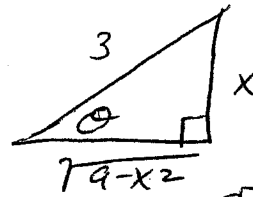
Version 1 - Alternate Solutions to Problem 2, 3, 4

(2) limits not changed - go back to x



$x = 3\sin\theta$

$\theta = \arcsin(x/3)$

$\cos\theta = \frac{\sqrt{9-x^2}}{3}$

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \frac{9}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]$$

$$= 9/2\left[\theta - \sin\theta\cos\theta\right]$$

$$= \frac{9}{2}\left[\arcsin(x/3) - \left(\frac{x}{3}\right)\frac{\sqrt{9-x^2}}{3}\right] + C$$

$$\int_0^{3/2} \frac{x^2}{\sqrt{9-x^2}}\,dx = \frac{9}{2}\left[\left(\arcsin\frac{1}{2} - \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right)\right] - 9/2\left(\arcsin 0 - 0\right)$$

$$= \frac{9}{2}\left[\pi/6 - \frac{\sqrt{3}}{4}\right]$$

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(3) $\int \sin^3 t\cos^3 t\,dt = \int \sin^3 t\cos^2 t\cos t\,dt$

Let $u = \sin t$

$du = \cos t\,dt$

$$= \int \sin^3 t(1-\sin^2 t)\cos t\,dt$$

$$= \int (\sin^3 t - \sin^5 t)\cos t\,dt$$

$$= \int (u^3 - u^5)\,du$$

$$= u^4/4 - u^6/6 + C$$

$$= \frac{1}{4}\sin^4 t - \frac{1}{6}\sin^6 t + C$$

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(4) $\int e^{-2x}\sin x\,dx$

$u = \sin x$, $du = \cos x\,dx$; $dv = e^{-2x}dx$, $v = -\frac{1}{2}e^{-2x}$

$$= -\frac{1}{2}\sin x\, e^{-2x} - \int -1/2\, e^{-2x}\cos x\,dx$$

$u = \cos x$, $du = -\sin x\,dx$; $dv = e^{-2x}dx$, $v = -1/2\, e^{-2x}$

$$= -1/2\sin x\, e^{-2x} + \frac{1}{2}\int e^{-2x}\cos x\,dx$$

$$= -1/2\sin x\, e^{-2x} + \frac{1}{2}\left[-1/2\cos x\, e^{-2x} - \int(-1/2\, e^{-2x})(-\sin x)\,dx\right]$$

Then $\int e^{-2x}\sin x\,dx = -\frac{1}{2}e^{-2x}\sin x - \frac{1}{4}e^{-2x}\cos x - \frac{1}{4}\int e^{-2x}\sin x\,dx$

$$\frac{5}{4}\int e^{-2x}\sin x\,dx = -\frac{1}{2}e^{-2x}\sin x - \frac{1}{4}e^{-2x}\cos x + C$$

$$\int e^{-2x}\sin x\,dx = -2/5\, e^{-2x}\sin x - \frac{1}{5}e^{-2x}\cos x + C$$

Mth Sc 108: Exam 2 Spring 2003 Version 1

Sections 8.1-8.5, 8.8

Note: For all problems, if the method of substitution is required be sure to write out u and du. For a definite integral give the exact value. For an improper integral determine if it is convergent or divergent.

For problems 1-5, evaluate the integral.

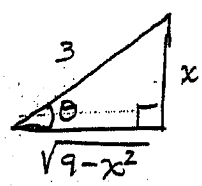
1. (9 points) $\int x e^{3x}\,dx$

$u = x \qquad dv = e^{3x}dx$

$du = dx \qquad v = \frac{1}{3}e^{3x}$

$$\int x e^{3x}dx = \frac{x}{3}e^{3x} - \frac{1}{3}\int e^{3x}dx = \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x} + C$$

2. (12 points) $\int_0^{3/2} \frac{x^2}{\sqrt{9-x^2}}\,dx$



$\sin\theta = \frac{x}{3}, \quad x = 3\sin\theta, \quad dx = 3\cos\theta\,d\theta$

$\cos\theta = \frac{\sqrt{9-x^2}}{3}, \quad \sqrt{9-x^2} = 3\cos\theta$

$\theta = \sin^{-1}(x/3), \quad \sin^{-1}0 = 0$

$\sin^{-1}(1/2) = \pi/6$

$$\int_0^{3/2}\frac{x^2}{\sqrt{9-x^2}}dx = \int_0^{\pi/6}\frac{9\sin^2\theta}{3\cos\theta}\,3\cos\theta\,d\theta$$

$$= 9\int_0^{\pi/6}\sin^2\theta\,d\theta = 9\int_0^{\pi/6}\frac{1-\cos(2\theta)}{2}d\theta$$

$$= \frac{9}{2}\int_0^{\pi/6} 1-\cos(2\theta)\,d\theta = \frac{9}{2}\left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\pi/6}$$

$$= \frac{9}{2}\left[\pi/6 - \frac{1}{2}\sin\pi/3 - 0\right]$$

$$= \frac{9}{2}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right] = \frac{3\pi}{4} - \frac{9\sqrt{3}}{8}$$

3. (9 points) $\int \sin^3 t \cos^3 t \, dt$

$$= \int \sin^2 t \sin t \cos^3 t \, dt$$
$$= \int (1-\cos^2 t) \sin t \cos^3 t \, dt$$
$$= \int (\cos^3 t \sin t) dt - \int \cos^5 t \sin t \, dt$$
$$= -\int u^3 du + \int u^5 du$$
$$= -u^4/4 + u^6/6 + C$$
$$= \frac{1}{6}\cos^6 t - 1/4 \cos^4 t + C$$

$u = \cos t$
$du = -\sin t \, dt$

4. (9 points) $\int e^{-2x} \sin x dx$

$u = e^{-2x}$ $\quad dv = \sin x \, dx$
$du = -2e^{-2x} dx$ $\quad v = -\cos x$

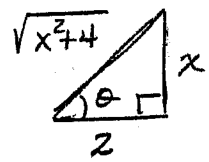
$$\int e^{-2x} \sin x \, dx = -e^{-2x}\cos x - 2\int e^{-2x}\cos x \, dx$$

$u = e^{-2x}$ $\quad dv = \cos x \, dx$
$du = -2e^{-2x} dx$ $\quad v = \sin x$

$$= -e^{-2x}\cos x - 2\left[e^{-2x}\sin x + 2\int e^{-2x}\sin x \, dx\right]$$
$$= -e^{-2x}\cos x - 2e^{-2x}\sin x - 4\int e^{-2x}\sin x \, dx$$
$$5\int e^{-2x}\sin x \, dx = -e^{-2x}\cos x - 2e^{-2x}\sin x$$
$$\int e^{-2x}\sin x \, dx = \frac{-1}{5}e^{-2x}[\cos x + 2\sin x] + C.$$

5. (9 points) $\int \frac{x^3}{\sqrt{4+x^2}}dx$

(Hint: Use the substitution $x = a\tan\theta$. You need to decide what a is!)



$\tan\theta = \frac{x}{2}$, $x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$

$\sec\theta = \frac{\sqrt{x^2+4}}{2}$, $\underline{\sqrt{x^2+4} = 2\sec\theta}$

$$\int \frac{x^3}{\sqrt{4+x^2}}dx = \int \frac{8\tan^3\theta}{2\sec\theta} 2\sec^2\theta\, d\theta$$

$$= 8\int \tan^3\theta \sec\theta\, d\theta$$

$$= 8\int \tan^2\theta \tan\theta \sec\theta\, d\theta$$

$$= 8\int (\sec^2\theta - 1)\tan\theta\sec\theta\, d\theta$$

$u = \sec\theta$
$du = \sec\theta\tan\theta\, d\theta$

$$= 8\int (u^2-1)\,du$$

$$= \frac{8}{3}u^3 - 8u + C$$

$$= \frac{8}{3}\sec^3\theta - 8\sec\theta + C.$$

$$= \frac{1}{3}(x^2+4)^{3/2} - 4(x^2+4)^{1/2} + C$$

6. (9 points) Write out the form of the partial fraction decomposition of the function. **Do not** determine the numerical values of the coefficients.

$$\frac{x^3+x+3}{x^2(x+1)^3(x^2+4)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}+\frac{E}{(x+1)^3}+\frac{Fx+G}{x^2+4}$$

7. (12 points) Evaluate the integral.

$$\int \frac{-3x^2+x+5}{x(x^2+1)}dx$$

$$\frac{-3x^2+x+5}{x(x^2+1)} = \frac{A}{x}+\frac{Bx+C}{x^2+1}$$

$$-3x^2+x+5 = A(x^2+1)+Bx^2+Cx$$

$\underline{x=0:}$ $\underline{5=A}$ & $-3x^2+x+5 = 5x^2+5+Bx^2+Cx$

$$-8x^2+x = Bx^2+Cx$$

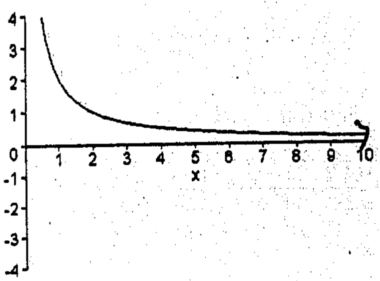
$$\underline{B=-8 \text{ \& } C=1}$$

$$\frac{-3x^2+x+5}{x(x^2+1)} = \frac{5}{x}+\frac{-8x+1}{x^2+1}$$

$$\int\frac{-3x^2+x+5}{x(x^2+1)}dx = \int\frac{5}{x}dx - 8\int\frac{x}{x^2+1}dx + \int\frac{dx}{x^2+1}$$

$$= 5\ln|x| - 4\ln|x^2+1| + \tan^{-1}x + c$$

8. (12 points) Set up and analytically find (if possible) the exact value for the definite integral that gives the area of the region bounded by the graph of $y=\dfrac{2}{x}$ and the x-axis for $1 \le x < \infty$. **Shade the region on the graph below and state the answer in a complete sentence.**



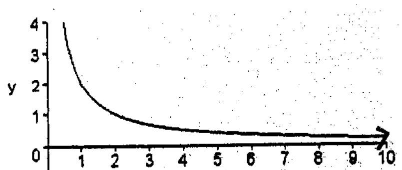
$$A=\int_1^{\infty}\frac{2}{x}\,dx=\lim_{t\to\infty}\int_1^{t}\frac{2}{x}\,dx = 2\lim_{t\to\infty}\Big[\ln|x|\Big]_1^{t}$$

$$=2\lim_{t\to\infty}\Big[\ln t-\ln 1\Big]$$

$$=2\lim_{t\to\infty}\ln t=\infty$$

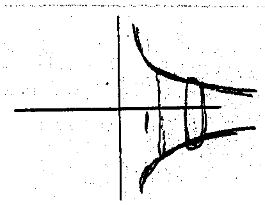
The area of the region is infinite.

9. (12 points) Set up and analytically find (if possible) the exact value for the definite integral that gives the volume of the solid formed when the region bounded by the graph of $y=\dfrac{2}{x}$ and the x-axis for $1 \le x < \infty$ is revolved around the x-axis. Use the disk method. **Give your answer in a complete sentence.**



$A_D=\pi r^2 = \pi\ 4/x^2$

$r(x)=2/x$



$$V=\int_1^{\infty}\pi\ 4/x^2\,dx = 4\pi\lim_{t\to\infty}\int_1^{t}x^{-2}\,dx$$

$$=4\pi\lim_{t\to\infty}\Big[-x^{-1}\Big]_1^{t}=-4\pi\lim_{t\to\infty}\Big[\frac{1}{t}-1\Big]$$

$$=-4\pi(-1)=4\pi.$$

The volume of the solid is 4π sq units

10. (12 points) Find $f(t)$ if $f'(t) = \ln(t^3)$ and $f(e) = \ln 2$. Show all work. Make sure to use an integration technique in this problem and not just a memorized formula.

$$f = \int \ln x^3 \, dx = 3 \int \ln x \, dx \qquad u = \ln x \quad dv = dx \qquad du = \frac{1}{x} \quad v = x$$

$$= 3\left[x \ln x - \int dx\right]$$

$$= 3x \ln x - 3x + C$$

$$f(e) = 3e \ln e - 3e + C = \ln 2 \Rightarrow C = \ln 2$$

$$f(t) = 3x \ln x - 3x + \ln 2.$$