# Mathematical Modeling using Partial Differential Equations (PDE's)

145. Physical Models: heat conduction, vibration.

146. Mathematical Models: why build them. The solution to the mathematical model will be an approximation to the physical solution. The accuracy of this approximation depends on the care with which the mathematical model was made and how well physical parameters are measured or estimated. If any important physical phenomena are not modeled, the accuracy may be poor. Mathematical models are an important design tool.

147. Example: Heat conduction in a rod of length $l$ insulated on its sides so that heat can only flow along the axis of the rod. Introduce a coordinate system so that the rod is positioned along an $x$ axis with the left end at $x = 0$ and the right end at $x = l$. The mathematical model is the following initial-boundary value problem (I-BVP) where $x$ is the position in the rod, $t$ is time, and $u$ denotes the temperature in the rod. Find $u = u(x,t)$ so that

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) - \alpha \frac{\partial^2 u}{\partial x^2}(x,t) &= F(x,t), \quad 0 < x < l, \ t > 0 \\ u(0,t) &= h_1(t), \quad t \geq 0 \\ u(l,t) &= h_2(t), \quad t \geq 0 \\ u(x,0) &= f(x), \quad 0 \leq x \leq l. \end{aligned}$$

148. Solving the mathematical model: Methods used for solving I-BVP's include separation of variables, finite element, and finite difference methods. The method generally learned first is the method of separation of variables. It is based on the concept of generalized Fourier series, which in turn is closely associated with the the minimum distance problem.

149. Separation of variables requires that the physical region have a boundary which is made up of pieces of coordinate curves or surfaces in an orthogonal system. Review of Cartesian, cylindrical, and spherical coordinates.

150. Heat conduction in a rod. Various boundary conditions (BC's): specifying temperature, specifying heat flux, and Newton's law of cooling. Fourier's law.

151. Heat conduction in a rod with temperature fixed at 0 at the ends and no source term in the heat equation. Separation of variables.

    - Finding nontrivial separable solutions to the homogeneous equations.
    - First example of a regular Sturm-Liouville (S-L) problem. Eigenvalues and eigenfunctions. Analogy with the eigenvalue problem for a symmetric matrix.

- Review of solution of second order, constant coefficient, homogeneous ordinary differential equations (ODE's) from sophomore differential equations.
- Solving the ODE involving time $t$.
- The trial solution, an infinite series of solutions to the homogeneous equations.
- Finding the coefficients. The Fourier sine series.
- Partial sums of the trial solution provide approximations.

152. The inner product space setting. Each partial sum of the trial solution is the solution to a minimum distance problem.

153. **Definition** Let $V$ be a normed, vector space. A sequence of vectors $\{\mathbf{x}_n\}$ in $V$ converges to $\mathbf{x}$ in $V$, if $\|\mathbf{x}_n - \mathbf{x}\| \to 0 \quad \text{as} \quad n \to \infty$. The vector $\mathbf{x}$ is called the limit of the sequence.

154. **Definition** Let $\{\mathbf{x}_n\}$ be a sequence of vectors in a normed, vector space $V$. If the sequence of partial sums, $\{\mathbf{y}_n\}$, defined by

$$\mathbf{y}_n = \sum_{k=1}^{n} \mathbf{x}_k \ ,$$

converges to a limit vector $\mathbf{y}$, we write

$$\mathbf{y} = \sum_{k=1}^{\infty} \mathbf{x}_k \ .$$

155. **Definition** Let $V$ be a normed, vector space A sequence of vectors $\{\mathbf{x}_n\}$ in $V$ is Cauchy if $\|\mathbf{x}_n - \mathbf{x}_m\| \to 0 \ \text{ as } \ n, m \to \infty$.

156. **Definition** Let $V$ be a normed, vector space. $V$ is called complete if every Cauchy sequence converges to a vector in $V$.

157. **Definition** A Hilbert space $H$ is an inner product space so that
    - $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}, \ \mathbf{x} \in H$,
    - $H$ is complete.

158. **Definition** Let $V$ be an inner product space.
    - A sequence of nontrivial vectors, $\{\mathbf{x}_n\}$, is called orthogonal if $\mathbf{x}_n \cdot \mathbf{x}_m = 0$ for $n \neq m$.
    - A sequence, $\{\mathbf{x}_n\}$, is called orthonormal if $\mathbf{x}_n \cdot \mathbf{x}_m = \delta_{nm}$.

159. **Definition** A sequence, $\{\mathbf{x}_n\}$, is called an orthogonal basis for a Hilbert space $H$ if

- it is orthogonal,
- $\mathbf{z} \cdot \mathbf{x}_n = 0$ for all $n$ implies that $\mathbf{z} = \mathbf{0}$.

160. **Theorem** Let $\{\mathbf{x}_n\}$ be an orthogonal sequence in a Hilbert space $H$. The the following are equivalent.

    - $\{\mathbf{x}_n\}$ is an orthogonal basis.
    - $\mathbf{x} \cdot \mathbf{x} = \sum_{n=1}^{\infty} |c_n|^2 (\mathbf{x}_n \cdot \mathbf{x}_n)$ for all $\mathbf{x} \in H$.
    - $\mathbf{x} = \sum_{n=1}^{\infty} c_n \mathbf{x}_n$for all$\mathbf{x} \in H$.

    where
    $$c_n = \frac{\mathbf{x} \cdot \mathbf{x}_n}{\mathbf{x}_n \cdot \mathbf{x}_n} .$$

161. **Definition** (The inner product space $L_r^2[a, b]$) Let $r$ be a nonnegative function defined on $[a, b]$. Define
    $$L_r^2[a, b] = \left\{ f \ : \ \int_a^b |f(x)|^2 r(x) dx < \infty \right\}.$$
    Let $f, g \in L_r^2[a, b]$. Then the inner product of $f$ and $g$ is defined by
    $$< f, g > = \int_a^b f(x) \overline{g(x)} r(x) dx$$
    and the norm of $f$ is defined by
    $$\| f \| = \sqrt{< f, f >}.$$

162. **Theorem** $L_r^2[a, b]$ is a Hilbert space.

163. **Definition** (Regular S-L problem) Find scalar $\lambda$ and nontrivial function $y = y(x)$ defined on $[a, b]$ so that
    $$\begin{aligned} (p(x)y(x)')' + q(x)y(x) + \lambda r(x)y(x) &= 0, \ a < x < b \\ \alpha y(a) + \beta y'(a) &= 0 \\ \gamma y(b) + \delta y'(b) &= 0 \end{aligned}$$
    where

    - $-\infty < a < b < \infty$,
    - $p$, $p'$, $q$, and $r$ are real-valued and continuous on $[a, b]$,
    - $p, r > 0$ on $[a, b]$,
    - $\alpha$ and $\beta$ are not both zero,

- $\gamma$ and $\delta$ are not both zero.

The function $r$ is called the weight function.

164. **Theorem** (Regular S-L problem) For the regular S-L problem, the following hold.

- There are infinitely (countable) many eigenvalues.
- The eigenvalues are real.
- There is a smallest eigenvalue $\lambda_1$ and the eigenvalues can be numbered so that

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots \quad .$$

- $\lambda_n \to \infty$ as $n \to \infty$.
- The eigenfunctions are in the Hilbert space $L_r^2[a, b]$ with $r$ the weight function from the S-L problem.
- The subspace of eigenfunctions corresponding to eigenvalue $\lambda_n$ has dimension one and is the span of a real-valued eigenfunction $y_n$. The sequence $\{y_n\}$ is an orthogonal basis for $L_r^2[a, b]$.
- Eigenfunctions corresponding to different eigenvalues are orthogonal.

165. **Definition** (Generalized Fourier Series) Let $f \in L_r^2[a, b]$ and let $\{y_n\}$ denote an orthogonal basis of eigenfunctions generated by a S-L problem with weight function $r$. The series

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x) \quad \text{where} \quad c_n = \frac{< f, y_n >}{< y_n, y_n >}$$

is called a generalized Fourier series (GFS) for $f$ and

$$f_N(x) = \sum_{n=1}^{N} c_n y_n(x)$$

is called a partial sum of the GFS.

166. **Theorem** (GFS) Using the notation of the previous definition, the following hold.

- $< f, f > \geq \sum_{n=1}^{N} |c_n|^2 < y_n, y_n >$ (Bessel's inequality).
- $< f, f > = \sum_{n=1}^{\infty} |c_n|^2 < y_n, y_n >$ (Parseval's identity).
- $\|f - f_N\|^2 = \sum_{n=N+1}^{\infty} |c_n|^2 < y_n, y_n >$.
- $\|f - f_N\| \to 0$ as $N \to \infty$ (convergence in the mean).

167. **Definition** A function $f$ is called piecewise continuous on an interval $[a, b]$ if it is continuous on $[a, b]$ except for a finite number of points at which the limit from the left and the limit from the right exist, but are not equal. At a point $x \in (a, b)$, these limits are denoted by $f(x^-)$ and $f(x^+)$.

168. **Theorem** (Pointwise Convergence of the GFS) If $f$ and $f'$ are piecewise continuous on $[a, b]$, the GFS converges to

$$\frac{f(x^+) + f(x^-)}{2}$$

at each $x \in (a, b)$.

169. **Theorem** (Uniform Convergence of the GFS) If $f$ is continuous and $f'$ is piecewise continuous on $[a, b]$ and if $f$ satisfies the boundary conditions of the S-L problem, then the GFS converges uniformly.

170. Examples of pointwise and uniform convergence. The Gibbs phenomenon. Behaviour of $c_n$ as $n \to \infty$.

171. Heat conduction in a rod insulated at the right end and with temperature set to 0 at the left end. No source term in the heat equation.

172. Heat conduction in a rod with Newton's law of cooling applied at the right end and the temperature set to 0 at the left end. No source term in the heat equation. Graphical analysis of the existence of eigenvalues.

173. Five homework problems involving heat conduction in a rod with various BC's and the five corresponding S-L problems.

174. 1-D wave equation. Model for a violin string. Homogeneous PDE, BC's, and initial velocity, nonhomogeneous initial displacement. A homework problem looks at the case of homogeneous initial displacement and nonhomogeneous initial velocity.

175. The Superposition Principle for linear equations. Decomposition of the full problem into subproblems.

176. Steady state heat conduction. Laplace's equation in a rectangular plate.

177. Steady state heat conduction. The circular disk. The periodic S-L problem. Review of Cauchy-Euler equations from sophomore differential equations.

178. Discussion of the periodic S-L problem which yields the classic Fourier (cosine-sine) series. The theory is the same as for the regular S-L problem except that the eigenspaces corresponding to the positive eigenvalues have dimension two.

179. (Optional) Separation of Variables: 2-D heat conduction in a rectangular plate.

180. (Optional) Double Generalized Fourier Series.

181. (Optional) Heat conduction in a circular plate. Bessel's equation. A singular S-L problem and associated theory. The Bessel-Fourier series (BFS).