These are brief notes for the lecture on Friday September 3, 2010: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

1.9. The Matrix of a Linear Transformation

THEOREM 11. *A linear transformation* $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *is 1-1 if and only if* $T\underline{x} = \underline{0}$ *has only the trivial solution.*

Proof:

THEOREM 12. *Let* $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *be a linear transformation and let* A *be its matrix.*

(1) T *is onto if and only if the span of the columns of* A *is all of* $\mathbb{R}^m$.
(2) T *is 1-1 if and only if the columns of* A *are linearly independent.*

Proof:

As a consequence of this theorem we can check whether T is 1-1 or onto by row-reducing its matrix A: if there is a pivot in every row, then T is onto: if there is a pivot in every column, then T is 1-1.

Example: Let $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{pmatrix}$. Is T 1-1, onto?

Let T be a linear transformation from $\mathbb{R}^n \longrightarrow \mathbb{R}^m$. We can immediately determine information about T if we know how large m and n are.

(1) If $m < n$ then it is impossible for T to be 1-1. So, for example, if T is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$, then there must be a vector $\underline{b}$ so that there are at least two (and hence infinitely many) different solutions to $T(\underline{x} = \underline{b}$.

(2) If $m > n$ then it is impossible for T to be onto. So, for example, if T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$, then there is at least one $\underline{b}$ so that the equation $T(\underline{x}) = \underline{b}$ has no solutions.

(3) If $m = n$ then we can't come to any conclusions without some more work.

Counting rows, columns and pivots: a little foreshadowing

Suppose that A is an $m \times n$ matrix, and that after we put it into reduced row echelon form, there are exactly p rows with pivots in. Then we also know that there are exactly p columns with pivots in.

So, if we are trying to solve the matrix equation $A\underline{x} = \underline{0}$, we see that we'll get p basic variables, and $n - p$ free variables.

The fact that there are p columns with pivots on is going to mean that if we look at the image of $\mathbb{R}^n$ under T, that is, the set of vectors that $\mathbb{R}^n$ gets mapped to, it is going to be p-dimensional (whatever that means).

Example: