These are brief notes for the lecture on Monday August 31, 2009: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

1.7. Linear dependence and independence

Recall from last time: the vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ are said to be
— linearly dependent if we can express $\underline{0}$ as a non-trivial linear combination

$$\underline{0} = c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + c_k\underline{v}_k$$

(that is, in which at least one c_j is not zero).
— linearly independent if $\underline{0}$ is only representable in the trivial way:

$$\underline{0} = 0\underline{v}_1 + 0\underline{v}_2 + \cdots + 0\underline{v}_k.$$

Equivalent to this: $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ is
— linearly independent if *every* vector in $\mathrm{Span}(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k)$ can be represented in exactly one way
— linearly dependent if *every* vector in $\mathrm{Span}(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k)$ can be represented in at least two ways.

DEFINITION. *If we have a set of vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ which spans $\mathbb{R}^n$, and which is linearly independent, we call the set a* basis.

A basis enables us to represent every vector in $\mathbb{R}^n$ uniquely. It gives us a *finite* way to deal with every vector in the space at once: all infinitely many of them. Note: the book doesn't actually introduce the concept of a basis until much later. Regard this as useful foreshadowing.

1.8. Linear Transformations

We are going to talk about special kinds of functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ (note: n comes before m here!)
First: what is a function? We write $f : A \longrightarrow B$ to denote that f is a function from the set A to the set B. What this means is that f is a rule, which assigns, for each element $a \in A$, a unique element $b \in B$ so that $b = f(a)$. We refer to A as the domain of f, and to B as the co-domain. The set

$$\{f(a) : a \in A\}$$

of values taken by the function is called the *image* or *range* of f.

However, this concept is too broad: there are lots and lots of functions (if A is a finite set, with n elements, and B is a finite set, with m elements, then there are already m^n functions from A to B: if A or B is $\mathbb{R}$, or $\mathbb{R}^n$ then things are much much more complicated!) When we think of functions on the reals, we tend to think of functions which have nice graphs. For $\mathbb{R}^n$, and important class of functions is the set of *linear transformations*. These are the functions which interact really nicely with vector addition and scalar multiplication:

DEFINITION. *A* linear transformation *from* $\mathbb{R}^n$ *to* $\mathbb{R}^m$ *is a function* $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *which satisfies the following two properties: whenever* $\underline{u}, \underline{v} \in \mathbb{R}^n$ *and* $c \in \mathbb{R}$*, then*

(1) $T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v})$
(2) $T(c\underline{u}) = cT(\underline{u})$.

Note: We've already encountered examples of this type of function! If A is a $m \times n$ matrix, then we can multiply a vector in $\underline{v} \in \mathbb{R}^n$ by A, obtaining a vector $A\underline{v} \in \mathbb{R}^m$. Hence multiplication by A is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. And we've seen that it satisfies both conditions, and hence it is a linear transformation.
Example: Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 5 & -1 \end{pmatrix}$$

Define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by $T(\underline{x}) = A\underline{x}$.
1. $T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) =$

2. Find all $\underline{x} \in \mathbb{R}^2$ so that $T(\underline{x} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$.

Example: Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by $T(\underline{x}) = A\underline{x}$ is called a projection:

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$$

Example: Shear transformation: $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

Note: If T is a linear transformation, then

(1) $T(\underline{0} = \underline{0}$.

(2) $T(c\underline{u} + d\underline{v}) = cT(\underline{u}) + dT(\underline{v})$.

(3) $T\left(\sum_{i=1}^{p} c_i \underline{v}_i\right) = \sum_{i=1}^{p} c_i T(\underline{v}_i)$

Proof:

Example: define a map $T_r : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $T\underline{x} = r\underline{x}$. If $0 < r < 1$, the map is called a *contraction*. If $r > 1$ it is called a dilation. Show that T_r is a linear transformation.