These are brief notes for the lecture on Wednesday October 7, 2009: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

4.2. Null Spaces, Column Spaces and Linear Transformations, continued

We now extend the definition of linear transformations to cover all vector spaces.

DEFINITION. *Suppose that U and V are vector spaces. A transformation $T : U \longrightarrow V$ is said to be linear if*

(1) *For all $\underline{u}, \underline{v} \in U$, $T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v})$*

(2) *For all $\underline{u} \in U$ and $c \in \mathbb{R}$, $T(c\underline{u}) = cT(\underline{u})$.*

We also extend the definition of the null space and the column space of a matrix to a general linear transformation.

DEFINITION. *Let U and V be vector spaces, and let $T : U \longrightarrow V$ be a linear transformation. The* kernel *of T is*

$$ker(T) := \{\underline{u} \in U \ : \ T(\underline{u}) = \underline{0}\}.$$

The range *of T is*

$$range(T) = im(T) := \{T(\underline{u}) \ : \ \underline{u} \in U\}.$$

Fact: Given a linear transformation $T : U \longrightarrow V$,

(1) $\ker(T) < U$.

(2) $\mathrm{im}(T) < W$.

Proof:

4.3. Linear Independence in Vector Spaces; Bases

Just as in the case of $\mathbb{R}^n$, we can define linear dependence and linear independence of sets of vectors.

DEFINITION. *Let V be a vector space and let $\underline{v}_1.\underline{v}_2, \dots, \underline{v}_p \in V$. If the only solution to the equation $x_1\underline{v}_1 + x_2\underline{v}_2 + x_p\underline{v}_p = \underline{0}$ is the trivial solution $x_1 = x_2 = \cdots = x_p = 0$ then the vectors $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p$ are said to be* linearly independent.

If there is *a non-trivial solution to the equation (i.e. one for which some of the x_j's are non-zero) then the vectors are said to be* linearly dependent.

THEOREM 4. *Let V be a vector space. Suppose that $\underline{v}_1, \underline{v}_2, \dots \underline{v}_p \in V$, $p \geq 2$, and that $\underline{v}_1 \neq \underline{0}$. Then $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p$ are linearly dependent if and only if there is a $j \leq p$ so that $\underline{v}_j$ is a linear combination of the vectors $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_{j-1}$.*

Example: Consider the vector space $\mathbb{P}_3$. Let

$$S = \{x^2 + 2x + 3,\ x^3 + 1,\ x^3 + 2x^2 + 4x + 7\}.$$

$S \subseteq \mathbb{P}_3$: is it linearly dependent or linearly independent?

Example: Consider the vector space

$$V = \{f : [0, 1] \longrightarrow \mathbb{R} \text{ so that } f \text{ is continuous}\}$$

and let $S = \{\sin(x), \cos(x)\} \subseteq V$. Is this set linearly dependent or linearly independent?

DEFINITION. *Suppose that V is a vector space, and that $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p \in V$. This sequence of vectors is a* basis *for V if it is linearly independent and spans V. A basis for a subspace $H < V$ is a sequence of vectors in H which is linearly independent and spans H.*

Note: often a *set* of vectors is described as a basis, but then the discussion uses that $\underline{b}_1$ is the first vector in the set, $\underline{b}_2$ is the second vector in the set, etc. This makes it slightly better to refer to a sequence of vectors as being a basis (or as the book puts it, an "indexed set", which is non-standard terminology). For our purposes, we will allow a basis to be either a sequence or a set as is most convenient.

Example: Let $A = [\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n]$ be an invertible $n \times n$ matrix. What does the Invertible Matrix Theorem say about $\{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\}$?

Example: $\{\underline{e}_1, \underline{e}_2, \dots, \underline{e}_n\}$ is a basis for $\mathbb{R}^n$. Why?

Example: $\{1, x, x^2, \dots, x^n\}$ is a basis for $\mathbb{P}_n$. Why?

Example: Let

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \underline{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

Show that $\mathrm{Span}(\underline{v}_1, \underline{v}_2, \underline{v}_3) = \mathrm{Span}(\underline{v}_1, \underline{v}_2)$.

THEOREM 5. *Let V be a vector space, and $S = \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p\} \subseteq V$, and let $H = Span(\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p)$.*

(1) *If there exists k so that $\underline{v}_k$ is a linear combination of the other vectors in S, then*
$$H = Span(\underline{v}_1, \underline{v}_2, \dots, \underline{v}_{k-1}, \underline{v}_{k+1}, \dots, \underline{v}_p)$$

(2) *If $H \neq \{\underline{0}\}$ then some subset of S is a basis for H.*

Proof:

(1) Suppose that $\underline{u} \in H$. We need to show that $\underline{u}$ is a linear combination of vectors in $\underline{v}_1, \dots, \underline{v}_{k-1}, \underline{v}_{k+1} \dots, \underline{v}_n$. We know that it is a linear combination of vectors in S.

(2) If S is linearly independent, then it is a basis. Otherwise, there is a non-trivial linear combination of vectors in S giving $\underline{0}$, and hence there is some vector in S which can be written as a linear combination of the others. Hence we can replace S by a smaller set S' which still spans H. Clearly we can continue this process, and it has to stop either with $S' = \emptyset$ (in which case $H = \{\underline{0}\}$) or with S' a linearly independent set spanning H, and hence a basis for H.

Bases for Nul A and Col A

We already have seen how to find a basis for Nul A: row reduce A to obtain a matrix in reduced row echelon form and use this to express the null space in vector parametric form. The vectors appearing will be the basis for Nul A.

Example: Let

$$B = \begin{pmatrix} 1 & 3 & 0 & 2 & 9 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Find a basis for Nul B.

For the same matrix B, find a basis for Col B.

Fact: If $A \sim B$, then the linear dependencies of the columns of A are exactly the same as the linear dependencies of the columns of B.

Theorem 6. *The pivot columns of a matrix A form a basis for Col A.*