Sections 7.2*-7.4*, 7.5, 7.7, 10.1, 4, 6.2, 6.3

For problems 1-3, find the derivative of the function. (6 points each)

1. $g(x) = \log_7(x^3 - 3)$ $= \dfrac{\ln(x^3-3)}{\ln(7)}$

$\therefore\ g'(x) = \dfrac{1}{\ln(7)(x^3-3)} \cdot 3x^2 = \dfrac{3x^2}{\ln(7)\cdot(x^3-3)}.$

2. $y = \ln\left(\sqrt[5]{\dfrac{x^2(x-1)}{(x+4)^2}}\right)$ $= \dfrac{1}{5}\ln\left(\dfrac{x^2(x-1)}{(x+4)^2}\right) = \dfrac{1}{5}\ln(x^2) + \dfrac{1}{5}\ln(x-1) - \dfrac{1}{5}((x+4)^2)$

$= \dfrac{2}{5}\ln(x) + \dfrac{1}{5}\ln(x-1) - \dfrac{2}{5}\ln(x+4).$

$\therefore\ y' = \dfrac{2}{5x} + \dfrac{1}{5(x-1)} - \dfrac{2}{5(x+4)}.$

3. $y = (3x)^{(1-2x)}$

$= \exp[\ln(3x^{(1-2x)})]$

$= \exp((1-2x)\ln(3x)]$

$\therefore\ y' = \exp[(1-2x)\ln(3x)] \cdot \left[(1-2x)\cdot\frac{1}{x} + (-2)\ln(3x)\right]$

$= (3x)^{(1-2x)}\left[\frac{1-2x}{x} - 2\ln(3x)\right]$

-or-

$\ln(y) = \ln(3x^{(1-2x)}) = (1-2x)\ln(3x)$

$\therefore$

$\dfrac{y'}{y} = \dfrac{1-2x}{x} + (-2)\ln(3x)$

$\Rightarrow y' = (3x)^{(1-2x)}\left[\dfrac{1-2x}{x} - 2\ln(3x)\right].$

For problems 4-7, evaluate the integral. If the method of substitution is required, be sure to **write out u and du**. For a definite integral give the **exact value**. (8 points each)

4. $\int_0^{\pi/2} \tan\frac{x}{2}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin(\frac{x}{2})}{\cos(\frac{x}{2})}\,dx$

Let $u = \cos(\frac{x}{2})$, $du = -\sin(\frac{x}{2})(\frac{1}{2})dx$,

$dx = \frac{-2\,du}{\sin(\frac{x}{2})}$,

$u(0) = \cos(0) = 1$

$u(\frac{\pi}{2}) = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$.

$$2\int_1^{1/\sqrt{2}} \frac{-1}{u}\,du = 2\int_{1/\sqrt{2}}^{1} \frac{1}{u}\,du$$

$$= 2\ln|u|\Big|_{1/\sqrt{2}}^{1} = \cancel{2\ln(1)}^{\,0} - 2\ln(\tfrac{1}{\sqrt{2}})$$

$$= 2\ln(\sqrt{2})$$

$$= \ln((\sqrt{2})^2) = \ln(2).$$

5. $\int \frac{3x+2}{\sqrt{1-x^2}}\,dx = \underbrace{\int \frac{3x}{\sqrt{1-x^2}}\,dx}_{\substack{\text{Let } u = 1-x^2 \\ du = -2x\,dx}} + \int \frac{2}{\sqrt{1-x^2}}\,dx$

$$= -\frac{3}{2}\int u^{-1/2}\,du + 2\arcsin(x) + C$$

$$= -3u^{1/2} + 2\arcsin(x) + C$$

$$= -3(1-x^2)^{1/2} + 2\arcsin(x) + C$$

6. $\int (1+e^{\tan\theta})\sec^2\theta\, d\theta$

$$\boxed{\begin{aligned} u &= \tan\theta \\ du &= \sec^2\theta\, d\theta \end{aligned}}$$

$$= \int 1+e^u\, du$$

$$= u + e^u + C$$

$$= \boxed{\tan\theta + e^{\tan\theta} + C.}$$

-or-

$$\int \sec^2\theta\, d\theta + \underbrace{\int e^{\tan\theta}\sec^2\theta\, d\theta}$$

$$\boxed{\begin{aligned} u &= \tan\theta \\ du &= \sec^2\theta\, d\theta \end{aligned}}$$

$$= \tan\theta + \int e^u du + C$$

$$= \tan\theta + e^u + C$$

$$= \boxed{\tan\theta + e^{\tan\theta} + C}$$

7. $\int \frac{5}{1+4x^2}dx$

Let $u = 2x$

$du = 2dx$.

$$= \int \frac{5}{1+(2x)^2}dx$$

$$= \frac{5}{2}\int \frac{1}{1+u^2}du = \frac{5}{2}\arctan(u) + C$$

$$= \boxed{\frac{5}{2}\arctan(2x)+C}$$

For problems 8-10, evaluate the limit. If the limit is indeterminate, give its form. Show all work. (6 points each)

8. $\lim_{x\to\infty} \frac{e^{3x}}{2x^2-1} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{3e^{3x}}{4x} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{9e^{3x}}{4} = \infty.$

9. $\lim_{x\to\pi/2} \overset{0}{(\pi/2-x)}\overset{\infty}{\tan x}$

$$= \lim_{x\to\pi/2} \frac{(\pi/2-x)}{\cot(x)} \overset{0/0}{=} \lim_{x\to\pi/2} \frac{-1}{-\csc^2(x)}$$

$$= \lim_{x\to\pi/2} \sin^2(x) = 1.$$

10. $\lim_{x\to 0}(1+5x)^{2/x}$ i.d. form of type 1^∞.

$$= \lim_{x\to 0} \exp\left(\ln\left[(1+5x)^{2/x}\right]\right)$$

$$= \lim_{x\to 0} \exp\left(\frac{2}{x}\ln(1+5x)\right) = \exp\left[\lim_{x\to 0} \overset{\infty}{\frac{2}{x}}\, \overset{0}{\ln(1+5x)}\right]$$

$$= \exp\left[\lim_{x\to 0} \frac{2\ln(1+5x)}{x}\right]$$

$$\overset{0/0}{=} \exp\left[\lim_{x\to 0} \frac{\frac{2}{1+5x}\cdot 5}{1}\right] = \exp\left[\lim_{x\to 0} \frac{10}{1+5x}\right]$$

$$= \exp[10] = e^{10}.$$

For the applications, 11-12, use **exact values**.

11. Find $f(t)$ if $f'(t) = \dfrac{e^t}{2+e^t}$ and $f(0) = \ln 6$. Show your work. (8 points)

$$f(t) = \int \frac{e^t}{2+e^t}\,dt$$

Let $u = 2+e^t$
$du = e^t\,dt$

$$= \int \frac{1}{u}\,du = \ln|u| + C$$
$$= \ln(2+e^t) + C.$$

Now find C.

$$f(0) = \ln(2+e^0) + C = \ln(6)$$
$$\Rightarrow C = \ln(6) - \ln(3) = \ln(2)$$
$$\therefore \quad f(t) = \ln(2+e^t) + \ln(2)$$

12. At sea level (h=0 km) atmospheric pressure is, on average, 1 atm. On Pike's Peak (h=4.3 km) the atmospheric pressure is about 6/10 atm. Assuming an exponential model for atmospheric pressure, what is the atmospheric pressure at the top of a mountain 3km above sea level? Give the answer in a complete sentence as an exact value. (8 pts)

$$P(h) = P_0 e^{kh}$$
$$P_0 = 1$$
$$P(h) = e^{kh}, \qquad P(4.3) = 6/10 = 3/5.$$

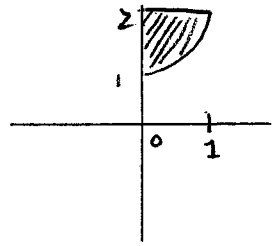
$$3/5 = e^{4.3 \cdot k}$$
$$\ln(3/5) = 4.3 \cdot k \Rightarrow \qquad k = \frac{\ln(3/5)}{4.3}.$$

$$\Rightarrow P(h) = e^{\ln(3/5)h/4.3}$$
$$P(3) = e^{\ln(3/5)\cdot \frac{3}{4.3}}$$
$$= (3/5)^{(3/4.3)} \text{ atm.}$$

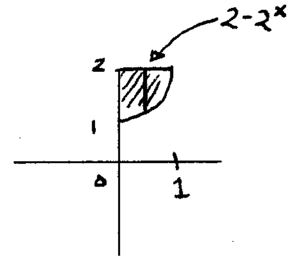
NOTE: When asked to set up an integral, do not simplify or evaluate the integral. All limits of integration must be written as exact values.

For problems 13-16, let R be the region bounded by $y = 2^x$, $y = 2$, and $x = 0$. Shade the region on the graph below. Set up the integral that gives the volume when the region is revolved around the

13. Shade (1 pt)



~~13.~~ 14. y-axis using the shell method. (5 points)

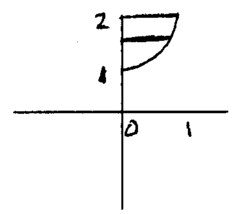


Representative height $= 2-2^x$ on $[0,1]$

$$V = \int_0^1 2\pi x\,(2-2^x)\,dx$$

~~14.~~ 15. y-axis using the disk/washer method. (5 points)

Representative area $= \pi(\log_2(y))^2$ on $[1,2]$



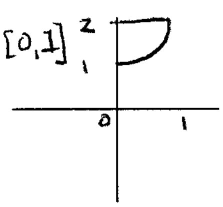
$$V = \int_1^2 \pi(\log_2(y))^2\,dy$$

~~15.~~ 16. x-axis. State the method that you use. (5 points)

Shell: Rep. width $= \log_2(y)$ on $[1,2]$

$$V = \int_1^2 2\pi y\,(\log_2(y))\,dy$$

Washer: Rep. Area. $= \pi(2)^2 - \pi(2^x)^2$ on $[0,1]$



$$V = \int_0^1 4\pi - \pi 4^x\,dx$$

~~16.~~ 17. line $y = 2$. State the method that you use. (5 points)

Disc: Rep area $= \pi(2-2^x)^2$.

$$V = \int_0^1 \pi(2-2^x)^2\,dx$$

Shell:

$$V = \int_1^2 2\pi(2-y)\,\log_2(y)\,dy$$