right)^{2\times 2} \times W^{1,r'}(\Omega) \times \left(W^{1,r}(\Omega)\right)^2$ with $div\,\boldsymbol{\tau} \in \left(W^{1,r'}(\Omega)\right)^2$, there exists $(\boldsymbol{\varsigma}_h, \boldsymbol{\tau}_h, q_h, \mathbf{v}_h) \in T_h \times T'_{div,\,h} \times P_h \times U_h$ satisfying

$$
\begin{aligned}
\|\boldsymbol{\varsigma} - \boldsymbol{\varsigma}_h\|_T &\leq Ch\|\boldsymbol{\varsigma}\|_{1,r,\Omega}\,, && \forall \boldsymbol{\varsigma} \in \left(W^{1,r}(\Omega)\right)^{2\times 2}\,, && (4.77)\\
\|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{T'} &\leq Ch\|\boldsymbol{\tau}\|_{1,r',\Omega}\,, && \forall \boldsymbol{\tau} \in \left(W^{1,r'}(\Omega)\right)^{2\times 2}\,, && (4.78)\\
\|div\,(\boldsymbol{\tau} - \boldsymbol{\tau}_h)\|_{T'} &\leq Ch\|div\,\boldsymbol{\tau}\|_{1,r',\Omega}\,, && \forall\,(div\,\boldsymbol{\tau}) \in \left(W^{1,r'}(\Omega)\right)^{2}\,, && (4.79)\\
\|q - q_h\|_P &\leq Ch\|q\|_{1,r',\Omega}\,, && \forall q \in W^{1,r'}(\Omega)\,, && (4.80)\\
\|\mathbf{v} - \mathbf{v}_h\|_U &\leq Ch\|\mathbf{v}\|_{1,r,\Omega}\,, && \forall \mathbf{v} \in \left(W^{1,r}(\Omega)\right)^{2}\,. && (4.81)
\end{aligned}
$$

Thus we have the following a priori error estimate, the proof of which follows directly from Theorem 4.2, Lemmas 4.1, 4.2, and 4.3, and the properties (4.77)-(4.81).

Theorem 4.3 *Let* $\mathbf{f} \in \left(L^{r'}(\Omega)\right)^2$ *and* $\mathbf{u}_\Gamma \in \left(W^{1-1/r\,,\,r}(\Gamma)\right)^2$. *Let* $(\boldsymbol{\phi}, \boldsymbol{\psi}, p, \mathbf{u}, \lambda) \in T \times T'_{div} \times P \times U \times \mathbb{R}$ *solve (2.13)-(2.15) and let* $(\boldsymbol{\phi}_h, \boldsymbol{\psi}_h, p_h, \mathbf{u}_h, \lambda_h) \in T_h \times T'_{div,\,h} \times P_h \times U_h \times \mathbb{R}$ *solve (4.5)-(4.7). Assume*

$(\boldsymbol{\phi}, \boldsymbol{\psi}, p, \mathbf{u}) \in \left(W^{1,r}(\Omega)\right)^{2\times 2} \times \left(W^{1,r'}(\Omega)\right)^{2\times 2} \times W^{1,r'}(\Omega) \times \left(W^{1,r}(\Omega)\right)^2$ *with* $div\, \boldsymbol{\psi} \in \left(W^{1,r'}(\Omega)\right)^2$. *Then there exists a positive constant* C *such that*

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T^2 \le C\Big\{ h^r \mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h)^r \|\boldsymbol{\phi}\|_{1,r,\Omega}^r + h^2 \Big(\|\boldsymbol{\phi}\|_{1,r,\Omega} + \|\mathbf{u}\|_{1,r,\Omega} + \|\boldsymbol{\psi}\|_{1,r',\Omega} + \|div\, \boldsymbol{\psi}\|_{1,r',\Omega} + \|p\|_{1,r',\Omega} \Big) \Big\}, \tag{4.82}$$

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{T'_{div}} + \|p - p_h\|_P \le C\, h \left(\|\boldsymbol{\psi}\|_{1,r',\Omega} + \|div\, \boldsymbol{\psi}\|_{1,r',\Omega} + \|p\|_{1,r',\Omega} \right) + \mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h) \left(\int_\Omega |\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)|\, |\boldsymbol{\phi} - \boldsymbol{\phi}_h|\, d\Omega \right)^{1/r'}, \tag{4.83}$$

$$\|\mathbf{u} - \mathbf{u}_h\|_U + |\lambda - \lambda_h| \le C\, \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T\,. \tag{4.84}$$

Remark 4.2 *The extension of Remark 4.1 to these approximation spaces is given by: If* $1/(|\boldsymbol{\phi}| + |\boldsymbol{\phi}_h|) \le C$ *for some constant* $C > 0$ *and* $\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_\infty \sim \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T$*, the estimates (4.82)–(4.84) may be written as*

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T + \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{T'_{div}} + \|p - p_h\|_P + \|\mathbf{u} - \mathbf{u}_h\|_U + |\lambda - \lambda_h| \le C\, h \Big\{ \|\boldsymbol{\phi}\|_{1,r,\Omega} + \|\mathbf{u}\|_{1,r,\Omega} + \|\boldsymbol{\psi}\|_{1,r',\Omega} + \|div\, \boldsymbol{\psi}\|_{1,r',\Omega} + \|p\|_{1,r',\Omega} \Big\}. \tag{4.85}$$

4.3 Higher Order Approximation

In this section, approximation spaces of higher order are considered. For $k \ge 1$, define the following discrete spaces:

$$\begin{aligned}
T_h &:= \left\{ \boldsymbol{\phi} \in T : \boldsymbol{\phi}|_K \in (\mathbb{P}_k(K))^{2\times 2}, \quad \forall K \in \mathcal{T}_h \right\}, \\
T'_{div,\,h} &:= \Big\{ \boldsymbol{\psi} \in T'_{div} : \boldsymbol{\psi} = (\boldsymbol{\psi}_1 \quad \boldsymbol{\psi}_2)^{\mathrm{T}}|_K \in (\mathbb{RT}_k(K))^2, \\
&\qquad (\psi_{i1} \quad \psi_{i2})^{\mathrm{T}}|_K \in \mathbb{RT}_k(K), \quad \forall i \in \{1,2\}, \quad \forall K \in \mathcal{T}_h \Big\}, \\
P_h &:= \left\{ p \in P : p|_K \in \mathbb{P}_k(K), \quad \forall K \in \mathcal{T}_h \right\}, \\
U_h &:= \left\{ \mathbf{u} \in U : \mathbf{u}|_K \in (\mathbb{P}_k(K))^2, \quad \forall K \in \mathcal{T}_h \right\}.
\end{aligned}$$

Remark 4.3 *Note that there is no interelement continuity requirement on the spaces* T_h*,* U_h*, and* P_h*.*

Let $s > 1$ and let $\mathcal{I}_h^k : \left(W^{1,s}(\Omega)\right)^{2\times 2} \longrightarrow T'_{div,\,h}$ be the k-th order Raviart-Thomas interpolation operator [7], defined by, for row $j = 1, 2$ of $\boldsymbol{\tau} \in T'_{div}$,

$$\begin{aligned}
\int_{e_i} (\boldsymbol{\tau}_j - \mathcal{I}_h^k \boldsymbol{\tau}_j) \cdot \mathbf{n}_{e_i} v_k\, ds &= 0, \quad \forall v_k \in \mathbb{P}_k(K), \quad \forall e_i \in \partial K, \quad i = 1,2,3, \quad \forall K \in \mathcal{T}_h, \\
\int_K (\boldsymbol{\tau}_j - \mathcal{I}_h^k \boldsymbol{\tau}_j) \cdot \mathbf{v}_{k-1}\, dK &= 0, \quad \forall \mathbf{v}_{k-1} \in (\mathbb{P}_{k-1}(K))^2, \quad \forall K \in \mathcal{T}_h.
\end{aligned}$$

Then, for $0 \le m \le k+1$, we have

$$\|\boldsymbol{\tau} - \mathcal{I}_h^k \boldsymbol{\tau}\|_{0,r',\Omega} \le C h^m |\boldsymbol{\tau}|_{m,r',\Omega}\,, \tag{4.86}$$

$$\|div\,(\boldsymbol{\tau} - \mathcal{I}_h^k \boldsymbol{\tau})\|_{0,r',\Omega} \le C h^m |div\,\boldsymbol{\tau}|_{m,r',\Omega}\,, \tag{4.87}$$

and, for $\mathbf{v} \in U$,

$$\int_\Omega \mathbf{v} \cdot div(\boldsymbol{\tau} - \mathcal{I}_h^k \boldsymbol{\tau})\, d\Omega = 0, \quad \forall \boldsymbol{\tau} \in T'_{div}\,. \tag{4.88}$$

In the lowest-order case, the special functions that were constructed to show the inf-sup conditions (4.49) and (4.58), for example

$$\boldsymbol{\phi}^* \;=\; \frac{-|q_h\mathbf{I} \,+\, \boldsymbol{\tau}_h|^{r'/r-1}\,(q_h\mathbf{I} \,+\, \boldsymbol{\tau}_h)}{\|q_h\mathbf{I} \,+\, \boldsymbol{\tau}_h\|_{T'}^{r'-1}}\,,$$

were readily available in the appropriate piecewise constant function spaces. However, for higher-order approximation, the analogous functions do not lie in polynomial spaces for $1 < r < 2$. Nevertheless, one can find functions in the appropriate polynomial spaces that share the same important features of these special functions, which are related to the norm and L^2 inner product.

Let $\Pi : T \to T_h = (\mathcal{P}_k)^{2\times 2}$ denote the L^2 projection operator, defined by $\Pi(\boldsymbol{\phi}^*) := \boldsymbol{\phi}_h$, where

$$\int_\Omega \boldsymbol{\phi}^* : \boldsymbol{\tau}_h \, d\Omega = \int_\Omega \boldsymbol{\phi}_h : \boldsymbol{\tau}_h \, d\Omega \qquad \forall \boldsymbol{\tau}_h \in T_h.$$

Lemma 4.4 *Let $\boldsymbol{\phi} \in T$ and $\boldsymbol{\phi}_h = \Pi\boldsymbol{\phi}$. Then there is a constant $C_* > 0$ such that*

$$\|\boldsymbol{\phi}_h\|_T \le C_* \|\boldsymbol{\phi}\|_T. \tag{4.89}$$

Proof: Note that, since T_h is the space of 2×2 tensors whose components are discontinuous piecewise polynomials of degree k on each $K \in \mathcal{T}_h$, we have that,

$$\phi_h = \Pi\boldsymbol{\phi} = \sum_{K\in\mathcal{T}_h} (\Pi\boldsymbol{\phi})|_K = \sum_{K\in\mathcal{T}_h} \Pi(\boldsymbol{\phi}|_K)\,, \tag{4.90}$$

where $\boldsymbol{\phi}|_K$ is the restriction of $\boldsymbol{\phi}$ to K. Let $\boldsymbol{\phi}_K = \boldsymbol{\phi}|_K$. Let $K \in \mathcal{T}_h$, and let $\widehat{K}$ denote the reference element in $\mathcal{T}_h$. Let χ represent the affine map from $\widehat{K}$ to K. Then $\widehat{\boldsymbol{\phi}} = \boldsymbol{\phi}_K \circ \chi$ is the representation of $\boldsymbol{\phi}_K$ on the reference element $\widehat{K}$.

Let $m = \dim((\mathcal{P}_k(\widehat{K}))^{2\times 2})$ and let $\{\widehat{\boldsymbol{\Phi}}_i\}_{i=1}^m$ be an L^2 orthonormal basis for $(\mathcal{P}_k(\widehat{K}))^{2\times 2}$. Then we can write

$$\widehat{\boldsymbol{\phi}}_h(\boldsymbol{\xi}) = \sum_{i=1}^m \phi_i \widehat{\boldsymbol{\Phi}}_i(\boldsymbol{\xi})$$

where the coefficients ϕ_i are given by

$$\phi_i = (\widehat{\boldsymbol{\phi}}, \widehat{\boldsymbol{\Phi}}_i)_{\widehat{K}} \tag{4.91}$$

where $(\cdot,\cdot)_{\widehat{K}}$ represents the L^2 inner product over $\widehat{K}$.

Now we have

$$\begin{aligned}
\|\boldsymbol{\phi}_h\|_{0,r,K} &= \left(\int_K |\boldsymbol{\phi}_h|^r \, dK\right)^{1/r} \\
&= \left(\int_{\widehat{K}} |\widehat{\boldsymbol{\phi}}_h|^r \frac{|K|}{|\widehat{K}|} \, d\widehat{K}\right)^{1/r} \\
&= \left(\int_{\widehat{K}} \left|\sum_{i=1}^m \phi_i \widehat{\boldsymbol{\Phi}}_i\right|^r \, d\widehat{K}\right)^{1/r} \left(\frac{|K|}{|\widehat{K}|}\right)^{1/r} \\
&\leq m^{(r-1)/r} \sum_{i=1}^m |\phi_i| \|\widehat{\boldsymbol{\Phi}}_i\|_{0,r,\widehat{K}} \left(\frac{|K|}{|\widehat{K}|}\right)^{1/r} . \qquad (4.92)
\end{aligned}$$

Now (4.91) implies

$$|\phi_i| \leq \|\widehat{\boldsymbol{\phi}}\|_{0,r,\widehat{K}} \|\widehat{\boldsymbol{\Phi}}_i\|_{0,r',\widehat{K}}. \qquad (4.93)$$

We also have

$$\|\boldsymbol{\phi}\|_{0,r,K} = \left(\int_K |\boldsymbol{\phi}|^r \, dK\right)^{1/r} = \left(\int_{\widehat{K}} |\widehat{\boldsymbol{\phi}}|^r \frac{|K|}{|\widehat{K}|} \, d\widehat{K}\right)^{1/r} = \|\widehat{\boldsymbol{\phi}}\|_{0,r,\widehat{K}} \left(\frac{|K|}{|\widehat{K}|}\right)^{1/r},$$

which implies

$$\|\widehat{\boldsymbol{\phi}}\|_{0,r,\widehat{K}} = \left(\frac{|\widehat{K}|}{|K|}\right)^{1/r} \|\boldsymbol{\phi}\|_{0,r,K} . \qquad (4.94)$$

Combining (4.92)–(4.94), we have

$$\begin{aligned}
\|\boldsymbol{\phi}_h\|_{0,r,K} &\leq m^{(r-1)/r} \sum_{i=1}^m |\phi_i| \|\widehat{\boldsymbol{\Phi}}_i\|_{0,r,\widehat{K}} \left(\frac{|K|}{|\widehat{K}|}\right)^{1/r} \\
&\leq m^{1/r'} \sum_{i=1}^m \|\widehat{\boldsymbol{\phi}}\|_{0,r,\widehat{K}} \|\widehat{\boldsymbol{\Phi}}_i\|_{0,r',\widehat{K}} \|\widehat{\boldsymbol{\Phi}}_i\|_{0,r,\widehat{K}} \left(\frac{|K|}{|\widehat{K}|}\right)^{1/r} \\
&= m^{1/r'} \sum_{i=1}^m \left(\left(\frac{|\widehat{K}|}{|K|}\right)^{1/r} \|\boldsymbol{\phi}\|_{0,r,K}\right) \|\widehat{\boldsymbol{\Phi}}_i\|_{0,r',\widehat{K}} \|\widehat{\boldsymbol{\Phi}}_i\|_{0,r,\widehat{K}} \left(\frac{|K|}{|\widehat{K}|}\right)^{1/r} \\
&= m^{1/r'} \|\boldsymbol{\phi}\|_{0,r,K} \left(\sum_{i=1}^m \|\widehat{\boldsymbol{\Phi}}_i\|_{0,r',\widehat{K}} \|\widehat{\boldsymbol{\Phi}}_i\|_{0,r,\widehat{K}}\right) \left(\frac{|K|}{|\widehat{K}|}\right)^{1/r} \left(\frac{|\widehat{K}|}{|K|}\right)^{1/r} \\
&= m^{1/r'} \|\boldsymbol{\phi}\|_{0,r,K} \left(\sum_{i=1}^m \|\widehat{\boldsymbol{\Phi}}_i\|_{0,r',\widehat{K}} \|\widehat{\boldsymbol{\Phi}}_i\|_{0,r,\widehat{K}}\right) . \qquad (4.95)
\end{aligned}$$

Now $\|\widehat{\boldsymbol{\Phi}}_i\|_{0,2,\widehat{K}} = 1$ and since $\mathcal{P}_k(\widehat{K})$ is finite-dimensional, the equivalence of finite dimensional norms implies there exist constants c_r and $c_{r'}$ such that

$$\|\widehat{\boldsymbol{\Phi}}_i\|_{0,r,\widehat{K}} \leq c_r \|\widehat{\boldsymbol{\Phi}}_i\|_{0,2,\widehat{K}} = c_r \quad \text{and} \quad \|\widehat{\boldsymbol{\Phi}}_i\|_{0,r',\widehat{K}} \leq c_{r'} \|\widehat{\boldsymbol{\Phi}}_i\|_{0,2,\widehat{K}} = c_{r'}$$

Thus (4.95) implies

$$\|\boldsymbol{\phi}_h\|_{0,r,K} \le C_* \,\|\boldsymbol{\phi}\|_{0,r,K} \tag{4.96}$$

for $C_* = m^{1+1/r'} c_r c_{r'}$, which is independent of K. Therefore

$$\|\boldsymbol{\phi}_h\|_T = \left(\sum_{K\in\mathcal{T}_h} \|\boldsymbol{\phi}_h\|^r_{0,r,K} \right)^{1/r} \le \left(\sum_{K\in\mathcal{T}_h} C^r_* \|\boldsymbol{\phi}\|^r_{0,r,K} \right)^{1/r}$$
$$= C_* \left(\sum_{K\in\mathcal{T}_h} \|\boldsymbol{\phi}\|^r_{0,r,K} \right)^{1/r} = C_* \,\|\boldsymbol{\phi}\|_T \,, \tag{4.97}$$

and thus the result is shown. ■

The constant C_* in Lemma 4.4 depends only on the constants c_r and $c_{r'}$, as the dimension m of $(\mathcal{P}_k)^{2\times 2}$ is fixed for k. The constants c_r and $c_{r'}$ that arise in the norm equivalences depend only on the dimension of the space (which is m as well) and not on the size of the domain. A result analogous to Lemma 4.4 holds for the L^2 projection from U onto U_h. Let $\Pi_U : U \to U_h$ be denoted by $\Pi_U \mathbf{u}^* := \mathbf{u}_h$, where

$$\int_\Omega \mathbf{u}^* \cdot \mathbf{w}_h \, d\Omega = \int_\Omega \mathbf{u}_h \cdot \mathbf{w}_h \, d\Omega \qquad \forall \mathbf{w}_h \in U_h.$$

Corollary 4.1 *Let $\mathbf{u} \in U$ and $\mathbf{u}_h = \Pi_U \mathbf{u}$. Then there is a constant $C_{**} > 0$ such that*

$$\|\mathbf{u}_h\|_U \le C_{**} \|\mathbf{u}\|_U \,. \tag{4.98}$$

The inf-sup conditions (4.49) and (4.50) are now shown to hold for $k \ge 1$.

Lemma 4.5 *For the choices of T_h, $T'_{div,h}$, P_h, and U_h above, there exists a positive constant c_1 such that*

$$\inf_{(\boldsymbol{\tau}_h, q_h)\in Z_{1h}} \ \sup_{\boldsymbol{\phi}_h \in T_h} \frac{[\mathbf{B}(\boldsymbol{\phi}_h)\,,\,(\boldsymbol{\tau}_h, q_h)]}{\|\boldsymbol{\phi}_h\|_T \, \|(\boldsymbol{\tau}_h, q_h)\|_{T'_{div}\times P}} \ \ge \ c_1 \ .$$

Proof: Note that for $(\boldsymbol{\phi}_h, q_h) \in Z_{1h}$, $div\, \boldsymbol{\tau}_h = \mathbf{0}$ implies $\boldsymbol{\tau}_h|_K \in (\mathbb{P}_k(K))^{2\times 2}$ for all $K \in \mathcal{T}_h$. We also have that $(\boldsymbol{\tau}_h + q_h\mathbf{I})|_K \in (\mathbb{P}_k(K))^{2\times 2}$ for all $K \in \mathcal{T}_h$. Thus $(\boldsymbol{\tau}_h, q_h) \in Z_{1h}$ implies $\boldsymbol{\tau}_h \in T_h$ and $(\boldsymbol{\tau}_h + q_h\mathbf{I}) \in T_h$.
Assume that $\|q_h\|_P \le \|\boldsymbol{\tau}_h\|_{T'_{div}}$. Let $\boldsymbol{\tau}^0_h = \boldsymbol{\tau}_h - \frac{1}{n} tr(\boldsymbol{\tau}_h)\mathbf{I}$, and

$$\boldsymbol{\phi}^* = -|\boldsymbol{\tau}^0_h|^{r'/r-1}\, \boldsymbol{\tau}^0_h / \|\boldsymbol{\tau}^0_h\|^{r'-1}_{T'}.$$

Note that $\|\boldsymbol{\phi}^*\|_T = 1$, and let $\boldsymbol{\varsigma}_h = \Pi \boldsymbol{\phi}^*$. From Lemma 4.4,

$$\|\boldsymbol{\varsigma}_h\|_T \le C_* \|\boldsymbol{\phi}^*\|_T = C_* \,.$$

Also $[\mathbf{B}(\boldsymbol{\varsigma}_h), (\boldsymbol{\tau}_h, p_h)] = [\mathbf{B}(\boldsymbol{\phi}^*), (\boldsymbol{\tau}_h, p_h)]$ for all $(\boldsymbol{\tau}_h, q_h) \in Z_{1h}$. Continuing as in (3.22), the result is shown as in Case 1 of Lemma 3.3, with the inclusion of the constant $1/C_*$.
Now assume $\|q_h\|_P \ge \|\boldsymbol{\tau}_h\|_{T'_{div}}$. Let

$$\boldsymbol{\phi}^* \ = \ \frac{-|q_h\mathbf{I} \, + \, \boldsymbol{\tau}_h|^{r'/r-1}\,(q_h\mathbf{I} \, + \, \boldsymbol{\tau}_h)}{\|q_h\mathbf{I} \, + \, \boldsymbol{\tau}_h\|^{r'-1}_{T'}} \,.$$

Again let $\varsigma_h = \Pi\boldsymbol{\phi}^*$ and note that $\|\varsigma_h\|_{0,r} \le C_* \|\boldsymbol{\phi}^*\|_T = C_*$. Continuing as in the proof of Case 2 of Lemma 3.3, the result is shown. ■

Lemma 4.6 *For the choices of T_h, $T^{'}_{div,h}$, P_h, and U_h above, there exists a positive constant c_2 such that*

$$\inf_{(\mathbf{u}_h,\lambda_h)\in U_h\times\mathbb{R}} \; \sup_{(\boldsymbol{\tau}_h,q_h)\in T^{'}_{div,h}\times P_h} \frac{[\mathbf{C}(\boldsymbol{\tau}_h,q_h)\,,\,(\mathbf{u}_h,\lambda_h)]}{\|(\boldsymbol{\tau}_h,q_h)\|_{T^{'}_{div}\times P}\,\|(\mathbf{u}_h,\lambda_h)\|_{U\times\mathbb{R}}} \ge \; c_2\,. \tag{4.99}$$

Proof: The result is shown in a manner identical to the proof of Lemma 4.2, with the k-th order interpolation operator $\mathcal{I}_h^k$. ■

Before showing the inf-sup condition (4.12) holds for the chosen approximation spaces, we first discuss some properties of the Raviart-Thomas elements of order $k \ge 1$. Let $K \in \mathcal{T}_h$ and let $\mathbf{r} \in \mathbb{RT}_k(K)$. Then $\mathbf{r}$ can be written as $\mathbf{r} = \mathbf{r}^k + \mathbf{r}^*$, where $\mathbf{r}^k \in (\mathbb{P}_k(K))^2$ and the components of $\mathbf{r}^*$ consist of polynomial terms of degree $k+1$ only. In fact, $\mathbf{r}^*$ can be written as

$$\mathbf{r}^* = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sum_{j=0}^{k} \gamma_j\, x_1^{k-j}\, x_2^{j} = \begin{bmatrix} \displaystyle\sum_{j=0}^{k} \gamma_j\, x_1^{k-j+1}\, x_2^{j} \\ \displaystyle\sum_{j=0}^{k} \gamma_j\, x_1^{k-j}\, x_2^{j+1} \end{bmatrix},$$

for some constants $\gamma_j, j = 0, \ldots, k$.

We can also write $div\,\mathbf{r} = div\,\mathbf{r}^k + div\,\mathbf{r}^*$, where $div\,\mathbf{r}^k$ is a polynomial of degree at most $k-1$ and $div\,\mathbf{r}^*$ is a polynomial with terms of degree k only. It is important to note that if $div\,\mathbf{r} = 0$, then $div\,\mathbf{r}^* = 0$ (as the polynomials in $\mathbf{r}^*$ are linearly independent of the polynomials in $\mathbf{r}^k$), and thus

$$\begin{aligned} 0 = div\,\mathbf{r}^* &= \frac{\partial r_1^*}{\partial x_1} + \frac{\partial r_2^*}{\partial x_2} \\ &= \sum_{j=0}^{k} (k-j+1)\gamma_j\, x_1^{k-j}\, x_2^{j} + \sum_{j=0}^{k} (j+1)\gamma_j\, x_1^{k-j}\, x_2^{j} = (k+2) \sum_{j=0}^{k} \gamma_j\, x_1^{k-j}\, x_2^{j}\,, \end{aligned}$$

which implies that $\gamma_j = 0$ for all $0 \le j \le k$. Hence $div\,\mathbf{r} = 0 \Rightarrow \mathbf{r} \in (\mathbb{P}_k(K))^2$ and thus $\mathbf{r}^* = \mathbf{0}$.

The following lemma is a result from the general theory of finite-dimensional normed spaces (see [19]).

Lemma 4.7 *Let $\{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$ be a linearly independent set of vectors in a normed space $\mathbf{X}$ of dimension at least $n+1$. Then, there is a constant $C_* > 0$ such that for every choice of scalars $\gamma_0, \ldots, \gamma_n$, we have*

$$\|\gamma_0 \mathbf{v}_0 + \cdots + \gamma_n \mathbf{v}_n\| \ge C_*(|\gamma_0| + \cdots + |\gamma_n|)\,.$$

The preceeding lemma is used to show that on each triangle, the norm of the gradient of the highest-degree terms of a Raviart-Thomas element can be bounded by the norm of the divergence, which will be used in establishing an approximation property.

Lemma 4.8 *Let $K \in \mathcal{T}_h$ and let $\mathbf{r} = \mathbf{r}^k + \mathbf{r}^* \in \mathbb{RT}_k(K)$ where the components of $\mathbf{r}^*$ consist of polynomial terms of degree $k+1$ only. Then there exists a constant $\tilde{C} > 0$, independent of K, such that*

$$\|\nabla \mathbf{r}^*\|_{0,r',K} \leq \tilde{C}\, \|div\, \mathbf{r}\|_{0,r',K} \,. \tag{4.100}$$

Proof: Let the finite-dimensional vector space $\mathbf{X}$ be defined by

$$\mathbf{X} = \text{ span } \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} x_1^{k-j}\, x_2^j \,, \quad j = 0, \ldots, k \right\} = \text{ span } \{\mathbf{v}_j \,, \quad j = 0, \ldots, k\} \,.$$

Define the functional $\|\cdot\|_{grad} : \mathbf{X} \to \mathbb{R}$ by, for $\mathbf{v} = (v_1 \quad v_2)^{\mathrm{T}} = \gamma_0 \mathbf{v}_0 + \cdots + \gamma_k \mathbf{v}_k$,

$$\|\mathbf{v}\|_{grad} = \int_K \left|\frac{\partial v_1}{\partial x_1}\right| + \left|\frac{\partial v_1}{\partial x_2}\right| + \left|\frac{\partial v_2}{\partial x_1}\right| + \left|\frac{\partial v_2}{\partial x_2}\right| \, dK \,. \tag{4.101}$$

We now show that $\|\cdot\|_{grad}$ defines a norm on the space $\mathbf{X}$. Of course, $\|\mathbf{v}\|_{grad} \geq 0$ for all $\mathbf{v} \in \mathbf{X}$. To show $\|\mathbf{v}\|_{grad} = 0$ if and only if $\mathbf{v} = \mathbf{0}$, suppose that there exists a nonzero $\mathbf{v} \in \mathbf{X}$ with $\|\mathbf{v}\|_{grad} = 0$. Then (4.101) requires all of the partial derivatives of the components of $\mathbf{v}$ to be zero. Note that since $k \geq 1$, the partial derivatives of the components of $\mathbf{v}$ are functions with polynomial terms of degree k only. But since $\mathbf{v} = \gamma_0 \mathbf{v}_0 + \cdots + \gamma_k \mathbf{v}_k \neq \mathbf{0}$, there is at least one nonzero $\gamma_j, 0 \leq j \leq k$, and thus the partial derivative $\partial v_1 / \partial x_1$ contains a term of the form $(k-j+1)\gamma_j x_1^{k-j} x_2^j$, a contradiction. It is easy to see that $\|\alpha \mathbf{v}\|_{grad} = |\alpha| \|\mathbf{v}\|_{grad}$ and that the triangle inequality holds for $\|\cdot\|_{grad}$. Thus $\|\cdot\|_{grad}$ defines a norm on $\mathbf{X}$. In fact, we have that $\|\mathbf{v}\|_{grad} = \|\nabla \mathbf{v}\|_{0,1,K}$.
Now define the functional $\|\cdot\|_{div} : \mathbf{X} \to \mathbb{R}$ by,

$$\|\mathbf{v}\|_{div} = \int_K \left| \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right| \, dK \,.$$

This functional also defines a norm, in particular because $div\, \mathbf{v} = 0$ implies that $\mathbf{v} = \mathbf{0}$. We also have that $\|\mathbf{v}\|_{div} = \|div\, \mathbf{v}\|_{0,1,K}$. By the equivalence of norms on a finite-dimensional vector space, we have that, there exist constants C_1, C_2, and C_3, depending only on the dimension of the space (in this case $k+1$), such that

$$\|\nabla \mathbf{v}\|_{0,r',K} \leq C_1 \|\nabla \mathbf{v}\|_{0,1,K} \leq C_2 \|div\, \mathbf{v}\|_{0,1,K} \leq C_3 \|div\, \mathbf{v}\|_{0,r',K}.$$

Thus there is a $C_K > 0$ such that

$$\|\nabla \mathbf{r}^*\|_{0,r',K} \leq C_K \, \|div\, \mathbf{r}^*\|_{0,r',K} \tag{4.102}$$

for all $K \in \mathcal{T}_h$. The dependence of C_K on $K \in \mathcal{T}_h$ is due to the integral over K. The condition (4.1) guarantees that $\mathcal{T}_h$ is a quasi-uniform triangulation of Ω, thus we can find a global constant C, independent of K, such that

$$\|\nabla \mathbf{r}^*\|_{0,r',K} \leq C \, \|div\, \mathbf{r}^*\|_{0,r',K} \tag{4.103}$$

for all $K \in \mathcal{T}_h$.
Now, let $\mathbf{X}^k$ be the finite dimensional vector space spanned by the polynomials of degree k only, and let $\overline{\mathbf{X}} = \mathbb{P}_k(K)$. Note that $\overline{\mathbf{X}} = \mathcal{P}_{k-1}(K) \oplus \mathbf{X}^k$, and that $div\, \mathbf{r} \in \overline{\mathbf{X}}$, $div\, \mathbf{r}^k \in \mathcal{P}_{k-1}(K)$, and

$div\,\mathbf{r}^* \in \mathbf{X}^k$. Let $\{\mathbf{v}_0, \ldots, \mathbf{v}_k, \ldots, \mathbf{v}_n\}$ be a basis for $\overline{\mathbf{X}}$ where $\{\mathbf{v}_0, \ldots, \mathbf{v}_k\}$ is also a basis for $\mathbf{X}^k$. From Lemma 4.7, there is a constant $C_* > 0$ such that, for all $\mathbf{v} = \gamma_0 \mathbf{v}_0 + \cdots + \gamma_n \mathbf{v}_n \in \overline{\mathbf{X}}$,

$$\|\mathbf{v}\|_{0,r',K} \geq C_*(|\gamma_0| + \cdots + |\gamma_n|)\,.$$

Define the functional $\|\cdot\|_* : \overline{\mathbf{X}} \to \mathbb{R}$ for $\mathbf{v} = \gamma_0 \mathbf{v}_0 + \cdots + \gamma_n \mathbf{v}_n$ by

$$\|\mathbf{v}\|_* = C_*(|\gamma_0| + \cdots + |\gamma_n|)\,.$$

It is straightforward to show that $\|\cdot\|_*$ defines a norm on $\overline{\mathbf{X}}$. Then, by the definition of $\mathbf{r}$, $\mathbf{r}^*$, the equivalence of norms on a finite-dimensional space, and the quasi-uniform triangulation $\mathcal{T}_h$, we have that there is a constant C_4, dependent only upon n (which itself is dependent only upon k) such that

$$\begin{aligned}\|div\,\mathbf{r}^*\|_{0,r',K} &\leq C_4 \|div\,\mathbf{r}^*\|_* = C_4 C_*(|\gamma_0| + \cdots + |\gamma_k|) \\ &\leq C_4 C_*(|\gamma_0| + \cdots + |\gamma_k| + \cdots + |\gamma_n|) = C_4 \|div\,\mathbf{r}\|_* \leq C_4 \|div\,\mathbf{r}\|_{0,r',K}\,. \end{aligned} \tag{4.104}$$

Combining (4.103) and (4.104) the result is shown. ■

The above results are easily extended to the tensor space $T'_{div,\,h}$ to obtain, for $\boldsymbol{\tau}_h = \boldsymbol{\tau}^k + \boldsymbol{\tau}^*$ where the components of $\boldsymbol{\tau}^*$ consist of polynomial terms of degree $k+1$ only,

$$\|\nabla \boldsymbol{\tau}^*\|_{0,r',K} \leq \tilde{C}\, \|div\,\boldsymbol{\tau}_h\|_{0,r',K}\,, \quad \forall\, K \in \mathcal{T}_h\,. \tag{4.105}$$

Let $\Pi_k : T'_{div,\,h} \longrightarrow T_h$ be the classical Lagrangian $\mathcal{P}_k$ interpolation operator ([9]) and define

$$\hat{\boldsymbol{\tau}} = \boldsymbol{\tau}^k + \Pi_k \boldsymbol{\tau}^*\,. \tag{4.106}$$

Note that $\hat{\boldsymbol{\tau}}|_K \in (\mathbb{P}_k(K))^{2\times 2}$ for all $K \in \mathcal{T}_h$, and $div\,\boldsymbol{\tau}_h = \mathbf{0}$ implies $\boldsymbol{\tau}^* = \mathbf{0}$ and $\hat{\boldsymbol{\tau}} = \boldsymbol{\tau}_h$. Then, using (4.105) and standard polynomial approximation properties [6, 9], the error associated in the approximation of $\boldsymbol{\tau}_h$ by $\hat{\boldsymbol{\tau}}$ is given by

$$\begin{aligned}\|\boldsymbol{\tau}_h - \hat{\boldsymbol{\tau}}\|_{0,r',\Omega} &= \|\boldsymbol{\tau}^* - \Pi_k \boldsymbol{\tau}^*\|_{0,r',\Omega} \\ &= \left(\sum_{K\in\mathcal{T}_h} \|\boldsymbol{\tau}^* - \Pi_k \boldsymbol{\tau}^*\|_{0,r',\Omega}^{r'} \right)^{1/r'} \\ &\leq C\,h \left(\sum_{K\in\mathcal{T}_h} \|\nabla \boldsymbol{\tau}^*\|_{0,r',K}^{r'} \right)^{1/r'} \\ &\leq C\,h \left(\sum_{K\in\mathcal{T}_h} \tilde{C}\, \|div\,\boldsymbol{\tau}_h\|_{0,r',K}^{r'} \right)^{1/r'} \\ &\leq C\tilde{C}\,h\,\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega} = \hat{C}\,h\,\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega}\,. \end{aligned} \tag{4.107}$$

Lemma 4.9 *For h sufficiently small, there is a constant $c_3 > 0$ such that*

$$\inf_{(\boldsymbol{\tau}_h, q_h)\in T'_{div,h}\times P_h} \ \sup_{(\boldsymbol{\varsigma}_h, \mathbf{v}_h, \eta_h)\in T_h\times U_h\times\mathbb{R}} \frac{[\mathbf{B}(\boldsymbol{\varsigma}_h), (\boldsymbol{\tau}_h, q_h)] + [\mathbf{C}(\boldsymbol{\tau}_h, q_h)\,,\,(\mathbf{v}_h, \eta_h)]}{\|(\boldsymbol{\varsigma}_h, \mathbf{v}_h, \eta_h)\|_{T\times U\times\mathbb{R}}\, \|(\boldsymbol{\tau}_h, q_h)\|_{T'_{div}\times P}} \geq \ c_3\,. \tag{4.108}$$

where $\|(\boldsymbol{\varsigma}_h, \mathbf{v}_h, \eta_h)\|_{T\times U\times\mathbb{R}} = \|\boldsymbol{\varsigma}_h\|_T + \|\mathbf{v}_h\|_U + \|\lambda_h\|_{\mathbb{R}}$.

Proof: The approach is similar to the proof of Lemma 4.3.

Case 1: ($\|\boldsymbol{\tau}_h\|_{T'_{div}} \leq \|q_h\|_P$)

Let $\hat{\boldsymbol{\tau}}$ be as defined in (4.106). Let

$$\boldsymbol{\phi}^* = \frac{-1}{\|q_h\mathbf{I} + \hat{\boldsymbol{\tau}}\|_{T'}^{r'-1}} |q_h\mathbf{I} + \hat{\boldsymbol{\tau}}|^{r'/r-1}(q_h\mathbf{I} + \hat{\boldsymbol{\tau}})\,.$$

Note that $\|\boldsymbol{\phi}^*\|_T = 1$, and let $\boldsymbol{\varsigma}_h = \Pi\boldsymbol{\phi}^*$, so $\|\boldsymbol{\varsigma}_h\|_T \leq C_*\|\boldsymbol{\phi}^*\|_T = C_*$. Then, since $q_h\mathbf{I} + \hat{\boldsymbol{\tau}} \in T_h$, we have that

$$\int_\Omega \boldsymbol{\varsigma}_h \,:\, (q_h\mathbf{I} + \hat{\boldsymbol{\tau}})\, d\Omega = \int_\Omega \boldsymbol{\phi}^* \,:\, (q_h\mathbf{I} + \hat{\boldsymbol{\tau}})\, d\Omega\,.$$

Then we have

$$\begin{aligned}
\frac{[\mathbf{B}(\boldsymbol{\varsigma}_h), (\boldsymbol{\tau}_h, q_h)]}{\|\boldsymbol{\varsigma}_h\|_T} &\geq \frac{-1}{C_*}\int_\Omega \boldsymbol{\varsigma}_h \,:\, (q_h\mathbf{I} + \boldsymbol{\tau}_h)\, d\Omega \\
&= \frac{-1}{C_*}\int_\Omega \boldsymbol{\varsigma}_h \,:\, (q_h\mathbf{I} + \hat{\boldsymbol{\tau}})\, d\Omega - \int_\Omega \boldsymbol{\varsigma}_h \,:\, (\boldsymbol{\tau}_h - \hat{\boldsymbol{\tau}})\, d\Omega \\
&= \frac{1}{C_*}\left(\int_\Omega \frac{|q_h\mathbf{I} + \hat{\boldsymbol{\tau}}|^{r'/r-1}}{\|q_h\mathbf{I} + \hat{\boldsymbol{\tau}}\|_{T'}^{r'-1}}(q_h\mathbf{I} + \hat{\boldsymbol{\tau}}) : (q_h\mathbf{I} + \hat{\boldsymbol{\tau}})\, d\Omega \right.\\
&\qquad \left. - \int_\Omega \boldsymbol{\varsigma}_h \,:\, (\boldsymbol{\tau}_h - \hat{\boldsymbol{\tau}})\, d\Omega\right) \\
&\geq \frac{1}{C_*}\left(\|q_h\mathbf{I} + \hat{\boldsymbol{\tau}}\|_{T'} - \|\boldsymbol{\varsigma}_h\|_T\|\boldsymbol{\tau}_h - \hat{\boldsymbol{\tau}}\|_{T'}\right) \\
&\geq \frac{1}{C_*}\left((n^{1/r'} - 1)\|q_h\|_P - 2\hat{C}h\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega}\right)\,. \qquad (4.109)
\end{aligned}$$

Let

$$\mathbf{u}_* = \frac{-(n^{1/r'} - 1)|div\,\boldsymbol{\tau}_h|^{r'/r-1}}{\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega}^{r'-1}}\,(div\,\boldsymbol{\tau}_h)\,,$$

and note that $\|\mathbf{u}^*\|_U = n^{1/r'} - 1$. Recall that $div\,\boldsymbol{\psi}_h \in U_h$ for all $\boldsymbol{\psi}_h \in T'_{div,\,h}$. Let $\mathbf{v}_h = \Pi_U\mathbf{u}^*$. Then $\|\mathbf{v}_h\|_U \leq C_{**}\|\mathbf{u}^*\|_U = C_{**}(n^{1/r'} - 1)$. Let $\eta_h = 0$, then we have

$$\begin{aligned}
\frac{[\mathbf{C}(\boldsymbol{\tau}_h, q_h)\,,\,(\mathbf{v}_h, \eta_h)]}{\|(\mathbf{v}_h, \eta_h)\|_{U\times\mathbb{R}}} &= \frac{1}{C_{**}}(n^{1/r'} - 1)\int_\Omega \frac{|div\,\boldsymbol{\tau}_h|^{r'/r-1}}{\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega}^{r'-1}}\,(div\,\boldsymbol{\tau}_h)\cdot(div\,\boldsymbol{\tau}_h)\, d\Omega \\
&= \frac{1}{C_{**}}(n^{1/r'} - 1)\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega}\,. \qquad (4.110)
\end{aligned}$$

Thus, from (4.109) and (4.110), we have

$$\begin{aligned}
&\frac{[\mathbf{B}(\boldsymbol{\varsigma}_h), (\boldsymbol{\tau}_h, q_h)] + [\mathbf{C}(\boldsymbol{\tau}_h, q_h)\,,\,(\mathbf{v}_h, \eta_h)]}{\|(\boldsymbol{\varsigma}_h, \mathbf{v}_h, \eta_h)\|_{T\times U\times\mathbb{R}}} \\
&\qquad \geq \frac{1}{C_* + C_{**}(n^{1/r'} - 1)}\left((n^{1/r'} - 1)\|q_h\|_P + (n^{1/r'} - 1 - 2\hat{C}h)\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega}\right)\,, \qquad (4.111)
\end{aligned}$$

and, for h small enough to satisfy $n^{1/r'} - 1 - 2\hat{C}h > 0$, we have that

$$\frac{[\mathbf{B}(\boldsymbol{\varsigma}_h), (\boldsymbol{\tau}_h, q_h)] + [\mathbf{C}(\boldsymbol{\tau}_h, q_h)\,,\,(\mathbf{v}_h, \eta_h)]}{\|(\boldsymbol{\varsigma}_h, \mathbf{v}_h, \eta_h)\|_{T\times U\times\mathbb{R}}} \geq C\|(\boldsymbol{\tau}_h, q_h)\|_{T'_{div}\times P}\,,$$

for some constant $C > 0$.
Case 2: ($\|\boldsymbol{\tau}_h\|_{T'_{div}} \geq \|q_h\|_P$)
Let

$$\tilde{\boldsymbol{\tau}} = \hat{\boldsymbol{\tau}} - \frac{1}{n}\left(\int_\Omega tr(\hat{\boldsymbol{\tau}})\, d\Omega\right)\mathbf{I}\,,$$

and

$$\boldsymbol{\tau}^0 = \tilde{\boldsymbol{\tau}} - \frac{1}{n} tr(\tilde{\boldsymbol{\tau}})\mathbf{I}\,.$$

Let

$$\boldsymbol{\phi}^* = \frac{-|\boldsymbol{\tau}^0|^{r'/r-1}}{\|\boldsymbol{\tau}^0\|_{T'}^{r'-1}}\,\boldsymbol{\tau}^0\,,$$

and let $\boldsymbol{\varsigma}_h = \Pi\boldsymbol{\phi}^*$. From (4.89), $\|\boldsymbol{\varsigma}_h\|_T \leq C_*\|\boldsymbol{\phi}^*\|_T = C_*$. Note that $\tilde{\boldsymbol{\tau}}, \boldsymbol{\tau}^0 \in T_h$. Let

$$\mathbf{u}^* = \frac{-2|div\,\boldsymbol{\tau}_h|^{r'/r-1}}{\|div\,\boldsymbol{\tau}_h\|_{0,r',\Omega}^{r'-1}}\,(div\,\boldsymbol{\tau}_h)\,,$$

and let $\mathbf{v}_h = \Pi_U\mathbf{u}^*$. From (4.98), $\|\mathbf{v}_h\|_U \leq C_{**}\|\mathbf{u}^*\|_U = 2C_{**}$. Let

$$\eta_h = sgn\left(\int_\Omega tr(\boldsymbol{\tau}_h)\, d\Omega\right)\left(1 + \frac{1}{C_0}\right) n^{-1/2}|\Omega|^{1/r'}.$$

Continuing in the manner of (4.73)-(4.76), if h is small enough to guarantee

$$C_0 > \hat{C}h(1 + 2C_0 + (1 + C_0)|\Omega|)\,,$$

then the result is shown. ■
From the standard approximation properties [9, 7], the following error estimate is derived.

Theorem 4.4 *Let* $\mathbf{f} \in \left(L^{r'}(\Omega)\right)^2$ *and* $\mathbf{u}_\Gamma \in \left(W^{1-1/r\,,\,r}(\Gamma)\right)^2$. *Let* $(\boldsymbol{\phi}, \boldsymbol{\psi}, p, \mathbf{u}, \lambda) \in T \times T'_{div} \times P \times U \times \mathbb{R}$ *solve (2.13)-(2.15) and let* $(\boldsymbol{\phi}_h, \boldsymbol{\psi}_h, p_h, \mathbf{u}_h, \lambda_h) \in T_h \times T'_{div,\,h} \times P_h \times U_h \times \mathbb{R}$ *solve (4.5)-(4.7). Assume* $1 \leq m \leq k+1$ *and* $(\boldsymbol{\phi}, \boldsymbol{\psi}, p, \mathbf{u}) \in (W^{m,r}(\Omega))^{2\times 2} \times \left(W^{m,r'}(\Omega)\right)^{2\times 2} \times W^{m,r'}(\Omega) \times (W^{m,r}(\Omega))^2$ *with* $div\,\boldsymbol{\psi} \in \left(W^{m,r'}(\Omega)\right)^2$. *Then there exists a positive constant* C *such that*

$$\begin{aligned}\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T^2 \leq C\Bigg\{ & h^{mr}\mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h)^r\|\boldsymbol{\phi}\|_{m,r,\Omega} \\ & + h^{2m}\Big(\|\boldsymbol{\phi}\|_{m,r,\Omega} + \|\mathbf{u}\|_{m,r,\Omega} + \|\boldsymbol{\psi}\|_{m,r',\Omega} + \|div\,\boldsymbol{\psi}\|_{m,r',\Omega} + \|p\|_{m,r',\Omega}\Big)\Bigg\}\,, \end{aligned} \tag{4.112}$$

$$\begin{aligned}\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{T'_{div}} + \|p - p_h\|_P \leq & \; C\, h^m\left(\|\boldsymbol{\psi}\|_{m,r',\Omega} + \|div\,\boldsymbol{\psi}\|_{m,r',\Omega} + \|p\|_{m,r',\Omega}\right) \\ & + \mathcal{E}(\boldsymbol{\phi}, \boldsymbol{\phi}_h)\left(\int_\Omega |\mathbf{g}(\boldsymbol{\phi}) - \mathbf{g}(\boldsymbol{\phi}_h)|\,|\boldsymbol{\phi} - \boldsymbol{\phi}_h|\, d\Omega\right)^{1/r'}\,, \end{aligned} \tag{4.113}$$

$$\|\mathbf{u} - \mathbf{u}_h\|_U + |\lambda - \lambda_h| \leq C\,\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T\,. \tag{4.114}$$

Remark 4.4 *The extension of Remark 4.2 to these approximation spaces is given by: If* $1/(|\boldsymbol{\phi}| + |\boldsymbol{\phi}_h|) \leq C$ *for some constant* $C > 0$ *and* $\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_\infty \sim \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T$*, the estimates (4.112)-(4.114) may be written as*

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T + \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{T'_{div}} + \|p - p_h\|_P + \|\mathbf{u} - \mathbf{u}_h\|_U + |\lambda - \lambda_h|$$
$$\leq C\, h^m \left\{ \|\boldsymbol{\phi}\|_{m,r,\Omega} + \|\mathbf{u}\|_{m,r,\Omega} + \|\boldsymbol{\psi}\|_{m,r',\Omega} + \|div\, \boldsymbol{\psi}\|_{m,r',\Omega} + \|p\|_{m,r',\Omega} \right\}. \quad (4.115)$$

5 Numerical Experiments

In this section we describe numerical experiments that support the theoretical results outlined in Sections 3 and 4. The first example illustrates the theoretical rate of convergence of the solution method and the second example illustrates the computed approximation for a benchmark physical problem. Computations are performed using the `FreeFEM++` finite element software package [18]. All computations below are performed in the lowest-order case ($k = 0$).

5.1 Example 1

For this example (similar to one in [15]) approximations are computed for a Ladyzhenskaya law fluid with $\nu_0 = 0$ and $\nu_1 = 1.0$. The computational domain is $\Omega = [0, 2] \times [0, 2]$, with $\mathbf{f}$ and $\mathbf{u}_\Gamma$ chosen so that the exact solution of (2.10)-(2.12) is given by

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad p = x_1 + x_2\,,$$

with

$$u_1 = -(4.0 - x_1 - x_2)^\alpha \quad \text{and} \quad u_2 = -u_1$$

for α just large enough to ensure $\mathbf{f} = -\,div\, \boldsymbol{\psi} \in W^{\mu-\varepsilon, r'}(\Omega)$. It should be noted that $\alpha = -\frac{2}{r} + r' + \frac{\mu}{r-1} + \varepsilon$ ensures $\mathbf{f} \in W^{\mu, r'}(\Omega)$ for $\varepsilon > 0$.

Computations are performed on uniform meshes of decreasing size h and for selected values of r, α, and μ. For $1 < r < 2$, the resulting system of equations is nonlinear, and a fixed-point iteration is used to compute approximations. The fixed-point iteration is terminated when the pointwise maximum absolute difference in successive approximations falls below 10^{-5}. Results for the velocity, $\mathbf{u}$, the gradient of the velocity, $\boldsymbol{\phi}\,(= \nabla \mathbf{u})$, and the total stress, $\boldsymbol{\psi}$, are shown in Table 5.1.

For this example, $div\, \boldsymbol{\psi} \in W^{\mu-\varepsilon, r'}(\Omega)$ is the most singular of the quantities to be approximated. The observed experimental convergence rate for $\|div\, \boldsymbol{\psi} - div\, \boldsymbol{\psi}_h\|_{0,r'}$ of Ch^μ is in agreement with that predicted by (4.44). The experimental convergence rates observed for $\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_T$ and $\|\mathbf{u} - \mathbf{u}_h\|_U$ are both better than that given by (4.44).

5.2 Example 2

This example is the benchmark driven cavity problem. Driven cavity flows of power law fluids were computed using a mixed method by Manouzi and Farhloul in [22]. (In [22] the authors explicitly

	h	$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{0,r}$	rate	$\|div\, \boldsymbol{\psi} - div\, \boldsymbol{\psi}_h\|_{0,r'}$	rate	$\|\mathbf{u} - \mathbf{u}_h\|_{0,r}$	rate
	1	2.5481		0.8014		37.3797	
$r = 3/2$	1/2	1.2633	1.01	0.4459	0.85	19.6284	0.93
$\mu = 1$	1/4	0.6218	1.02	0.2426	0.88	9.8677	0.99
$\alpha = 11/3$	1/8	0.3080	1.01	0.1299	0.90	4.9294	1.00
	1/16	0.1534	1.01	0.0687	0.92	2.4623	1.00
	1	1.3341		0.2556		10.5023	
$r = 3/2$	1/2	0.6899	0.95	0.1824	0.49	5.3111	0.98
$\mu = 1/2$	1/4	0.3405	1.02	0.1294	0.49	2.6503	1.00
$\alpha = 8/3$	1/8	0.1677	1.02	0.0917	0.50	1.3223	1.00
	1/16	0.0832	1.01	0.0648	0.50	0.6605	1.00
	1	2.6967		1.3410		4721.1800	
$r = 5/4$	1/2	1.3109	1.04	0.7234	0.89	2553.9800	0.89
$\mu = 1$	1/4	0.6325	1.05	0.3833	0.92	1285.9000	0.99
$\alpha = 37/5$	1/8	0.3094	1.03	0.2007	0.93	635.6200	1.02
	1/16	0.1533	1.01	0.1042	0.95	315.0940	1.01
	1	1.4671		0.1661		363.2130	
$r = 5/4$	1/2	0.7461	0.98	0.1176	0.50	191.1110	0.93
$\mu = 1/2$	1/4	0.3604	1.05	0.0832	0.50	94.7585	1.01
$\alpha = 27/5$	1/8	0.1746	1.05	0.0588	0.50	46.8215	1.02
	1/16	0.0860	1.02	0.0416	0.50	23.2479	1.01

Table 5.1: Approximation errors and rates of convergence for Example 1.

inverted the constitutive equation to obtain $\Phi_\alpha(\boldsymbol{\sigma}) = \nabla \mathbf{u}$, which was used in their formulation.)

For $\Omega = [0,1] \times [0,1]$, we have that $\mathbf{f} = \mathbf{0}$ in Ω, $\mathbf{u}_\Gamma = \mathbf{0}$ on $\Gamma \setminus \Gamma_{\text{top}}$ and $\mathbf{u}_\Gamma = [1 \quad 0]^{\mathrm{T}}$ on Γ_{top}, where Γ_{top} is the portion of the boundary satisfying $0 \leq x_1 \leq 1$ and $x_2 = 1$. Computations were performed for a power law fluid with $\nu_0 = 1.0$ and selected values of r. Figures 5.1, 5.2, and 5.3 show plots of the streamlines computed for $h = 1/32$ for $r = 2$, $r = 1.5$, and $r = 1.1$, respectively. As the power r in the constitutive law is decreased, we see a movement of the central vortex toward the top of the cavity, corresponding to an increase in viscosity.

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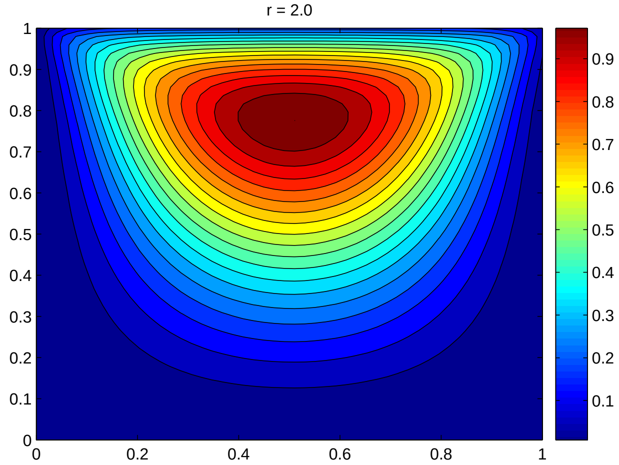

Figure 5.1: Streamlines for $r = 2.0$, driven cavity

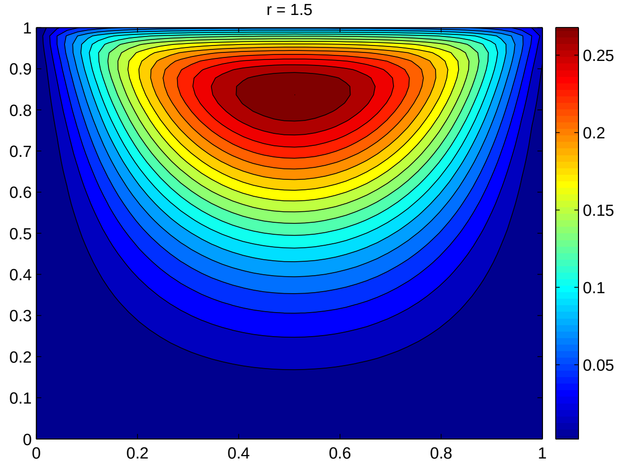

Figure 5.2: Streamlines for $r = 1.5$, driven cavity

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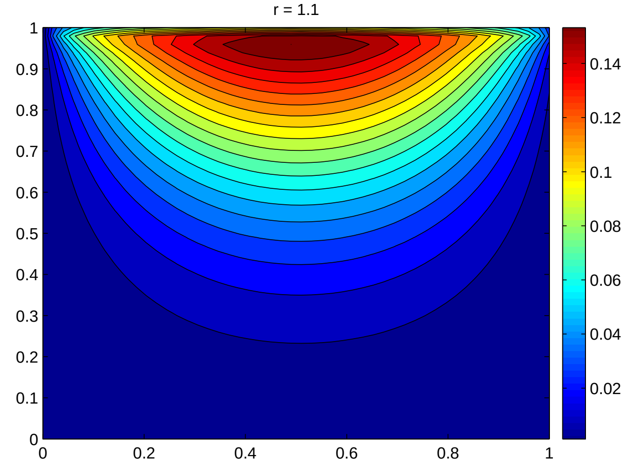

Figure 5.3: Streamlines for $r = 1.1$, driven cavity

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