General Directions: Show all work analytically. When determining the convergence or divergence of a series, state the test (one on the handout) you use and show in detail how you use it. Summarize your conclusion in sentence form.

For problems 1-4, determine if the series converges absolutely, converges conditionally, or diverges. State the test (one on the handout) you use and show in detail how you use it. Show all work analytically. Summarize your conclusion in sentence form. (~~10~~ 12 points each)

1. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$

Using the Ratio Test, we see:

$$\lim_{n\to\infty} \left| \frac{(-2)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-2)^n} \right| = \lim_{n\to\infty} \frac{2^{n+1} \cdot n!}{(n+1)!\, 2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0. < 1.$$

The series $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$ converges absolutely by the Ratio Test.

2. $\sum_{n=1}^{\infty} \frac{5-\sin n}{3^n}$

Notice that $0 \le 5 - \sin n \le 5+1 = 6$. Thus $0 \le \frac{5-\sin n}{3^n} \le \frac{6}{3^n}$.

Now the series $\sum_{n=1}^{\infty} \frac{6}{3^n} = \frac{6}{3} + \frac{6}{3^2} + \frac{6}{3^3} = \frac{6}{3}\left[1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots\right]$

is a geometric series with $a = 2$ and $r = \frac{1}{3}$. Since $|r| = \frac{1}{3} < 1$, the Geometric Series Test implies $\sum_{n=1}^{\infty} \frac{6}{3^n}$ converges.

Thus $\sum_{n=1}^{\infty} \frac{5-\sin n}{3^n}$ converges by the Direct Comparison Test with $\sum \frac{6}{3^n}$. Since $\sum \left| \frac{5-\sin n}{3^n} \right| = \sum \frac{5-\sin n}{3^n}$, the series converges absolutely.

3. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3}{\sqrt{n\ +1}}$

We first consider $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}3}{\sqrt{n+1}} \right| = \sum_{n=1}^{\infty} \frac{3}{\sqrt{n+1}}$.

Notice that

$$\lim_{n\to\infty} \frac{\frac{3}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{3}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{1} = \lim_{n\to\infty} \frac{3}{\sqrt{1+1/n}} = 3.$$

The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a p-Series with $p = \frac{1}{2} \le 1$, and so by the p-Series Test, diverges. Thus the Limit Comparison Test implies $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n+1}}$ must also diverge. (so $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3}{\sqrt{n+1}}$ does not converge abs.).

Check for conditional convergence:

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3}{\sqrt{n+1}}$ is an alternating series with $b_n = \frac{3}{\sqrt{n+1}} \geq 0$

Notice: $\lim_{n\to\infty} \frac{3}{\sqrt{n+1}} = 0$, and $n+1 \geq n \Rightarrow \sqrt{n+1} \geq n \Rightarrow$

$$\frac{1}{\sqrt{n+1}} \le \frac{1}{\sqrt{n}} \rightarrow \frac{3}{\sqrt{n+1}} \le \frac{3}{\sqrt{n}} \Rightarrow b_{n+1} \le b_n.$$

Hence $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 3}{\sqrt{n+1}}$ converges by the Alternating Series Test. The convergence is conditional.

4. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^{1/3}}$

Let $f(x)=\dfrac{1}{x(\ln x)^{1/3}}$. Notice that f is nonneg on $[2,\infty)$

and continuous on $[2,\infty)$.

$$f'(x) = -\frac{\left((\ln x)^{1/3} + x\cdot\frac{1}{3}(\ln x)^{-2/3}\cdot\frac{1}{x}\right)}{\left[x(\ln x)^{1/3}\right]^2} = \frac{-(3\ln x+1)}{3x^2(\ln x)^{4/3}} < 0$$

for x in $[2,\infty)$. Hence f is decreasing on $[2,\infty)$.

So the Integral Test applies.

$$\int_2^{\infty}\frac{1}{x(\ln x)^{1/3}}\,dx = \lim_{c\to\infty}\int_2^{c}\frac{1}{x(\ln x)^{1/3}}\,dx \qquad \left(\begin{array}{l}\text{Let } u=\ln x\\ du=\frac{1}{x}\,dx\end{array}\right)$$

$$= \lim_{c\to\infty}\int_{\ln 2}^{\ln c} u^{-1/3}\,du$$

$$= \lim_{c\to\infty} \frac{3}{2}u^{2/3}\Big|_{\ln 2}^{\ln c}$$

$$= \lim_{c\to\infty} \frac{3}{2}(\ln c)^{2/3} - \frac{3}{2}(\ln 2)^{2/3}$$

$$= \infty.$$

Thus the improper integral diverges.

The Integral Test implies that the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^{1/3}}$ also diverges.

For problems 5-11 consider the power series

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n (x-3)^n = 1-\frac{1}{2}(x-3)+\frac{1}{4}(x-3)^2+\cdots+\left(-\frac{1}{2}\right)^n(x-3)^n+\cdots$$

5. What is the center of the power series? (~~2~~ 3 points)

The center of the power series is 3.

6. Analytically find the radius of convergence. (~~10~~ 8 points)

Notice that $\sum_{n=0}^{\infty}(-\frac{1}{2})^n(x-3)^n$ is a geom. series with $r = -\frac{(x-3)}{2}$. The Geom. Series Test says the series will converge if $|r| = \left|-\frac{(x-3)}{2}\right| < 1 \Rightarrow |x-3| < 2$.

So the radius of convergence is 2.

or:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-3)^{n+1}}{2^{n+1}}\cdot\frac{2^n}{(-1)^n(x-3)^n}\right| = \lim_{n\to\infty}\frac{|x-3|^{n+1}\,2^n}{2^{n+1}\,|x-3|^n} = \lim_{n\to\infty}\frac{|x-3|}{2} = \frac{|x-3|}{2}.$$

The Ratio Test $\Rightarrow$ the series will converge (abs) whenever $\frac{|x-3|}{2} < 1 \Rightarrow |x-3| < 2$.

The radius of convergence is 2.

7. Determine the interval of convergence. Be sure to include a check for convergence at the endpoints of the interval. (6 points)

Geom. Series:

The geom. series test $\Rightarrow$ the series will diverge if $\left|-\frac{(x-3)}{2}\right| \geq 1 \Rightarrow$ The interval of convergence is $(1,5)$.

(Ratio Test above)

If $x=1$: the series becomes $\sum_{n=0}^{\infty}(-\frac{1}{2})^n(-2)^n = \sum_{n=0}^{\infty}(1)^n$. This is a geometric series with $|r| = |1| = 1 \geq 1$, and so diverges by the Geom. Series Test.

If $x=3$: the series becomes $\sum_{n=0}^{\infty}(-\frac{1}{2})^n(2)^n = \sum_{n=0}^{\infty}(-1)^n$. This is a geometric series with $|r| = |-1| = +1 \geq 1$, and so diverges by the Geom. Series Test.

The interval of convergence is $(1,5)$.

8. What is the sum of the power series? (4 points)

Using the Geom. Series Test, the sum

is $\frac{1}{1-\left(-\frac{(x-3)}{2}\right)} = \frac{2}{2+x-3} = \frac{2}{x-1}$.

for $1 < x < 5$.

9. What series do you get if you differentiate the power series term by term? (4 points)

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n(x-3)^n\right] = \sum_{n=1}^{\infty} n\left(-\frac{1}{2}\right)^n(x-3)^{n-1}$$

is the series.

10. What is the radius of convergence of the series in number 9? (4 points)

The radius of convergence is still 2.

11. What is the sum of the series in number 9? (4 points)

The sum of the series is $\frac{d}{dx}\left[\frac{2}{x-1}\right] = (-2)(x-1)^{-2} = \frac{-2}{(x-1)^2}$.

For problems 12-16, let $f(x) = \cos x$ and $g(x) = f(\sqrt{x})$. The Maclaurin series for $f(x)$ is

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

12. Using the Maclaurin series for $\cos x$, determine the Maclaurin series for $g(x)$. (4 points)

$$g(x) = \cos(\sqrt{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}$$

13. Calculate the series representation for the general antiderivative of $g(x)$. (5 points)

$$\int g(x)\,dx = \int \cos(\sqrt{x})\,dx = \sum_{n=0}^{\infty}\left[\int \frac{(-1)^n x^n}{(2n)!}\,dx\right] = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(2n)!(n+1)}\right] + C.$$

14. Assume that the series you found in problem 13 converges for all x. Determine a series that converges to the exact area under the curve $y = g(x)$ from $x = 0$ to $x = 1$. (5 points)

$$\int_0^1 g(x)\,dx = \int_0^1 \cos(\sqrt{x})\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(2n)!\,(n+1)}\Bigg|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(n+1)}$$

15. Estimate the area in problem 14 using the first 3 non-zero terms of the series. State your estimate as an exact value. (5 points)

The area is estimated to be $1 - \frac{1}{4} + \frac{1}{4!\cdot 3} = \frac{55}{72}$.

16. Give a bound on the error in your estimate in problem 15. State your bound as an exact value. (5 points)

The error has bound

$$|s - s_2| \le b_3 = \frac{1}{6!\cdot 4} = \frac{1}{2880}.$$