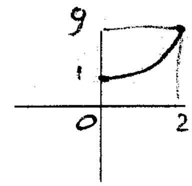
NOTE: When asked to set up an integral, do not simplify or evaluate the integral. All limits of integration must be written as exact values.

For problems 1-2, let $f(x) = 3^x$ on the interval $0 \le x \le 2$.

$y = 3^x$, $y' = 3^x \ln 3$

$x = \log_3 y$, $x' = \dfrac{1}{y \cdot \ln 3}$



1. Set up (DO NOT EVALUATE OR SIMPLIFY.) the integral that gives the arc length of the curve on the interval.
   a. With respect to x. (5 pts)

$$\ell = \int_0^2 \sqrt{1 + (3^x \ln 3)^2}\, dx$$

   b. With respect to y. (5 pts)

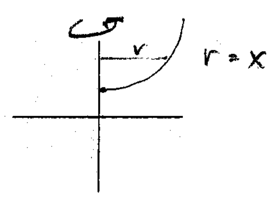
$$\ell = \int_1^9 \sqrt{1 + \left(\frac{1}{y \ln 3}\right)^2}\, dy$$

2. Set up (DO NOT EVALUATE OR SIMPLIFY.) the integral that gives the surface area of the solid generated when the curve on the given interval is rotated around the y-axis. (5 pts)

$$S = 2\pi \int_0^2 x \cdot \sqrt{1 + (3^x \ln 3)^2}\, dx$$

or

$$S = 2\pi \int_1^9 \log_3 y \sqrt{1 + \left(\frac{1}{y \ln 3}\right)^2}\, dy$$



For problems 3-4, suppose that 3 J of work are needed to stretch a spring from its natural length of 30 cm to 40 cm. Give the answer with proper units.

3. Find the spring constant. Show your work. (5 pts)

10 cm = 0.1 m

$$W = \int_a^b f(x)\, dx$$

where

$f(x) = k \cdot x$

by Hooke's Law

$$3 = \int_0^{0.1} kx\, dx = \frac{kx^2}{2}\Big|_0^{0.1} = \frac{0.01k}{2}$$

hence $k = 600$ $\left(\frac{N}{m}\right)$

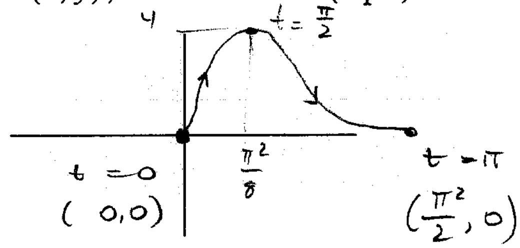
4. Determine how much work is needed to stretch the spring from 35 cm to 45 cm. Show your work. (5 pts)

$$W = \int_{0.05}^{0.15} 600 \cdot x\, dx = 300x^2 \Big|_{0.05}^{0.15} = 6\ (J)$$

Answ: The work of 6 J is needed to stretch the spring from 35 cm to 45 cm.

For problems 5-10 a curve is defined by the parametric equations $x = \frac{t^2}{2}$ **and** $y = 2 - 2\cos(2t)$ **for** $0 \le t \le \pi$.

5. Use your calculator to sketch the graph of the parametric curve. Indicate direction and label the initial and terminal points, with both t and (x, y), as exact values. (5 pts)



6. Analytically (without the use of your calculator) find $\frac{dy}{dx}$ in parametric form. Show all work. (5 pts)

$$\left.\begin{aligned} \frac{dy}{dt} &= 4\sin 2t \\ \frac{dx}{dt} &= t \end{aligned}\right\} \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4\sin 2t}{t}$$

7. For the interval $(0, \pi)$, analytically (without using your calculator) find all points, both t and (x, y) as exact values, on the curve where the tangent line is horizontal. Show your work. (5 pts)

horizontal tangent $\Leftrightarrow$ $\frac{dy}{dt} = 0 \wedge \frac{dx}{dt} \neq 0$

$\frac{dy}{dt} = 4\sin 2t = 0 \Rightarrow 2t = 0, \pi \rightarrow t = 0, \frac{\pi}{2}$

no horizontal tangent here, b/c $\left.\frac{dx}{dt}\right|_{t=0} = 0$

so There is a horizontal tangent at $t = \frac{\pi}{2}$, $(\frac{\pi^2}{8}, 4)$

8. Analytically find the equation of the line tangent to the curve at $t = \frac{\pi}{6}$. **Express all coefficients as exact values.** Show your work. (5 pts)

eqn. of the tangent line is:

$y - y_0 = m(x - x_0)$

$$y_0 = y\left(\frac{\pi}{6}\right) = 2 - 2\cos\left(2 \cdot \frac{\pi}{6}\right) = 1$$

$$m = \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{6}} = \frac{4\sin\frac{\pi}{3}}{\frac{\pi}{6}} = \frac{4\frac{\sqrt{3}}{2}}{\frac{\pi}{6}} = \frac{12\sqrt{3}}{\pi}$$

$$x_0 = x\left(\frac{\pi}{6}\right) = \frac{\frac{\pi^2}{36}}{2} = \frac{\pi^2}{72}$$

$$y - 1 = \frac{12\sqrt{3}}{\pi}\left(x - \frac{\pi^2}{72}\right)$$

eqn. of the tangent line at $t = \frac{\pi}{6}$

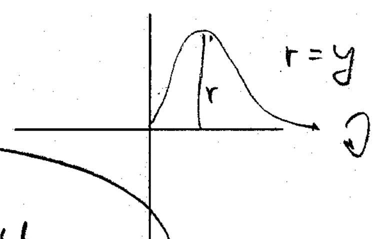
9. Set up (DO NOT EVALUATE OR SIMPLIFY.) the integral that gives the arc length of the curve on the interval. (5 pts)

$$l = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$l = \int_0^{\pi} \sqrt{t^2 + 16\sin^2 2t}\, dt$$

10. Set up (DO NOT EVALUATE OR SIMPLIFY.) the integral (in terms of t) that gives the surface area of the solid generated when the curve on the given interval is rotated around the x-axis. (5 pts)

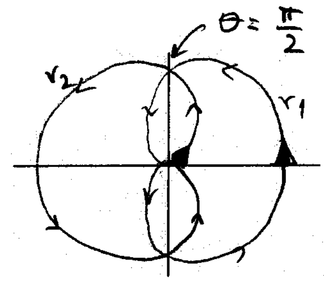
$$S = 2\pi \int_a^b y\sqrt{x'^2 + y'^2}\, dt$$



$$S = 2\pi \int_0^{\pi} (2 - 2\cos 2t)\sqrt{t^2 + 16\sin^2 2t}\, dt$$

For problems 11-14, use the polar equations $r_1 = 1+\cos\theta$ **and** $r_2 = 1-\cos\theta$ **on** $[0,2\pi]$.

11. Accurately sketch and label r_1 and r_2. (3 pts)



12. a. Analytically find and **label** the polar coordinates (r,θ), as exact values, of all points of intersection (**collision**) of the two curves, r_1 and r_2 on the interval $[0,2\pi]$. Show your work. (5 pts)

finding the points of collision:

$r_1 = r_2$

$\cancel{1}+\cos\theta = \cancel{1}-\cos\theta$

$2\cos\theta = 0$

$\theta = \frac{\pi}{2}, \frac{3}{2}\pi$

$r_1(\frac{\pi}{2}) = 1+\cos\frac{\pi}{2} = 1 = r_2(\frac{\pi}{2})$

$r_1(\frac{3}{2}\pi) = 1+\cos\frac{3}{2}\pi = 1 = r_2(\frac{3}{2}\pi)$

$\therefore$ Collision points: $(1, \frac{\pi}{2})$, $(1, \frac{3}{2}\pi)$

b. Analytically find the Cartesian coordinates (x, y), as exact values, of the polar points you found in part (a). Show your work. (4 pts)

$$\begin{cases} x = r\cdot\cos\theta \\ y = r\cdot\sin\theta \end{cases}$$

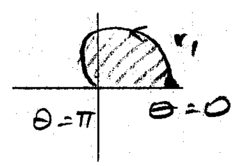
$(r,\theta) = (1, \frac{\pi}{2}) \longrightarrow \begin{cases} x = 1\cdot\cos\frac{\pi}{2} \\ y = 1\cdot\sin\frac{\pi}{2} \end{cases} \Rightarrow (x,y) = (0,1)$

$(r,\theta) = (1, \frac{3}{2}\pi) \longrightarrow \begin{cases} x = 1\cdot\cos\frac{3}{2}\pi = 0 \\ y = 1\cdot\sin\frac{3}{2}\pi = -1 \end{cases} \Rightarrow (x,y) = (0,-1)$

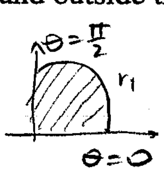
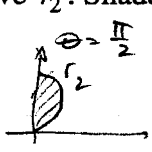
13. Set up (DO NOT EVALUATE OR SIMPLIFY.) the integral that gives the area enclosed by the curve r_1 above the horizontal axis. (6 pts)

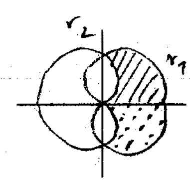
$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$$

$$A = \frac{1}{2}\int_{0}^{\pi} (1+\cos\theta)^2\, d\theta$$



14. Set up (DO NOT EVALUATE OR SIMPLIFY.) the integral that gives the area that lies inside the curve r_1 and outside the curve r_2. Shade this region on the graph. (8 pts)

$\frac{1}{2} A =$  $-$ 



$$A = 2\cdot\left[\frac{1}{2}\int_0^{\frac{\pi}{2}} r_1^2 - r_2^2\, d\theta\right] = \int_0^{\frac{\pi}{2}} \left((1+\cos\theta)^2 - (1-\cos\theta)^2\right) d\theta$$

Mth Sc 108: Exam 3 Spring 2003 Version 1

Sections 6.4; 9.1-2; 11.1-5; 12.1-2

Note: When discussing convergence, please state (in words) whether you are talking about a sequence or a series.

For problems 15-19, let $a_n = \dfrac{4}{2n+3} - \dfrac{4}{2n+5}$ **for** $n \geq 1$.

15. List the first four terms of $\{a_n\}$. Do not simplify. (2 points)

$a_1 = \frac{4}{5} - \frac{4}{7}$ $\qquad$ $a_2 = \frac{4}{7} - \frac{4}{9}$

$a_3 = \frac{4}{9} - \frac{4}{11}$ $\qquad$ $a_4 = \frac{4}{11} - \frac{4}{13}$

16. Analytically determine if $\{a_n\}$ converges. (3 points)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{4}{2n+3} - \frac{4}{n+5} \right) = 0 - 0 = 0$$

The limit of this sequence is 0

17. Determine a formula for $s_n = a_1 + a_2 + a_3 + \dots + a_n$. (3 points)

$$s_n = \frac{4}{5} - \cancel{\frac{4}{7}} + \cancel{\frac{4}{7}} - \cancel{\frac{4}{9}} + \cancel{\frac{4}{9}} - \cancel{\frac{4}{11}} + \dots + \cancel{\frac{4}{2n+3}} - \frac{4}{2n+5} = \frac{4}{5} - \frac{4}{2n+5}$$

$$S_n = \frac{4}{5} - \frac{4}{2n+5}$$

18. Analytically determine if $\{s_n\}$ converges. (3 points)

$$\lim s_n = \lim \left(\frac{4}{5} - \frac{4}{2n+5} \right) = \frac{4}{5} - 0 = \frac{4}{5}$$

the limit of the sequence $\{s_n\}$ is $\frac{4}{5}$

19. Does $\sum_{n=1}^{\infty} a_n$ converge? Explain why or why not. (3 points)

The series $\sum a_n$ converges

b/c $\sum a_n = \lim s_n = \frac{4}{5}$

$\therefore \sum a_n = \frac{4}{5}$

For problems 20-21, state if the expression is a sequence or series and determine if it converges or diverges. If a sequence converges, find its limit. If a series converges analytically find its exact sum. Show all work analytically. Summarize your conclusion in sentence form. (5 points each)

20. $\sum_{n=0}^{\infty} \frac{(-4)^{n+2}}{5^n}$ $= 16 - \frac{4^3}{5} + \frac{4^4}{5^2} - \frac{4^5}{5^3} + \dots$

This is a geometric series

$a = 16, \; r = \frac{4}{5}, \quad |r| < 1$

it converges

$s = \frac{a}{1-r} = \frac{16}{1-\frac{4}{5}} = 5 \cdot 16 = 80$

$\therefore \sum_{n=0}^{\infty} \frac{(-4)^{n+2}}{5^n} = 80$

21. $\sum_{n=0}^{\infty} \frac{4n^2+1}{1+n}$

a series

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4n^2+1}{1+n} = \infty \neq 0$

By the Divergence Test

the above series diverges